Rational Approximations of Transfer Functions of Some Viscoelastic Rods, with Applications to Robust Control

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Abstract

We study rational approximations of the transfer function $\widehat{P}$ of a uniform or nonuniform viscoelastic rod undergoing torsional vibrations that are excited and measured at the same end. The approximation is to be carried out in a way that is appropriate, with respect to stability and performance, for the construction of suboptimal rational stabilizing compensators for the rod. The function $\widehat{P}$ can be expressed as $\widehat{P}(s) = s^{-2}g(\beta^2(s))$, where g is an infinite product of fractional linear transformations and β is a (generally transcendental) function that characterizes a particular viscoelastic material. First, $g(\beta^2)$ is approximated by its partial products $g_N(\beta^2)$. For relevant values of β^2, convergence rates for g_N are analyzed in detail. Convergence suitable for our problem requires the introduction of a new irrational convergence factor, which must be approximated separately. In addition, the fractional linear factors in $\beta^2(s)$ that appear in $g_N(\beta^2(s))$ must be replaced by something rational. When the damping is weak it is possible to do this by separating the oscillatory modes from the "creep" modes and ignoring the latter; in general, this step remains incomplete. Some numerical data illustrating all the stages of the process as well as the final results for various viscoelastic constitutive relations are presented.

Key words: input-output system, boundary feedback, vibrations, transfer function, viscoelastic, control, rational approximation, compensator, sensitivity, stability, optimal, infinite product.

AMS classification: Primary: 93B36 H^∞ control, 93C22 Control systems governed by integral equations, Secondary: 73F15 Visocelasticity–time dependent problems.

Summary

We examine a scalar input-output system that models a boundary feedback scheme for the damping of torsional vibrations in a cylindrical rod of circular cross section, consisting of a linear viscoelastic material. The open loop transfer function for the system is irrational, and we study the problem of approximating some ideal compensator by a proper rational one. The approach is in the spirit of [2], where bending vibrations in an Euler-Bernoulli beam with Kelvin-Voigt damping were studied (and where a discussion of potential applications of the method is found). In particular, a compensator derived from the full distributed parameter model is approximated. Here we emphasize estimates for a wide range of viscoelastic materials, but we examine only one simple mechanical structure: the case of torsional vibrations (i.e. the viscoelastic wave equation) with actuator and sensor collocated at one end of the rod. For separated sensors and actuators, new issues arise that will not be addressed here, see [1] or [2].

In the particular (collocated) cases that we discuss, the open-loop transfer functions $\widehat{P}(s)$ have no zeros or poles in the open right half-plane, and no zeros or poles on the imaginary axis apart from a pole at zero and a (fractional order) zero at infinity. As in [2] our objective is to find a compensator $\widehat{C}(s)$ that minimizes the mixed sensitivity norm

$$\mu = \left\| \begin{bmatrix} W_1 \widehat{S} \\ W_2 \widehat{T} \end{bmatrix} \right\|_{H^\infty} .$$

Here the functions W_1 and W_2 are simple rational weights, and $\widehat{S} = 1/(1+\widehat{P}\widehat{C})$ and $\widehat{T} = \widehat{P}\widehat{C}/(1+\widehat{P}\widehat{C})$ are the sensitivity and complementary sensitivity functions, respectively. The optimal compensator given by $\widehat{C}_{\mathrm{opt}}(s) = W_1(s)/(W_2(s)\widehat{P}(s))$ cannot be used for three reasons: since it effectively inverts the plant it is both irrational and improper, and there is a forbidden zero-pole cancellation at zero.

The forbidden zero-pole cancellation and improperness are dealt with in an easy preliminary step. Namely, following [2], we simply replace $\widehat{P}$ in the definition of $\widehat{C}_{\mathrm{opt}}$ by a regularized plant $\widehat{P}_{\mathrm{sub}}(s) = s^2(\zeta_1 s+1)^2(s+\zeta_0)^{-2}\widehat{P}(s)$ where ζ_0 and ζ_1 are appropriately chosen small positive constants. The loss of performance (as measured by the mixed sensitivity norm) by using $\widehat{C}_{\mathrm{sub}}$ obtained in this way in place of $\widehat{C}_{\mathrm{opt}}$ can be made arbitrarily small by taking ζ_0 and ζ_1 sufficiently close to zero; however, $\widehat{C}_{\mathrm{sub}}$ will still be irrational since $\widehat{P}_{\mathrm{sub}}$ is irrational. In order to get a compensator that can be physically implemented, one has to approximate $\widehat{P}_{\mathrm{sub}}$ by a rational function, preferably of low degree.

The method of approximation that we propose will not be optimal with respect to any of the standard minimization problems. However, it will give fairly good results with respect any such minimization problem. Our solution is to approximate $\widehat{P}_{\mathrm{sub}}$ by $\widehat{P}_N$ in such a way that

- $\tau_N(s) = \widehat{C}_N(s)/\widehat{C}_{\rm sub}(s) = \widehat{P}_{\rm sub}(s)/\widehat{P}_N(s) \to 1$ uniformly on compact subsets of the right half-plane, and
- $\limsup_{N\to\infty, |s|\to\infty} \left|\widehat{T}_{\rm sub}(s)\tau_N(s)\right| = 0.$

Here $\widehat{T}_{\rm sub} = \widehat{P}\widehat{C}_{\rm sub}/(1+\widehat{P}\widehat{C}_{\rm sub})$ is the suboptimal complementary sensitivity. Of course, our compensators will not have the lowest possible order corresponding to a given accuracy, but the order may be further reduced by means of some standard order reduction scheme.

The transfer functions that we are interested in are of the type

$$\widehat{P}(s) = s^{-2}\beta(s)f(\beta(s)),$$

where $\beta f(\beta)$ is irrational and even (hence a function of β^2), and β^2 is either rational or irrational. The functions f depend solely on the geometry of the problem, and the function β describes the viscoelastic material damping properties of the the rod. In the the case of torsional vibrations in a rod of length one and uniform density one the function $\beta f(\beta)$ is given by

$$\beta f(\beta) = \beta \coth(\beta).$$

The function β is given by

$$\beta(s) = \frac{s}{\sqrt{s\hat{A}(s)}}$$

where the relaxation modulus A is a completely monotone function on $(0,\infty)$ with $A(\infty) < A(0+) \leq \infty$, and $\hat{A}$ is the Laplace transform of A. We show, in particular, that β is analytic in the whole complex plane minus the negative real axis, with the exception of a pair of complex poles in the left half plane. The most important qualitative differences between viscoelastic materials show up in the behavior of the relaxation modulus A at the origin, and this is reflected in the behavior of β at infinity.

To illustrate our approximation scheme we carry out our computations for four different model kernels (in decreasing order of structural damping):

1. $\widehat{A}_1(s) = E/s + \epsilon$; Kelvin-Voigt damping, where formally A_1 is the sum of a constant and a constant times the unit point mass at zero.
2. $A_2(t) = E + (\epsilon\delta^\mu/\Gamma(\mu))\, t^{\mu-1}e^{-\delta t}$, $\widehat{A}_2(s) = E/s + \epsilon(1+s/\delta)^{-\mu}$; $0 < \mu < 1, \epsilon, \delta > 0, \Gamma =$ gamma function; a modified "fractional derivative" model (see [3]) of order $1-\mu$ with exponential decay as $t \to \infty$.
3. $A_3(t) = E + \left(\epsilon\delta^{\mu+1}/\Gamma(\mu+1)\right) \int_t^\infty \tau^{\mu-1}e^{-\delta\tau}\, d\tau$,
   $\widehat{A}_3(s) = E/s + (\epsilon\delta/(\mu s))\,(1-(1+s/\delta)^{-\mu})$; an intermediate model of order $1-\mu$ with $A(0+) < \infty$ and $A'(0+) = -\infty$.
4. $A_4(t) = E + \epsilon\delta e^{-\delta t}$, $\widehat{A}_4(s) = E/s + \epsilon/\,(1+s/\delta)$; standard linear solid model.

In the case of the standard linear solid model the function β behaves approximately like the function $s + \kappa$ for some constant κ; in the other cases β resembles more a fractional power of $s + \kappa$, with exponent between 1/2 and 1.

Our solution to the problem of finding a rational approximation of $\widehat{P}_{\text{sub}} = (\zeta_1 s + 1)^2 (s + \zeta_0)^{-2} \beta(s) f(\beta(s))$ consists of three steps. First, we expand $\beta f(\beta)$ into an infinite product, where each factor is a linear fractional transformation, corresponding to a zero-pole pair of $\beta f(\beta)$. We identify the leading part of the error as a particular square root factor, and by dividing out this factor we get a significantly better convergence rate. We prove that the expansion converges uniformly in a region that is big enough so that the same expansion can be used for all different functions β. This means that we can use the same expansion in all the different cases; only the number of factors that one has to use to get a sufficiently good fit needs to be varied from one case to another.

The preceding step leads to an approximation of $\widehat{P}_{\text{sub}}$ which consists of linear fractional transformations of $\beta^2(s)$ and a square root factor. For models where β^2 is irrational we must approximate the linear fractional transformations of $\beta^2(s)$ by linear fractional transformations of s. For this step we suggest a very simple, low order approximation, and show with some numerical examples that this approximation can lead to good results. The idea behind this approximation is to separate the dynamic modes from the creep modes, and to ignore the latter type of modes. This seems to work quite well when the internal structural damping is small.

Finally, we have to take care of the square root factor that resulted from the product expansion of $\beta f(\beta)$. We show with asymptotic estimates and numerical experiments that we get good results by replacing the square root factor by a linear factor $(1 + \epsilon_N s)$ where $\epsilon_N \to 0$ as $N \to \infty$ at a rate that is determined by the product expansion for $\beta f(\beta)$.

Throughout the paper we give analytic estimates on the convergence rates at each approximation stage, and, in addition, we show with some numerical examples how large the actual error is at that stage. As these examples show, our theoretical estimates are very sharp.

References

[1] C. Foias, H. Özbay, and A. Tannenbaum. *Robust Control of Infinite Dimensional Systems, Frequencey Domain Methods*, volume 209 of *Lecture Notes in Control and Information Sciences.* Springer-Verlag, Berlin, 1996.

[2] K. E. Lenz, H. Özbay, A. Tannenbaum, J. Turi, and B. Morton. Frequency domain analysis and robust control design for an ideal flexible beam. *Automatica*, 27:947–961, 1991.

[3] P. J. Torvik and D. L. Bagley. Fractional derivatives in the description of damping materials and phenomena. In L. Rogers and J. C. Simonis, editors, *The Role of Damping in Vibration and Noise Control*, volume 5, pages 125–135, New York, 1988. The American Society of Mechanical Engineers.