Chapter 1

Certainty in Mathematics

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I wished to believe that some knowledge is certain and I thought that the best hope of finding certain knowledge was in mathematics.¹

The kind of certainty is the kind of language-game.²

A traditional question in philosophy concerns what, if anything, we can know with certainty. The only field of knowledge, of which there has been some rough consensus that it actually constitutes certain knowledge, is mathematics. Even though some philosophers have preferred not to limit the range of certainty to mathematics alone, this discipline is often put up as an ideal: “the paragon of truth and certitude”, as David Hilbert put it.³ A natural follow-up question is “Why is mathematical knowledge certain?” Another perspective that may shed as much light on the question as the investigation of the question itself is to ask why certainty has such a strong appeal to us.

In philosophy, one speaks of “certain knowledge” and about mathematical knowledge being “certain”, or one speaks simply about the certainty of mathematics. What does one mean by such expressions? An apparent problem with a claim to the effect

that mathematics or mathematical knowledge is certain is that we hardly ever have occasion or reason to say it – outside philosophy that is. I would like to suggest that it is not clear how one should understand “certainty” here. Even so, one feels that the claim that mathematics is certain is justified, and one feels that it makes sense, as is illustrated by Norbert Weiner in 1915 – “[t]he place most people would look for absolute certainty is in pure mathematics or logic.” Still, he too has “become somewhat suspicious of the absolute certainty of mathematics through hearing it continually dwelt upon.” But instead of questioning the meaning of the phrase, he frames his critical point in a sceptical question: “Is, then, mathematics absolutely certain?”\(^4\) In this respect John Stuart Mill, although opting for the sceptical alternative in the end, was better equipped to deal with the problem since he at least asked himself:

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\text{[W]herein lies the peculiar certainty always ascribed to the sciences which are entirely, or almost entirely, deductive? Why are they called the Exact Sciences? Why are mathematical certainty, and the evidence of demonstration, common phrases to express the very highest degree of assurance attainable by reason? Why are mathematics by almost all philosophers . . . characterized as systems of Necessary Truth?}^5
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The questions I have taken as my task to clarify are the same ones already posed by Mill in 1843. I have to try to see, first, How one should understand mathematical certainty, and second, which characteristics of our dealings with mathematics and the world around us that prompt the claim that mathematics is certain.

**Approaching the problem**

In philosophical texts, in contrast to everyday discussions, one finds the matter of certainty discussed frequently. Even though it is seldom pointed out explicitly, the discussion is usually situated within a certain aspect of mathematics: be it a discussion of the reliability of mathematical methods, the truth of mathematical propositions, the relation between mathematics and other disciplines, etc. In assessing certainty’s place in mathematics, these differences in emphasis of the discussion must be kept in mind.


The starting point in dealing with questions like Mill’s, have usually taken the form of a search for some quality of mathematics, of mathematical knowledge, or propositions, that will supposedly explain why mathematics is certain or show that this indeed is the case. Such a quality is often located in the nature of mathematical propositions or in the nature of some kind of mathematical objects. One can think of the logicist thesis that true mathematical propositions rest solely on logic, or on the standpoint of Hilbert’s in “The New Grounding of Mathematics” that “the objects of number theory are . . . the signs themselves, whose shape can be generally and certainly recognized by us”.  

Searching for some quality in order to show that mathematics is certain is a natural enough way to approach the problem if there seems to arise a possibility of doubt. Arguably, this was an ingredient in the research into the foundations of mathematics set off by the discovery of the antinomies in set theory and logic in the beginning of the twentieth century. Paradoxes, such as Russell’s, upset the mathematically minded philosophers and philosophically minded mathematicians. Be that as it may, if one is searching for some kind of quality, how does one come to know that it has this quality, and what does its having this quality entail for those dealing with mathematics? Furthermore, how does it affect the issue of certainty? If, as I am suggesting, the understanding of certainty with regard to mathematics is not sufficiently clear, will not a project whose aim is to establish such certainty suffer from this unclarity?

Another problem is that searching for some quality or other of, e.g. mathematical objects, may be an expression, not so much to an observation or insight into the nature of mathematics, as to a conclusion. One must take care that the motivation for the investigation is not a result of an inference from the premise that mathematics is certain paired with the assumption that this is in need of a justification. Hilbert’s remark imme-

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6David Hilbert. “The New Grounding of Mathematics: First Report”. In: From Brouwer To Hilbert. The Debate on the Foundations of Mathematics in the 1920s. Ed. by Paolo Mancosu. Oxford & New York: Oxford University Press, 1998, p. 202. Of course, according to Hilbert the certainty of mathematics is not secured until the consistency of analysis has been proven, but the starting point that Hilbert thinks will enable him to prove this – without using the problematic notions of analysis that call for a proof in the first place – is found in the simple signs. Hilbert’s thinking on this subject is complex and contains many valuable insights. In his attempt to prove consistency by using reliable methods one can indeed distinguish two different conceptions of certainty. One is the idea of a mathematical theory being consistent, and the other concerns his idea that some of our mathematical methods (the ones he calls finitary) are completely certain. For Hilbert, the certainty of these methods is, in turn, grounded in the surveyability of the simple signs mentioned in the quote. I will discuss both of these conceptions in this chapter, and I think Hilbert, in stressing the surveyability of the simple signs is onto something important, but his tendency to view this as an explanation the possibility of certainty seems to me to give it the wrong place. I will return to the issue of surveyability in subsequent chapters.
Immediately before declaring his point of view suggests as much: “If logical inference is to be certain, then these objects must . . .” The situation is reminiscent of Immanuel Kant’s inference to the existence of das Ding an sich; “. . . otherwise we should be landed in the absurd conclusion that there can be appearance without anything that appears.” This is a common enough situation in philosophy: One feels thoroughly confident that one has understood what the problem demands and thinks that one sees what kind of solution is needed. Yet, “confident” sounds as if one has made up one’s mind as to the nature of the problem, whereas one might simply have taken for granted an understanding of the issue it that comes naturally to oneself. This understanding in turn appears to force one to a certain kind of solution, regardless of the fact that it may sometimes seem odd.

Ludwig Wittgenstein has captured this tension:

“But this isn’t how it is!” – we say. “Yet this is how it has to be!”

“But this is how it is—” I say to myself over and over again. I feel as though, if only I could fix my gaze absolutely sharp on this fact, get it in focus, I must grasp the essence of the matter.

If I were a fisherman and were very dependent on the weather conditions for catching fish, and if I had noted many times that there was a change in the weather shortly after I saw cirrus clouds in the sky, I would soon start wondering whether there is some kind of connection between the clouds and the wind and the rain. I might start wondering if I should take measures when I see cirrus clouds in the evening. In this case it is clear what kinds of phenomena I want to explain and connect to each other. This is not so when I wish to understand why mathematics gives me certain knowledge. I have, to begin with, no clear grasp of the notion of certainty here, and I have furthermore no clear-cut idea of how the understanding I might gain will affect how I deal with mathematics or mathematical results.

To begin with it is worth distinguishing between expressions such as “I am certain of . . .”; “It is certainly . . .”; “It is certain that . . .”; and “. . . is certain”. With regard to this kind of phrases, Norman Malcolm remarks that “it is an individual who asserts that something is certain. If I am certain about the truth of something it is I who am certain.” As Malcolm points out it is always someone who claims to be certain of something or that something is certain. While it is true that it is always somebody who makes

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a claim to certainty, there seems to me to be a significant difference between claiming that I am certain of something and claiming that something is certain. The latter claim, so to speak, looks past my feeling of conviction (or perhaps: makes a gesture past my conviction). Perhaps one could say that it concerns the justifications one could have for being certain. Claiming “X is certain” can be a way of underlining the soundness of ones convictions, perhaps in the face of doubt. [example?]

With regard to the certainty of mathematics, I would describe a (my) spontaneous approach in the following way. When one takes into consideration the above distinction, the certainty of mathematics is, of course, not somebody's personal conviction or feelings of persuasion, but perhaps something like the interpersonal grounds for being convinced. One wants to say that mathematics is certain and that this has nothing with me and my opinions to do. Indeed, one cannot chose how one calculates (whatever that could be), but what the positive claim amounts to is still not clear.

Another important question which gives an indication of this unclarity is that it is not clear what opposite alternative would be. What it would mean for mathematics not to be certain? Especially if one – as were the logicists and formalists in the beginning of the twentieth century – is in the business of trying to prove that mathematics is indeed reliable and certain one would do well to consider what kind of possibility one is thereby disproving, and what consequences this opposite alternative would have. It think it is interesting, and telling, that there does not even seem to be no suitable word to describe this “not-certainty” of mathematics.

### 1.1 The infallibility of mathematical methods

I shall now examine some suggestions that perhaps could constitute a plausible understanding of the certainty in mathematics. Regardless of whether they, in the end, will be reasonable ways of understanding mathematical certainty or not, the suggestions may still point to features of mathematics that make it seem meaningful and justified to say that mathematics is certain. The first suggestion is:

*By the certainty of mathematical knowledge one means the impossibility of reaching erroneous results when using mathematical methods.*

Actually, this may be seen as two slightly different suggestions, the first being that mathematical methods are infallible when applied in science, trade, construction, etc. The
second is that the results of calculations and deductions in mathematics are always correct.

What then about the impossibility of reaching erroneous results when applying mathematics to empirical matters? It is true that mathematical tools are useful in a great deal of different situations and this is in itself an interesting thing to investigate. The advantages are of many different kinds: sometimes one saves time, sometimes the probability of making mistakes is smaller, sometimes a greater degree of accuracy is obtained, and sometimes there simply is no other alternative to using a mathematical method in order to solve a problem. The infallibility under scrutiny here has, in contrast to the more practical benefits mentioned above, an absolute air to it. One wants to say that mathematics, in contrast to other ways of finding a value (approximation, direct measurement, etc.), gives the correct result, whereas other methods give a more or less accurate estimate. For practical purposes, however, the result of a calculation will be no better than the in-data. If the calculation proceeds from some simple measurements, the inaccuracies of these values are reflected in the result. This is, of course, not considered a shortcoming of the mathematical method, which may be taken to say only: “If this is the correct value (of the length, weight, amount, voltage) then this is the result.”

This infallibility is not, however, the same thing as a total absence of erroneous results. One must not forget that one has to choose carefully which mathematical method to use in order to get the desired results. The wrong method will, if applicable at all, give incorrect results. Sometimes one has to adjust the details of an existing method to adapt it to a new problem, to make it give us the right results. A mathematical method does not by itself give either correct or incorrect results with regard to applications.

The supposed impossibility of erroneous results appears in a different light if one considers the fact that we do not use methods that do not work, that we have no use for, and we improve the ones that do work (in the sense that we alter them so as to allow us to take into account more and more relevant data, if, that is, there is room for improvement).

There is also the other proviso: the method will give a correct result if one has calculated correctly. As in the case of inaccuracies in measurements, the possibility of faulty results due to mistakes on part of the one doing the calculations is not considered a flaw of the mathematical method. It does, however, show that correct results do not emerge by themselves once a mathematical method is employed.
It is also worth asking if this infallibility of mathematics is not connected with the fact that we often use basic mathematics as, so to speak, a measure for ordinary experience. If measurements happen to diverge from what one has predicted through calculation, this kind of discrepancy is usually attributed to sloppiness on part of the one who has measured, or to some feature of the thing measured that is not taken into account in making the predictions, e.g. thermal expansion of the object measured. The discrepancies, unless they are taken to show that this particular mathematical method was inappropriate for the application, will not be assumed to be caused by flaws in the rules of arithmetic (whatever that means). Rather, a major incentive to look for inaccuracies in measurements is the occurrence of a discrepancy between the calculation and the measurement. The calculation is, so to speak, taken as the norm. It guides one in the search for errors in other places. In Language, Truth and Logic, A. J. Ayer, discusses an example where one has five pairs of objects but only nine objects when one counts them one by one. With regard to the attempts to come to terms with this disagreement, he remarks that

[one would say that I was wrong in supposing that there were five pairs of objects to start with, or that one of the objects had been taken away while was counting, or that two of them had coalesced, or that I had counted wrongly. One would adopt as an explanation whatever empirical hypothesis fitted in best with the accredited facts. The one explanation which would in no circumstances be adopted is that ten is not always the product of two and five.]

Ayer sees this as evidence for his claim that mathematical truths cannot be refuted by experience. Our reactions in this kind of case show that mathematical propositions are not analogous to empirical considerations, but in addition to this, I think that Ayer’s observation reveals this difference is connected with the normative character of mathematical propositions. Thus, it might be worth considering if the infallibility of mathematical methods is not a consequence of our judging the world of experience through our mathematical methods. Cora Diamond, in her article “Wittgenstein, mathematics and ethics: Resisting the attractions of realism” gives contrasts mathematics with practices where descriptive propositions are made by emphasising the normative character of mathematics:

Mathematics is integrated into the body of standards for carrying out methods of arriving at descriptive propositions, for locating miscounts (for example), or mistakes or inaccuracies of measurement.\textsuperscript{12}

Most importantly however, my suspicion towards the expression “mathematical methods are infallible” is linked with my impression that we have no clear conception of what this infallibility allows us to avoid. As mentioned above, the methods of mathematics have all kinds of advantages in comparison with other ways of estimating a value, but the idea of absolute correctness leads on to expect that there is another kind of failure that is excluded by the mathematical methods. It is as if our mathematics could have been a mathematics where following the correct rules and procedures sometimes led to one result and sometimes to another – only our mathematics happens, luckily, not to be such.

When one says: “Mathematical methods cannot go wrong” – what is it that they cannot do? If one thinks about calculation, something “going wrong” might mean that a mistake is made. In measuring something it might mean that one reads the wrong number from, e.g., the calliper. We have in these cases a clear conception of what “going wrong” could be. But, it is as if the infallibility of mathematics ensured protection from some mystical kind of error. This error would be one where following the rules of calculation correctly led to wrong results. Now, what kind of situation is this? If one tries to apply a certain mathematical method and arrives at a result that is not usable or one that diverges strongly from what seems reasonable, one would probably assume that one made a mistake and try again. If one is successful the next time one would probably not think more about it. If, however, the same result occurs once more and one feels confident that no mistakes were made the conclusion would probably be that this method was inappropriate for the application. These outcomes are not yet examples of the mystical error that I am trying to make sense of. Assume now that one calculates once more, checks the steps thoroughly, and the new result is a different one. Assume furthermore that one also checks the first calculation and finds no mistakes and thus confirms the original, unusable result. This seems to be a description of a fallible mathematics – doing the same thing, following the same rules in the same way leading to different results, and this will never occur in our infallible mathematics. Nevertheless,

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this does not seem to be a genuine description. Is it not a criterion for “having done the same thing”, for “having followed the rules in the same way” in mathematics that one arrives at the same result? The two identical calculations would have to part ways at some particular step, be it the last one or some intermediate step. Having done the same thing at this particular step means writing the same thing. There seems to be no logical space for “going wrong” other than through deviation from the rules, i.e. through mistakes made by the calculator. Doing the same thing means getting the same result, otherwise one has not done the same thing.

Since mathematical methods do not guarantee that no mistakes are made, nor that the correct method is employed, the claim that mathematical methods are infallible seems rather to be a conflation of the normative character of mathematical results with regard to empirical data and the practical benefits of mathematical methods in comparison to measuring, approximating and guessing when solving problems.

What about the case of using mathematical methods generally? This becomes the second suggestion

Mathematics will never produce an erroneous result.

Again, to this must be added the proviso: “given that one has calculated correctly”. David Hume, in *A Treatise of Human Nature*, argues that “[i]n all demonstrative sciences the rules are certain and infallible”, but he seizes on “our fallible and uncertain faculties” and concludes that we “are very apt to depart from them and fall into error.”\(^{13}\) This possibility means for Hume that mathematical knowledge is after all not certain. It is obvious that wrong results will ensue if the rules are not followed correctly. Even so, this is not considered a flaw in mathematics, but rather ascribed the human proneness to errors.

This suggests that mathematics – almost as a mechanism – produces results in a causal manner.\(^{14}\) However, if there is any production going on in mathematics it is what mathematicians and others produce when they do mathematics, what they write on paper or screen, what they say to each other etc. Yet, it is clear that when one produces results one occasionally makes mistakes. With the added clause one is left with: “Mathematicians will not produce erroneous results when they calculate correctly.” This could,

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\(^{14}\)The machine analogy will be discussed from the perspective of an other suggestion below, cf. section 1.3.
at best, be understood as an explanation of what it means to calculate correctly. It is in any case not surrounded by the mystical air that the original suggestion was.

This suggestion faces similar problems as the previous one. It lives, so to speak, in the fear of a kind of error that was shown above not to be meaningful: what would it look like if a mathematical method suddenly produced an erroneous result? Do we have a clear grasp of what mathematics does not produce if it produces only correct results? What I am trying to point out is that the notion of wrong result has no application in mathematics unless it is due to common mistakes, and the certainty of mathematics does not, as we are all aware of, allow us to avoid these.

If the subject is mathematics in general or pure mathematics, the idea of a correct result, in an absolute sense, is also delicate. Namely, the suggestion implies that there is some other standard for the correctness of the result than the calculation and the rules one has followed in completing it.

For a Platonist, this presents no problems at first sight. If mathematics produces truths that correspond to what can be truthfully said about the entities in the mathematical realm, then it produces correct results, otherwise incorrect results. Even so, the impossibility of somehow describing mathematical objects without actually doing mathematics, makes it highly questionable that one could establish true descriptive propositions against which to judge the ones reached through calculation or inference. As will, hopefully, be evident in the next chapter, the Platonist just as everybody else is left with the ordinary techniques of mathematics in order to determine what is correct and what is not. And, that this is no shortcoming. Sometimes the term intuition is brought in to compensate for the lack of a mathematical counterpart of sensory experience. Arguably, one develops an ability, a skill – usually spoken of as intuition – which allows one to judge what sounds plausible in mathematics, which method of proof, which rule, or what theorem one should rely on in order to solve some particular problem. This in-

\[15\] A Platonist as she or he is traditionally conceived at least. Whether there are any such Platonists is doubtful. Kurt Gödel is often put up as a paradigm Platonist, but I think Penelope Maddy has given good grounds for not labeling Gödel uncritically. Penelope Maddy. “Set Theoretic Naturalism”. In: The Journal of Symbolic Logic 61 (1996), 490-514, § 2.

\[16\] C.f. Kurt Gödel’s well-known defense of intuition as a kind of perception: “[T]he objects of transfinite set theory . . . clearly do not belong to the physical world . . . But, despite their remoteness from sense experience, we do have something like a perception also of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don’t see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception . . .” Kurt Gödel. “What is a Cantor’s continuum problem?” In: Philosophy of mathematics. Selected readings. Ed. by Paul Benacerraf and Hilary Putnam. 2nd ed. Cambridge: Cambridge University Press, 1983, p. 483-4.
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tuition allows one to judge, e.g., if the outcome of a calculation or if a purported proof seems reasonable. This is not, however, the same as judging if it is correct by means of this intuition. As William Tait – in “Truth and Proof: The Platonism of Mathematics” – points out: “What we call ‘mathematical intuition’, it seems to me, is not a criterion for correct usage.”\(^7\) In applied mathematics, the correspondence between results of calculations and measurements might make sense as an external criterion of correctness but this is not an option in pure mathematics. Proofs (taken in a wide sense, including, e.g, a proof that \(354 + 183 = 537\) by the addition algorithm) are the supreme courts in mathematics and if a proof is correct then what it proves is \textit{eo ipso} correct.

1.2 Eternal truths

The third suggestion is similar to the second:

\textit{Mathematical propositions are always true, they express eternal truths.}

To this suggestion must again be added the clause “provided that they are correct”. As in the former case we are left with a tautology. But there is more to this suggestion. It is frequently stated that mathematical truths, in contrast to other truths, are timeless and everlasting. In Plato’s \textit{The Republic}, Socrates expresses that “the knowledge at which geometry aims is knowledge of the eternal, and not of aught perishing and transient \ldots”\(^8\)

There are many ways of understanding this latter proposal. Importantly, it evokes precisely the kind of feeling needed to make the claim that mathematical knowledge is certain in the grandiose manner that philosophers often favour. The choice of the word “eternal” gives the suggestion a mystical air, and I doubt that this way of putting it can be separated from a will to confer a mystical air onto mathematics.

Still, there seems to be something to the idea that mathematics is not dependent on particular situations and the change of circumstances. I can e.g. use the same mathematical rule of calculation to draw conclusions about the amount of salt to add to the dough when baking bread on Saturday – “Sixteen grams of salt? But I’m doubling the recipe. Twice sixteen is thirty-two, so I’ll take thirty-two grams. I hope it won’t be too much” – as when on Monday predicting how many students I will have in my class when


\(^8\)Plato. \textit{The Republic}. New York: P F. Collier & Son, 1901, p. 527b.
my colleague suddenly catches a cold and asks me to teach both groups simultaneously – “On my course there are sixteen students and on hers there are as many. Twice sixteen is thirty-two, so we’ll be thirty-two persons in the seminar room if every one shows up. I wonder if there are enough chairs.” The validity of the inference based on a rule of calculation is not affected by the time of day, week or year.

In this respect mathematics is timeless. But, as Sören Stenlund notes, the use of “timeless” makes it sound “as though it made sense to talk about a mathematical fact as being a fact, before, after or simultaneous with something else, which it obviously does not. We do not ask questions like: ‘When did 2 + 2 = 4 become true and who made it true?’” One could compare this with the fact that issues of weight do not enter into my understanding of the answer “It’s four acres” to my question “How big is this field?” (Or perhaps with the fact that issues of cost does not enter into moral considerations, although some might disagree.) On the other hand, if I say: “It’s raining, you’d better bring an umbrella.” – it does matter when I say it, and it might be replied: “Oh, when did it start?” Consequently, the intuition that there is something to the claim that mathematics is “always true” would according to the above discussion not be misguided. There is, however, nothing about this understanding of non-temporality that implies the eternal existence of mathematical objects. Anyway, the claim that mathematical knowledge is certain does not seem to get a proper illumination by it.

Another way of understanding “eternal” here could be that the results of past mathematicians are as good today as they were thousands of years ago. In The Development of Mathematics, E. T. Bell remarks that “Euclid’s Propositions I, 47 stands, as it has stood for over 2,200 years. Under the proper assumptions it has been rigorously proved.” In view of the fact that many ideas of the ancient Greeks in e.g. astronomy have been revised since then, it may seem remarkable that their mathematics still stands. However, as Bell notes they have been proved “under the proper assumptions”. Considered as a theory of physical space the geometry of Euclid has been replaced by non-Euclidean geometry. Its timelessness is therefore restricted to the inferences from the assumptions stated in the axioms and postulates of Euclid, that is to Euclidean geometry considered as a system of pure mathematics. This is of course in line with the argument that mathematics is eternal, since it did not concern physical theories after all. However, that we

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still view the inferences of Euclid as valid can also be seen as an expression of a relative stability in the human life-form. We still reason in the same way as the ancient Greeks, or our reasoning is at least sufficiently close to theirs. We can also appreciate the philosophy of Plato and the plays of Sophocles. Sometimes it is indeed said that the plays of antiquity are timeless, and the mathematics of the Greeks is timeless in this sense too. I doubt that there is any stronger claims to be made as to the timelessness of mathematics, and I do not think mathematical certainty stands and falls with the possibility of finding a stronger sense of “timeless”.

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The fourth suggestion is that certainty in mathematics is to be understood as freedom from contradictions:

*Mathematics will never produce a contradiction.*

This suggestion is a natural one in face of the antinomies that arose in set theory in the late nineteenth and early twentieth centuries. The best known of these is the one called Russell’s paradox and it was discovered, independently of each other, by Ernst Zermelo and Bertrand Russell.\(^2\)

According to this conception it is not clear that one is entitled to claim that mathematical knowledge is certain after all, since antinomies did arise. One might, however, be “certain” and try to prove it as Hilbert and his followers did. Moreover, Kurt Gödel’s proof that consistency is impossible to prove, using Hilbert’s finitary methods, for large portions of mathematics, seems to entail that there can be no absolute certainty – in spite of the confidence that laymen and professional mathematicians alike feel towards the subject. Marcus Giaquinto’s *The Search for Certainty* might be said to give voice to

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this conception. He asks: “When we cannot be certain of the reliability, hence consistency, of some mathematics, can we none the less have a high degree of confidence in it?”

In his famous defense of Cantor’s paradise, Hilbert too seems to understand certainty along these lines. “[w]e must establish throughout mathematics the same certitude for our deductions as exists in ordinary elementary number theory, which no one doubts and where contradictions and paradoxes arise only through our own carelessness.”

What must be excluded are the contradictions and paradoxes, and the certitude that we find in elementary number theory (Hilbert’s finitary methods) seems in this passage to be due to the absence of contradictions in this region. However, it is also possible to view this in the light of the second suggestion. He identifies one region of mathematics, a set of methods which he considers to be completely reliable – these methods will not give rise to any troubles.

With regard to this fourth suggestion, and especially in light of the quote from Giaquinto, one can see a distinction between certainty in the sense of the confidence that I feel and an impossibility of reaching contradictory results. It is the latter that is important according to this fourth suggestion. The contrast is evident in fact that set-theoretical antinomies could be derived although there was a strong confidence in set theory among a large part of the mathematicians working around the turn of the twentieth century.

The “uncertainty” that the possibility of antinomies gives rise to is related to the impression that the antinomies in set theory were unexpected. One has the feeling that one might be doing mathematics in the ordinary fashion and – through the correct application of rules of inference or calculation – end up with contradictory results. Perhaps the result contradicts something one has arrived at at some other point, and perhaps one does not even notice the contradiction, and thereby invalidate one’s future work, by the principle of explosion. Maybe it is possible to imagine a situation where one would could complete a proof if one could only prove the existence of a function whose domain consists of functions that do not occur as elements in their own domains. Could one draw any conclusions about the existence of such a function when one cannot determine whether it is itself an element in its own domain?

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The disastrous effects of Russell’s paradox has sometimes been illustrated by the librarian who is left at a dead end in the task of completing two different catalogues of all the books in the library.\textsuperscript{24} The catalogues are themselves two separate books, the first listing all the books that contains at least one reference to their own titles, the second listing the books that lack such reference. The completion of the task would for the meticulous librarian be to include the two catalogues in one of the catalogues, making note of whether they make reference to their respective titles or not. The first catalogue is properly included in itself since this makes it into a book that makes reference to itself and then it belongs in the first catalogue. The second catalogue is another matter, however. If it is included in the first it ought to contain a reference to itself, otherwise it belongs in the second one, in itself, but if it is listed in the first there is no mention of the second catalogue in the second catalogue. Thus, an inclusion in the first catalogue excludes precisely that. But, if it is included in the second catalogue, then it contains a reference to itself and should properly be included in the first and not in the second, which contains only books that do not contain references to themselves.

At this point I would, however, wish to raise some critical points about the idea that mathematical certainty is about freedom from contradictions. The thought that mathematics will not leave one at a contradictory junction – whether the contradiction is obvious or hidden – is again an expression of the machine conception of mathematics. It seems to me that the idea that the occurrence of contradictions makes mathematical knowledge uncertain, lives within a fear that contradictions might, as it were, start popping up where one would not expect them to – and not because of an error on the part of human beings, but on part of mathematics itself. Georg Henrik von Wright expresses this uneasiness that the paradoxes gave rise to in \textit{Logik, filosofi och språk} (Logic, philosophy and language): “Could one be confident that one would not one day run into contradictions in arithmetic, algebra, or geometry too?”\textsuperscript{25} Uncertainty acquires an air of distrust of mathematics. It is as if mathematics, like a machine produces results and if certainty is found in mathematics one can trust it not to produce a contradiction. With this idea goes the further worry that one would not necessarily recognise a contradictory result or fail to realise that the result contradicts another that one is depending on.

Again it is worth pointing out that the ones who do the calculations are the only ones

\textsuperscript{24}Thanks to Martin Gustafsson who suggested using this example.

producing anything. As was mentioned earlier, these do occasionally produce erroneous or contradictory results. The usual response when this occurs is to go over the calculations once more to see if any errors were made. If none are found and the contradiction remains, a layman will perhaps ask a mathematician or somebody more skilled in mathematics for an explanation of the unexpected result. A mathematician might turn to colleagues for advice or conclude that a contradiction arises if one follows certain rules in a certain way and perhaps make this publicly known by publishing it. There is much prestige in finding a contradiction. It is then up to the community of mathematicians to decide what to do about it.

In the history of mathematics one finds many cases where mathematicians have had to deal with paradoxical situations. In the eighteenth century there were problems with the sums of infinite series, e.g. \(1 + 1 - 1 + 1 - 1 + \ldots\). Do they equal out and yield zero or do all but one equal out and yield one? A third alternative can be reached by considering the series \(1 - x + x^2 - x^3 + \ldots\) which by the division algorithm can be proven to be equal to equal to the fraction \(\frac{1}{1+x}\). If one in the equation:

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1 - x + x^2 - x^3 + \ldots = \frac{1}{1+x}
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substitutes 1 for \(x\) then one reaches the conclusion that the sum is equal to 1/2. In this case the solution was to stipulate criteria for the convergence of infinite series. The criteria are such as to allow only sums where no ambiguity arises. In the example discussed the radius of convergence is \(|x| < 1\), which means that the substitution which yielded the sum 1/2 gives rise to a divergent series.

Another case could perhaps be the exclusion of division by zero. In this case the solution was simply to exclude zero from the permissible divisors.

If one looks at this phenomenon from an “anthropological” point of view one can see that in mathematics one does not let a contradiction pass without attending to it somehow. When applying mathematics one will not know what to do when faced with a contradiction and in mathematics generally they are excluded from the permissible applications of the rules. Sometimes it takes a long time for the community to reach a satisfactory solution.

Now, the occurrence of contradictions, and the set-theoretical ones in particular, are seldom seen as a reason to abandon mathematics or even a part of it, as the malfunction-
1.3 Consistency

Assume that a machine produces strange results, a computer, e.g., that the same commands at one time gives one output, and at another a different output, when it is supposed to give the same output every time. There seems to be two kinds of problems to look for. Either a part of the machine is worn down or the machine was not properly constructed from the start. If the mathematical machine produces contradictions, there does not seem to be any possibility that the “parts” have broken, since mathematics is not a “real” machine made of physical parts. If it produces contradictions it must therefore be the case that the machine is somehow constructed wrongly. If the machine analogy is followed strictly—that is, not merely when one is thinking about the dangers of an inherent contradiction—the natural solution would be to abandon the machine.26 But of course only the intuitionists thought that a “reconstruction” was necessary. The majority of philosophers dealing with the foundations of mathematics were concerned instead to show that mathematics was in perfect order. Moreover, the majority of the ones who would actually be affected by the uncertainty brought in by the contradictions, the mathematicians, seems not to have bothered too much about it. It was an issue mainly for philosophers and philosophically minded mathematicians.27

In conclusion, one could say that the existence of contradictions does not seem to introduce uncertainty into mathematics, even though it does call for clarification. Maybe it would be a fair description of a good deal of the foundational work to call it clarificatory instead of foundational—although I suspect that many of those engaged in it were not themselves attentive to what kind of work they did. Many of them probably thought of their work as establishing certainty. I am also tempted to conjecture that what gave work on the foundations of mathematics much of its drive was an unclarity with regard to the nature of the problems. What urgency must not a project concerned with the

26A perhaps more serious problem with this picture is the nature of “wrong answer”. If I have a machine I usually have some idea of what kind of job it will accomplish for me if it functions adequately. “Giving out a result/product that it is supposed to” becomes meaningful in the light of the purpose of the machine. This purpose, is moreover, determined by matters external to the machine. The output of “the machine of mathematics”, on the other hand, cannot be judged in the same manner. True, the purpose of mathematics is, among other things application, but “the result it is supposed to give” cannot be understood extramathematically. Therefore, on the machine analogy one would simply have to accept whatever it pleases to give out; the assessing of the result is also a piece of mathematics and does therefore belong to the working of the machine.

27Haskell B. Curry comments on Hilbert’s insistence on consistency proofs that inconsistency does not make a theory useless for applications and “[e]ven if an inconsistency is discovered this does not mean complete abandonment of the theory, but is modification and refinement.” Haskell B. Curry. “Remarks on the Definition and Nature of Mathematics”. In: Philosophy of mathematics. Selected readings. Ed. by Paul Benacerraf and Hilary Putnam. 2nd ed. Cambridge: Cambridge University Press, 1983, p. 206
In the heyday of foundational research, Weiner described the attitude towards Gottlob Frege’s definition of the concept of number:

The average mathematician neither knows, nor, I grieve to say, cares, what a number is. You may say if you like that his analysis is blunted and his work rendered unrigorous by this deficiency, but the fact remains that not only can he attain to a very great degree of comprehension of his subject, but he can make advances in it, and discover mathematical laws previously unknown.28

If we are to understand anything by the certainty of mathematical knowledge, it cannot be freedom from contradictions.

1.4 Deductive vs. empirical sciences

The claim that mathematics is certain can, of course, be a simple reminder that there is a difference between sciences. Perhaps one wants to say that whereas in empirical sciences there is an inherent uncertainty, in mathematics we can have complete certainty. Sometimes this difference is underlined with a reference to proof. The fact that in mathematics we prove things deductively sets it apart from other disciplines. Stewart Shapiro, e.g., points to this difference: “Unlike science, mathematics proceeds via proof. A successful, correct proof eliminates all rational doubt, not just all reasonable doubt.29 What kind of difference this is is thereby not clear, nor what the nature of the certainty found in mathematics. A problem with this comparison is that, while I think it is in some sense correct, it still invites the idea that the difference in certainty is one of degree. This could be seen in the first and second suggestions too. It is as if one had a clear grasp of what certainty is, and that it is attainable only partially in the science, while it is attainable in full in mathematics, and this is what sets mathematics apart from science.

In the remaining section of this chapter I will try to show that the difference between certainty in mathematics is not one of degree but of kind. As a result, I hope that it will be clear in what way the suggestions discussed above try to capture different aspects of this certainty.

29Stewart Shapiro. Thinking about mathematics. The philosophy of mathematics. Oxford & New York: Oxford University Press, 2000, p. 22. I do not know what, according to Shapiro, the difference is between rational and reasonable doubt, and it still remains to be investigated what it is about proofs that allow for this. The relation between proof, doubt and conviction will be dealt with in chapters ?? and ??.
1.5 The status of elementary arithmetic

The remainder of this chapter will discuss something of a gut reaction of, I guess anybody familiar with basic mathematics. Someone not familiar with philosophical discussions on this subject may perhaps react to the question of certainty in mathematics by exclaiming that such things as “2 + 2 = 4” are always indubitably and absolutely true.\(^{30}\)

An obvious problem with this spontaneous reaction is that it is as unclear as the whole question of mathematical knowledge being certain. If not as a reaction to a philosophical challenge I doubt that one would ever have the occasion to claim that “2 + 2 = 4” is indubitable. Perhaps if somebody questions my claim that I have finished my dissertation, knowing that I hardly ever manage to finish any of my projects, I may reply as an extravagant gesture: “It is as certain as ‘2 + 2 = 4’.”\(^{31}\) However, the certainty that “2 + 2 = 4” is supposed to pass onto my claim that I have finished my dissertation may well suffer from the comparison. If I have reason to try to convince somebody – what I try to convince her of is, naturally, not self-evident. Besides, this way of speaking does not tell me anything about the character of the certainty, in virtue of which, the mathematical proposition has a rhetorical effect in this case.

Now if one is considering a more complex mathematical proposition, the truth or falsity of which is not immediately seen, one might perhaps make sense of this suggestion. Imagine two builders who are building a cottage. One of them, a man of mathematical bent, insists on using the theorem of Pythagoras to calculate the length of the slanting beam on the roof in order to get a steep enough roof to let the snow slide off of its own weight in the winter.

“Well, according to the calculation, we have to make the slanting beams at least 4.8 metres long otherwise they won’t reach up to the ridge that rises two metres above the outer walls.”

“Are you sure?” asks the other – not too sure of his capacity to judge the calculations.

“Yes, I’m quite certain.”

The other might be satisfied with his assurance and they go along, sawing off the

\(^{30}\)Sometimes one hears the claim that mathematics is just something we have invented and that it is therefore not certain either. I will not deal with this claim here since I am interested in the meaning of the claim that mathematical knowledge is certain.

\(^{31}\)I am not sure if the example is realistic, but [at least in Swedish] one does say “It is as certain as the fact that my name is . . . ” when trying to persuade someone of something, which one finds it important that the other person acknowledges.
planks at 4.8 metres. But, the other builder might not be happy with this assurance, because, say, the one doing the calculations made a mistake last time and the beams were too short and they had to discard some. He then insists:

“Are you really sure? I’d like to get it right this time!”

The first one now slightly annoyed at this questioning of his abilities, of which he is quite proud, answers:

“It’s perfectly certain that 4.8 metres is the right measure. I double-checked the calculations, they’re just fine.”

If there in this case is a difference between “I’m certain . . . ” and “It’s certain that . . . ”, as was discussed in the beginning of this chapter, it has to do with the urgency of the will to silence the doubts of the other. Furthermore, when as in this case a meaningful use of the phrases is considered, the meaningfulness of making claims to certainty is bound up with the fact that there were quite intelligible reasons to doubt the correctness of the mathematical result. And, the threat to correctness is in this case the possibility that somebody made a mistake, which makes the claim to certainty into claim about the certainty of having followed the rules correctly. This seems like a very ordinary kind of certainty and not one that somebody interested in the enigmatic certainty of mathematics might search for. Furthermore, this claim to certainty brings us further away from the suggestion, which turned on the indubitability of a very simple mathematical proposition like “2 + 2 = 4”. If certainty or indubitability is brought in when speaking of such simple rules of calculation, if I tried to tell somebody: “‘2 + 2 = 4’ is certain”, the reaction would perhaps be: “Why do you say that?”

It might, then, be thought that the certainty of mathematical knowledge lies in the fact that this and other simple rules of calculation are impossible to doubt. But, if this is indeed a logical impossibility, and not merely a psychological one, “being convinced” seems equally out of place. What I mean by logical impossibility is that it does not enter into the range of possibilities that I could doubt them, but also: one would not recognise any kind of behaviour as a doubt concerning, e.g., 2 + 2 = 4. [Någon typ av referens till Om visshet behövs här.]

Let me try to illustrate this by exploring the example further. As they try to determine the length of the slanting beam the first time the beam is too short. The theorem of

\[32\text{Interestingly, the reaction would probably be similar if one instead of the true proposition said: ‘‘2 + 2 = 5’ is certain.” If I insist: ‘Because it is certain, absolutely certain’, the other might well shake her head and think I’m trying to be funny or respond: ‘Well, has anybody doubted it?”} \]
Pythagoras will tell them how long to make it when they know the distance between the outer walls and the ridge, and the height of the ridge compared to the walls. These two distances will form the legs, while the beam forms the hypothenuse of a right triangle. Now they double-check the calculations but they get differing results a couple of times. When they finally settle for what they think is the right answer, the beam is too long. Now they just shorten it somewhat and use this one as a measure for the rest and leave the calculations altogether. When trying to fill out the details of their discussion as they tried to come to terms with their difficulties several alternatives are conceivable. First they will probably assume that some mistake was made in the calculations and go over them again to see if they can spot the error. If no errors are found in the calculations they might start wondering if they perhaps measured wrongly and therefore got faulty input data in the equation, so they measure once more. If they still can find no explanation as to why the beam did not fit as it should they might suspect that the walls are leaning and therefore that the angle between the ridge and the ceiling is not right. This would mean that they treated an angle as right although it was not and that the formula therefore does not work. If there is no problem at this point either they might start wondering: "Isn't this a suitable problem to apply the Pythagorean theorem to after all? But this is a right triangle and that is what the theorem is about." At some point they give up trying to solve this problem and see if they can solve it by other means.

If they are very keen on finding out why their approach did not work they might at some point doubt their memory, did we not remember the measures or did they perhaps something caused them to change? Was the weather extreme in some way that could have caused them to expand or bend? The last two alternatives are very unlikely, but one thing they will probably not question are basic arithmetical rules of calculation. In trying to locate the error they may even use the calculations as a guide. That is, they will probably not try calculating with other rules, unless of course they are aware of some other method which would also serve the purpose. But they would not try to calculate the square of the length of the legs of the triangle differently, whereas they might, e.g., use another measuring-tape. I would say that this is what Wittgenstein is after with his remark that a mistake is logically excluded – the language game is such that the rules of calculation are not considered as a possible source of error. One does revise one's opinion of a lot of things, but not of which are the correct rules of adding. The main point here is that this shows in the way they are used. It is not a question about what I
can conceive myself of being mistaken about or doubting, but rather a question of how people go along, and what kinds of attitudes one can discern towards different ways of speaking and doing.

Concerning the certainty about propositions one has arrived at through inference or calculation, there does not seem to be any room for doubt about these either once one has settled that they have indeed been reached through a correct procedure (at least as long as one stays within established mathematics). Or, perhaps one should say: doubt enters, not regarding any descriptive content of the propositions, but only rather regarding the proper following of the rules or adhering to the correct practise in reaching them. If two persons agree that one has arrived at the propositions in a correct way, there is no room for any further doubt as to its correctness – which there might well be in other areas of discourse. It is, e.g., possible to doubt the correctness of a sentence given in a trial although it has been reached through correct procedures. It seems to me that the philosophical use of “is certain” with regard to mathematics is pointing to this absence of alternatives once correct procedures have been followed. This kind of certainty which is related to being convinced that one has done something accurately, is in turn only possible against the background of a firmly established practise. Otherwise it would not be possible to distinguish correct from incorrect.

This might perhaps be a way of making sense of the claim that mathematics gives us certain knowledge, but one has of course not pointed to some quality of mathematics, but rather to how we relate to it. Importantly, one has not given an explanation of why mathematics holds this place. In a related discussion Wittgenstein points out that he has “not said why mathematicians do not quarrel, but only that they do not.”

In the his article “The Kind of Certainty is the Kind of Language Game”, Lars Hertzberg discusses the uncertainty that one sometimes feels when trying to figure out what another person is feeling or thinking. This uncertainty is often invoked as an argument in favour of a scepticism about other minds. It seems to me that the certainty one has about particular mathematical results is taken as an argument in favour of the certainty of mathematics in a way that is analogous to the contrary conclusion about other minds. As Hertzberg shows however, this occasional uncertainty and the occasional mistake when judging other people’s emotions does not warrant scepticism. It does not undermine our talk of emotions although it is part of it.

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Here we see what room there is for the notion that uncertainty might belong to the character of the language game. The uncertainty does not reside in a relation between the language game and something external to it; it is not, as it were, a comparative notion at all. It consists, rather, in the manner in which discussions about the right and wrong application of words are carried on, in the forms that disagreement and criticism may take.

In a similar fashion I think one could see the claim that mathematics is certain, taken as a labelling of the discipline with a neutral label “certain” as mistaken. That would amount to a comparison of mathematics with other disciplines after one has found arguments in favour of its certainty. Instead, I think one must study our dealings with mathematics and mathematical methods in order to see what concept mathematical certainty is. Then, I think one could say in: “Here we see what room there is for the notion that certainty might belong to the character of the language game.”

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