This is the second chapter of my dissertation. The first chapter deals with the concept of certainty. I take it that a common way into philosophical though about mathematics starts in a will to understand how mathematics can give us certain knowledge, how its truths can be necessary truths. In the first chapter I argue what we are to understand by certainty in mathematics is not clear. This chapter follows up the previous one by considering if we are reading too much into the phrase "mathematical knowledge." Perhaps the term knowledge can be misleading in itself when we talk about mathematics?

I believe that the numbers and functions of analysis are not the arbitrary product of our spirits: I believe that they exist outside of us with the same character of necessity as the objects of objective reality; and we find or discover them and study them as do the physicists, chemists and zoologists.

– Charles Hermite

I have so far been paying attention to the concept of certainty with regard to mathematical knowledge. Occasionally the phrase “mathematical certainty” has occurred, but in general, the discussion has taken for granted that certainty concerns some piece of (mathematical) knowledge, or indeed all of mathematical knowledge.
“This piece of knowledge ‘2 + 2 = 4’ is beyond doubt, and absolutely certain.” It is, however, worth considering what the consequences are of this assumption, which might turn out not to be as innocent or even self-evident as it seems. As the above discussion shows it might also be the case that one did not reflect on the nature of the object of certainty at all.

It seems to me that this assumption is tied up with the view of mathematics as a body of truths. One’s relation to this body of truths is, supposedly, one of knowledge or ignorance. To claim that this body of truths as such is certain seems somewhat odd, whereas the idea that the knowledge about this corpus is certain and infallible seems to conform better to common usage of “certainty”. Otherwise it appears that one has to say that mathematics is in itself certain. How could anything be certain in itself? A phrase that at least sounds better would be that the corpus is eternal. Accordingly, the mathematician gains “knowledge of the eternal, and not of aught perishing and transient”, as Plato wrote.\(^1\) Another alternative would be that it is a body of necessary truths, but this will be discussed later.

If, however, the fact that mathematics is also a technique is taken into consideration, the notion of mathematical knowledge will have to be put alongside notions such as \textit{skill} and \textit{craftsmanship}.

1

Does mathematics give us knowledge? Two spontaneous answers are (1) “Sure it does! Knowledge about theorems, calculi, axiomatic systems, and applications.” (2) “Of course – about numbers, functions, sets, and structures.” Sometimes one also hears “about the structure of the world.” A consideration of this latter answer would lead me too far astray and will be left for later.

The first part of the first answer – that mathematics teaches us about theorems, calculi, axiomatic systems, and applications – obviously goes together with the idea that mathematics is a body of truths, which the mathematician studies. During courses in mathematics these are naturally part of what we learn. A research mathematician might be said to find new theorems, new calculi, etc. Still, it is not clear that this is the best way to formulate it. To know what theorems there are might

\(^1\) (Plato 1901, p. 527b.)
seem like knowing what one will likely find if one opens a mathematics book, but
that can hardly be all there is to mathematical knowledge. I am sure many would
be inclined to say that the theorems express mathematical knowledge, but that the
theorems are not themselves the objects of knowledge. Wittgenstein draws attention
to a similar distinction:

\[T\]he propositions ‘Human beings believe that twice two is four’ and
‘Twice two is four’ do not mean the same. The latter is a mathe-
matical propositions; the other, if it makes sense at all, may perhaps
mean: human beings have arrived at the mathematical proposition.
The two propositions have entirely different uses.”

Whether he would have been prepared to go along with the above suggestion that
theorems express knowledge, I cannot say, but I suspect he would not. In this
chapter I hope to show why one would not want to say that. For now it might
be enough to conclude that knowing which theorems there are is a part of what
one acquires in the study of mathematics, but that it is not what one refers to as
mathematical knowledge and which has such a strong allure for philosophers.

It seems, then, to be along the lines of the picture I am sketching to say that
the body of truths, which presumably constitutes mathematics, is a collection of
knowledge about such things as was mentioned in the second suggestion above:
numbers, functions, sets, structures, etc.

The obvious question is then: “How does one gain knowledge about these things?”
This question has received quite a fair amount of attention, especially since the
publication of Paul Benacerraf’s “Mathematical Truth” in 1973. In his article Be-
acerraf draws attention to a tension in the philosophy of mathematics, a tension
which has its roots in the problem of explaining the concept of truth in mathe-
matics. If one wishes to understand truth in mathematics as one understands it
in other contexts then one runs into the problem just mentioned. In spite of this
problem, Benacerraf argues that a unified semantics for mathematics and “the rest
of language” ought to be our choice. He admits that there is no general account of

\[Wittgenstein 2001, pp. 192-3.\]
how truth works in the rest of language, but the option he prefers is Alfred Tarski’s, in essence a “referential semantics” according to Benacerraf.  

This is the problem that the “standard” view, as Benacerraf calls it, faces. If, on the other hand, one equates truth with derivability in an axiomatic system, it is easy to see how one comes to know the truths of mathematics – we can prove many things and thus we have knowledge about them. According to Benacerraf it is not clear, however, that we actually want to call derivability in a system truth. This is the problem facing “combinatorial” views. Benacerraf talks about “standard” and “combinatorial” views, but makes it clear that the more widely used labels Platonism and realism as well as constructivism and conventionalism fall under these headings respectively. William W. Tait, among others, have pointed out that Platonism is an unfortunate label since it originates in a misinterpretation of Plato. Tait uses the term “super-realism”, while reserving “realism” for the common attitude, which relates to mathematics as if propositions where either true or false without necessarily underpinning this belief with some kind of metaphysical assumptions about mathematical entities (Tait 2001, section II). A modern pair of positions related to the two overarching ones discussed by Benacerraf is naturalism, which is inspired by W. V. O. Quine, and fictionalism, which counts Hartry Field as its founder.

The reasons for scepticism about equating truth and derivability due in part to the incompleteness proofs of Kurt Gödel. A common way of describing the first one of these is that it shows that in a formal system that is complex enough to enable one to deduce all true propositions of arithmetic, there are propositions that are true but, nevertheless, not provable in the system. Thus, one would have to distinguish between truth and provability. Perhaps it ought not to be surprising when one takes this into account that, e.g., Gödel in “What is a Cantor’s continuum problem?” adopts a kind of Platonist position:

> [T]he objects of transfinite set theory . . . clearly do not belong to the physical world . . . But, despite their remoteness from sense experience, we do have something like a perception also of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don’t see any reason why we should have less confidence in

---

3 (Benacerraf 1983, pp. 403, 408, 413, 418.)
Benacerraf’s essay has had a significant influence on the modern discussion in the philosophy of mathematics. What one finds are different approaches, such as naturalism, fictionalism, and structuralism, which can be viewed as attempts to solve Benacerraf’s problem from the perspective of one of the two overarching positions portrayed by him. I must not get caught up in this debate, however. I fear that once one accepts the starting points of that discussion as a meaningful way of trying to understand knowledge and truth with regard to mathematics, one has already lost sight of things, which the philosopher interested in clarity should not oversee.

What I think is lacking is an awareness of matters relating to the “technique” aspects of mathematics. A look at the difficulties involved in learning what propositions in mathematics mean – in particular for the learner but also to some extent for mathematicians – may illustrate what I am after. The question “How does one gain knowledge about such objects?” seems clear-cut and seems to demand a simple answer. However, what it is to have knowledge in mathematics and what it is to realise that something is true or false cannot be understood as a matter of either observing it or proving something about it. A greater sensitivity towards the historical development may also make one realise that this dichotomy is somewhat coarse.

Let me start with a simple example. Take the equation $x^2 = 1$. Seen from the perspective of a pupil it may be taken as an exercise. Possibly, it looks like other tasks the pupil has performed and so she assumes that she understands it, and that she only needs to do some calculations to find its roots. If imaginary numbers have not yet been introduced to her she will have to revise her understanding of the equation. Her teacher might have wanted to introduce the notion of equations without solutions by allowing the pupils themselves to draw the conclusion that some equations lack roots themselves after working on such an example.

---

4 (Gödel 1983a, p. 483–4.)
The students’ broadening of the concept of equation from something that has a solution to something that may or may not have a solution has consequences for the truth of claims about equations. One cannot simply say that the pupils’ first belief that equations have solutions was wrong. It is more a matter of the meaning of the word “equation.” One of the pupils may even object to the teacher’s new notion of equation: “That is not an equation because the two sides will never be equal.”

Later, the teacher introduces imaginary numbers and shows the pupils how this extension of the concept of number allows one to solve equations like $x^2 = 1$. Again, they have to revise old truths about equations and roots. I do not think it would do justice to the teaching of mathematics to say that they were wrong when prior to their learning of imaginary numbers they claimed that such equations lack a solution. The teacher might, of course, have been of the careful kind and advised them to say that what is lacking are real roots. However, as long as they know only real numbers this precaution has, I would say, no meaning to them.

“What I taught you earlier about equations lacking a solution was wrong, …” A teacher may perhaps introduce a lesson about imaginary numbers in this fashion, perhaps to catch the attention of the pupils but he will not correct his earlier teaching only introduce a new concept, a new way of calculating, which will have consequences for how one sees the old knowledge. But, as long as one is speaking of real numbers only, nothing will have changed. If the teacher is not trying to awaken the paradoxical feeling as the above mentioned statement may have done, a more truthful introductory remark could be: “What I taught you earlier about equations lacking a solution was not the whole truth, …”

This development of the pupils’ knowledge of mathematics finds parallels in the history of the subject too. When considering the discovery or invention of the imaginary numbers by Girolamo Cardano in the 16th century, it is even less intelligible to say that his predecessors were simply wrong in supposing that certain equations did not have roots. Even the word “suppose” is misleading here, they did not suppose anything, they saw clearly that, e.g., $x^2 = 1$ could not have any solutions. Considering its graphical representation in a cartesian coordinate system this is also clear. A parable, which is completely located above the $x$-axis, does not intersect it.
Cardano did not discover that earlier mathematicians had been wrong, he extended the concept solution. The controversy that surrounded the introduction of complex numbers lasted roughly until the time of Carl Friedrich Gauß, that is, almost three hundred years. This I believe is telling of the difficulties involved in determining the meaning and place in the surrounding theory of mathematical concepts and propositions. One must also be aware of the false impressions that the reading of a greater continuity into the historical development gives rise to. As E. T. Bell warns us: “Nothing is easier ... than to fit a deceptively smooth curve to the discontinuities of mathematical invention.” Afterwards it is easy to get the impression that past mathematicians were dealing with the same concepts as we are and that they viewed them in a similar manner.

The example of imaginary numbers is especially striking since these were probably invented out of a will to unify the treatment of polynomial equations. When considered from the perspective of real numbers, such an equation has anything between zero and \(n\) roots if its degree is \(n\). Once imaginary numbers are introduced one can state that a polynomial equation (single-variable and non-constant) of degree \(n\) has \(n\) roots, imaginary or real, some possibly appearing more than once. (This is the fundamental theorem of algebra.) That mathematics may develop through this kind of extension of concepts earlier thought to be clearly grasped with regard to content and consequences, is illustrated nicely by Lakatos. Moreover, in the dialogue in *Proofs and Refutations* the parallel between learning mathematics as a student and the historical development of the discipline concerning the extension of concepts is also hinted at.

My point is that the idea of studying an object in mathematics in order to learn about its features, cannot be viewed in a manner analogous to the weighing, measuring and observing of a concrete object, like an orange that one is about to buy. I would like to say things like: “There is nothing there to study,” “The features of a number, a

---

5 As an aside, I think Wittgenstein’s controversial remark that “[t]he mathematician is an inventor, not a discoverer” should be considered in the light of this kind of extension of mathematical knowledge. (Wittgenstein 1978, I § 168.)

6 (Bell 1945, p. viii.)

7 (Lakatos 1976, ch. 1.)
function, etc. are not given by its constitution in advance,” and “Its features emerge in the interplay with other mathematical objects and techniques.” But, in a sense its features are given in advance, but one has to be very careful to see what aspects of mathematical discourse one is thereby referring to, and the same goes for the claim that they are not given in advance.

Consider someone who is studying an ancient coin. There are many kinds of questions one can ask in relation to the coin. One kind is related to the physical aspects of the coin: “What are its measurements?” “What is its weight?” “What material is it made of?” Another kind is questions that concern the coin’s place in the monetary system that it presumably was a part of. One could be interested in knowing whether its value was high or not, whether people in general used this kind of coin or if it was a privilege of higher classes in the ancient society, what kinds of goods it could be used to buy, etc. The answers to these questions do not depend on the answers to the first kind of questions, and one could certainly not find an answer to them by looking at the answers to the first ones. In mathematics, I would say, there is trade but no coins. Another comparison could be an investigation of a piece in chess, where the physical aspects of the piece are irrelevant for what features it has in the game of chess. If chess was played by drawing or writing down the positions of the pieces, it would be a game without physical pieces. My earlier claim that there is nothing there to study concerning the study of mathematical objects should be understood in the same way as a potential claim that there is nothing to study if someone wanted to study one of the pieces in the game of chess that is played entirely with pen and paper. On the other hand, the piece is thoroughly determined by the rules for its movement and its relations to the other pieces. And, what one wants to know about the piece is, I take it, precisely what can be gathered from that. This in turn means that there is something there to study, and that this something is in fact given in advance.

Moreover, to gain knowledge of a function, class of functions, theorem, or theory, one must spend time with it – calculate, compare, deduce, conjecture, etc. One cannot simply study it. Or better: to study it is to spend time with it in this way. To learn something about a function, say \( f(x) = \sin x \), one cannot simply read a list of its features and thereby “know” it. (I assume that this is the closest one gets the study of a physical object, which is not to say that there is much similarity.) If one is familiar with similar functions the list may well make sense, but if one is
not, it is hard to see how such a list could be of any use. That the function in question is, e.g., periodic means something only when one realises that the function values must recur at a certain interval \((2\pi)\) of values of the argument, and that this is a consequence of how the function is defined. A graphical representation of this definition or of the function curve give a further idea of what periodicity means. Its meaning is also enriched if one tries to determine the maximum or minimum of this function, perhaps as part of an attempt to make use of it in application. As one realises that it has an infinite number of periodically recurring maxima, either one has to give up the idea of determining a maximum or to restrict oneself to a certain interval of the function’s domain, i.e., a certain interval \([a, b] \in \mathbb{R}\). This exercise in turn reveals something about periodicity – that such a function has no definite extreme value for instance.

This is not all, however. What it means to learn things about a mathematical object, say a particular function, is not limited to spending time with it and finding out more about it. When new concepts are introduced into mathematics, one may thereby have to revise one’s idea of a familiar function although no investigation of the function has taken place. This was the case with the introduction of imaginary numbers in the example above, when one could conclude that an equation has roots without studying this particular equation. Another example might be the concept set, which allowed a thorough revision of the concept of function in the 19th century. Earlier a function had been thought of as an expression by means of which one could calculate a value when an argument (or arguments) was supplied and substituted for the variable (or variables) in this expression. This means, among other things, that since different expressions agree in value when evaluated for the same argument, it seems that the same function could be expressed in different ways. Afterwards a function is seen as a relation between two sets. The function takes one member of the first set (the domain of the function) and associates it with exactly one member of the second set (the codomain). This new concept of function entails that one does not need to know the expression, the rule, by which one calculates the value of the function, one can postulate that such a relation exists. This in turn means that the concept of function is expanded.
I would argue that a consequence of this discussion is that the referential semantics called for by Benacerraf is ill-suited for an understanding of mathematical truth. On top of that, it is ill-suited for an understanding of “the rest of language” too. As was noted above a supposed merit of the referential view was that it gave hope of a unified account of meaning that would incorporate all areas of human discourse. To be sure, we do speak of objects around us, especially if we are trying to do something with an object, say when making use of an ingredient when preparing supper, painting the walls of a house, or analysing a blood sample. But if one thinks of the discussions and exchanges we often have, quite a fair share of them cannot be understood in terms of reference to things.\(^8\) The charm of the referential view may thus be a result of a false appearance. We find Benacerraf explaining:

My bias for what I call a Tarskian theory stems simply from the fact that he has given us the only viable systematic general account we have of truth. So, one consequence of the economy attending the standard view is that logical relations are subject to uniform treatment: they are invariant with subject matter.\(^9\)

Now, what Benacerraf means by “uniform treatment” is that sentences such as “There is a Finnish city between Helsinki and Stockholm,” and “There is a prime number between 12 and 14” can be understood to share the same logical form (from the perspective of predicate logic):

\[
\text{There is } x \text{ such that } F(x) \text{ and } R(a, x, b). \tag{1}
\]

This in turn means that their truth conditions are given along the lines of satisfaction of a formula of predicate logic. The formula (1) is true precisely when there is at least one element in our universe of discourse from which \(x\) is chosen that has the properties \(F\) and stand in the relation \(R\) to the individuals \(a\) and \(b\) of the universe of discourse, which would be the set of cities and the set of natural numbers, respectively. This is what is Tarskian about Benacerraf’s view. A technical problem with this attempt at explaining the notion of truth in mathematics is that Tarski’s

\(^8\) There is much to be said about language and meaning in general, but it would be beside the point here. There are also an abundance of good discussions of this kind of problems. [nämna några]

\(^9\) (Benacerraf 1983, p. 411.)
account concerns the definition of a predicate (a truth predicate) in a meta language which takes as arguments propositions of a formalised theory. A condition for the success of this project, was according to Tarski, that the meta theory is “stronger” than the object theory; it should not be possible to express the truth predicate in the object theory, only in the meta theory.\(^\text{10}\) Now, if we are considering mathematics in general as our object theory it is difficult to see how one should compare its expressive strength to any supposed meta theory since one has not limited it in any way. And at the same time, any limitations would seem as a flaw, since one was, after all, interested in mathematics.

Be that as it may, there are perhaps more serious problems if we consider the matter from a philosophical point of view. Just because it is formally possible to paraphrase a proposition in the idiom of predicate logic, with its explicit definitions for truth and falsity, it does not mean that one has thereby captured the essence of the proposition – if we can meaningfully speak of such a thing. The fact that a calculus or formal method is applicable to something, does not tell us the whole truth about it. It is worth asking if this possibility of framing the propositions of mathematics in predicate logic does not in fact conceal *philosophically* important things about mathematical discourse. Of course, it should not come as a surprise to us that this paraphrasing is possible, Frege’s intention was, after all, precisely to develop the *Begriffsschrift* for the purpose of formalising and checking deductions in arithmetic.\(^\text{11}\) To be sure, paraphrasing propositions in predicate logic seems to work reasonably well only in the case of mathematics. Whether this is a consequence of the predicate calculus’ originally being intended to formalise mathematics, or of the jargon of formal logic having permeated and changed mathematical practice, I cannot say. Anyhow, this outward unity in the logical structure of propositions in mathematics, as well as in “the rest of language,” seems to licence a “uniform treatment.” In this situation the philosopher faces a struggle between the demands of clarity and economy in thinking. Sören Stenlund comments on the relation between the regularity of forms of expressions that predicate logic gives rise to and metaphysical realism in mathematics: “In order to justify the importance assigned to regularity in forms of expression, contentual or metaphysical correlates to regularity in linguistic forms are needed.”\(^\text{12}\)

\(^{10}\) (Hodges 2008, 1.3.)

\(^{11}\) (Frege 1967, Preface.)
Benacerraf remarks with reference to the uniform treatment of logical relations and the economy of attending the “standard” view: “Indeed, they [the logical relations] help define the concept of ‘subject matter’.” But, one must not let economy decide a matter where clarity is the goal, one has to investigate what is being said when reference is made to numbers and other mathematical objects, and what is taking place when a mathematical discovery is made.

I have devoted some space to criticising the referential picture of mathematical propositions, although my principal aim is to discuss the body-of-truths conception. I found this necessary since the idea of explaining truth and meaning in mathematics in terms of reference can seem very appealing. It is also intimately connected to the body-of-truths conception.

What the above considerations show, among other things, is that the “body-of-truths” conception is best suited to describe a mathematics, where development is always a cumulative growth in which old truths remain intact, their meanings unaltered, as new ones are added. Bearing this in mind, I think that one does not do the positions in the contemporary debate too much injustice by saying that although they fall on different sides of the realism/conventionalism divide, they can still be seen as expressions of the body-of-truths conception.

Let me try to convey another picture. If I study a triangle there may arise the question of whether it is a right triangle or not. I might for example wonder whether the Pythagorean theorem applies, which would enable me to calculate the length of one of the sides as the length of the other two are given. A first assessment has probably already taken place since I consider proving that it is a right triangle; if it were an obvious impossibility I would not even consider a proof. Of course, I cannot inspect a drawing of the triangle to determine its status, I have to prove it mathematically. If I am only interested in finding a rough approximation of the unknown side it might be enough to look a figure and be satisfied if it looks right, but let us leave this option aside. Now, I cannot perform anything like the examination of fruit at the market place. I have to go by the theory of geometry and the facts

---

12 (Stenlund 2009, p. 499.)
13 (Benacerraf 1983, p. 411.)
that are known about this triangle. These facts are known either because of some earlier proof or because the triangle was defined in a certain way. If it is known that two of the angles are 30° and 60° respectively, I can conclude that it must be a right triangle. It can be proven that the sum of the interior angles of any triangle is 180° (it may also be considered to be a defining feature of a triangle), and if there is one angle which is 30° and another which is 60°, there remain 90° for the last one. There are also other facts about the triangle that may give me reason to conclude the same, but in an other case I might not have enough information to conclude anything about its rightness. To study this mathematical object is to consider how it is defined and how these concepts in terms of which it is defines are in their turn defined and infer different things from these facts. It is an object given in advance to the extent that it is defined in advance. If I know only that it is a triangle and that one of the angles is 30°, I cannot conclude anything about its rightness. It is often considered a feature of a realistic position in the philosophy of mathematics that one considers properly formulated propositions to be either true or false. In this case I could ask myself if this triangle still is right or not although I happen to have no means of finding the answer. A similar question that one often finds in discussions of realism is whether there occurs a certain sequence of numbers, e.g. 7777, in the infinite decimal expansion of π. Since the number π is well defined, if irrational, one feels that there must be a definite answer although one has not been able to prove it yet. In the case of the triangle, which is not sufficiently defined, the temptation to think that there must be an answer is not as strong as in the case of π. The question whether it is right or not and even more so the claim that it is right or not even though I cannot prove it either way, seems meaningless.

There may in the future arise a situation, perhaps in the light of a new mathematical theory, where one will find it necessary to stipulate that a triangle will always be assumed to be right unless it defined to be otherwise. This does of course not change the situation for me, when I want to study a triangle for a particular purpose today. I guess one could say that my purpose for studying the triangle could not even be the same as that which will lead future geometers to lay down this stipulation.

Perhaps it could be illuminating to say that the question: “What is that?” is not possible in mathematics. One is never faced with an object that one knows nothing about. Everything one studies is already defined in some way as something. And
what is possible to find out about this object depends entirely on this definition, and on other things that are already established about such objects as the object is defined to be.

Another example could be that I want to show that there is prime number between 56 and 60. How do I prove this? By pointing it out? I might simply point it out if I am familiar with prime numbers and know, say, all prime numbers between 1 and 100 by heart. But if I am do not know them by heart, what does pointing it out mean in this case? I have to consider the definition of prime number, which says that a number is prime if it is divisible only by 1 and by itself. This automatically rules out all even numbers since even numbers are all divisible by 2 (this is even taken as the defining feature of even numbers), and that leaves me with 57 and 59. I could check if one of them is prime by dividing them first by 3 and then if this does not give a natural number continue checking with odd numbers greater than three. In the case of 57, I find that $57/3 = 19$ and conclude that it is not prime, and stop checking it. In the case of 59 I continue until I have tested all odd numbers up till 19. Then I can safely stop and conclude that 59 is prime, and that there exists a prime number between 56 and 60. The reason why I stop at 19 is that $59/3$ was a number between 19 and 20 and any numbers greater than 19 could only give quotients less than 3, and since 59 is odd, 2 cannot divide it, thus it is not necessary to check numbers greater than 19. The result could be formalised in predicate logic by considering the subset $M = \{56, 57, 58, 59, 60\}$ of the domain of natural numbers and state that there is an element in M satisfying the claim, i.e. satisfying:

$$\exists x \forall y : (x \in M) \& [(y|x) \rightarrow (y = 1 \lor y = x)].$$

What was proved fits the picture of referential semantics very neatly. I am, however, sceptical of the idea that one understands this proposition when one understands its truth conditions. Firstly, I think one can understand it even if one does not understand the truth conditions as they are expressed in predicate logic. Perhaps a reply to this would be that understanding the proposition and thus its truth conditions need not be expressed formally, that one can have an intuitive understanding of the same thing. Well, perhaps one could argue that way, but what is a more serious is my second objection. I would spontaneously phrase it this way: To understand that there is a prime number between 56 and 60, it cannot be enough to simply know that 59 is a number between 56 and 60 and satisfies the second conjunct.
But of course this is not a good way of putting it. To know that it satisfies the second conjunct one has to understand the concept of divides and the fact that the concept prime number is usually defined in predicate logic in this way, etc. But this is exactly my point: one needs to have a certain skill and familiarity with these concepts in order to understand it, and has one not thereby stepped outside the domain of understanding something through reference? Perhaps this is what I want to say: Is it not true that in order to understand the proposition one would have to be able to run through the proof oneself? As Wittgenstein writes: “If you want to know what is proved, look at the proof.”

If mathematics could be described as a body of knowledge it would in principle be possible to look up what one needed to know in some kind of collection of these truths. Needless to say, there are many such collections. There are tables of values for certain functions, lists of formulas, books describing axiomatic theories by stating theorems and proving them one by one. It is obvious though, that what might properly be described as “mathematical knowledge”, or “knowing mathematics” does not end here, quite the opposite. To understand the point of the tables and lists of formulas one has to be able to apply the formulas. The lists can be regarded as aid for the memory, at best, they can in no way supplant the knowledge of how to use them and make them bear on the case one is interested in. Furthermore, to understand the proofs of the theorems in a book of mathematics usually requires some familiarity with the techniques used and some effort filling out details that are left out. The theorems usually open up only when one tries to solve exercise problems, and when one realises which problems led to the formulation of the theorems.

The learning of mathematics finds parallels in the learning of a language. What I have in mind is the learning of a new word or a new phrase, but not necessarily the translation of a foreign word into a well-known word of one’s own language. The comparison with language may allow for a fruitful shift of focus. Instead of concentrating on the – often misleading – analogy with finding new features of a physical object, one can think about learning when a word is appropriate to use, when it is not, which connotations it has, etc.

14 (Wittgenstein 1974, p. 369.)
What does the discovery consist of? The “discovery” is often about establishing a connection between definitions. The motivation for attempting to construct this connection is sometimes found in wild guesses, sometimes in a hunch that it will work, perhaps because one has done something analogous or that one is otherwise familiar with the concepts.

The need to sharply determine the wordings of definitions and their extensions, I would say, goes together with the fact that there are no mathematical objects, at least not as one might find it intelligible to speak about pieces of evidence in an investigation of a crime. Everything that goes for an object of a certain kind has to be written down in the definition because there is no object to investigate. One could not look at the object to see if one has forgotten something in the definition. One can check if the definition matches one’s ideas about the concept, but this is more closely related to the question: “Is this what I want to say?” rather than: “Is this thing like this?”

Stenlund has suggested that the concept of knowledge as such leads us to think in terms of knowledge about something, and that the phrase “mathematical knowledge” might be misleading. I agree with him although I have in this chapter tried to see what sense one could make of it, rather than avoiding it. Anyway, Stenlund’s suggestion finds an interesting parallel in Wittgenstein’s remark: “What I want to say is: mathematics as such is always measure, not thing measured.” It might be fruitful to compare mathematics to, say, a ruler, the activities that involve a ruler, and the relation between the ruler and what we measure with it. If knowledge is seen as “things measured” then mathematics naturally acquires another status, as it is involved in our tools for gaining knowledge about other things. From this perspective work in pure mathematics could be seen as a work on our tools for gaining knowledge. This is not meant to be an alternative explanation of what mathematics deep down is, but more like a proposal that may draw attention to features of mathematics that often are forgotten in philosophy.

References


15 Personal correspondence.
16 (Wittgenstein 1978, III § 75.)