# A Backward Characterization of Adjoint Strong Stability

Olof J. Staffans
Åbo Akademi University
Department of Mathematics
FIN-20500 Åbo, Finland
http://www.abo.fi/~staffans/

April 27, 2005

**Abstract**

We prove the following theorem: Let $A$ be a bounded linear operator on a reflexive Banach space with the property that all forward trajectories are bounded. Then the adjoint of $A$ is strongly stable if and only if $A$ does not have a nontrivial bounded backward trajectory. The same result is also valid in continuous time.

Let $A$ be a bounded linear operator on a reflexive Banach space $\mathcal{X}$. We call $A$ *stable* or *forward bounded* if it is true for every $x \in \mathcal{X}$ that the sequence $\{A^n x\}_{n=0}^{\infty}$ is bounded. It is *strongly stable* if it is true for every $x \in \mathcal{X}$ that $\lim_{n\to\infty} A^n x = 0$ (in the norm of $\mathcal{X}$). We shall refer the sequence $x_n = A^n x$, $n \in \mathbb{Z}^+ := \{0, 1, 2, \ldots\}$ as a *forward trajectory of* $A$ (with initial value $x$). Thus, $A$ is stable if and only if all forward trajectories are bounded, and $A$ is strongly stable if and only if all forward trajectories tend to zero at infinity.

By a *bounded backward trajectory of* $A$ we mean a bounded sequence $\{x_n\}_{n=-\infty}^{0}$ satisfying $x_n = Ax_{n-1}$ for all $n \in \mathbb{Z}^- := \{\ldots, -2, -1, 0\}$. This trajectory is *nontrivial* if it is not identically zero. Note that if $A$ is stable, then a nontrivial bounded backward trajectory of $A$ cannot tend to zero at $-\infty$. Backward trajectories appear naturally in, e.g., optimal control.

**Theorem 1.** *Let $A$ be a stable bounded linear operator on a reflexive Banach space $\mathcal{X}$. Then $A^*$ is strongly stable if and only if $A$ does not have a nontrivial bounded backward trajectory.*

The significance of this theorem is that it makes it possible to characterize the strong stability of $A^*$ entirely in terms of the original operator $A$, without any formal reference to $A^*$.

*Proof of Theorem 1.* We denote the adjoint of $\mathcal{X}$ by $\mathcal{X}^*$, and the value of $x^* \in \mathcal{X}^*$ applied to $x \in \mathcal{X}$ by $\langle x, x^* \rangle$.

Assume first that $A^*$ is strongly stable. Let $x := \{x_n\}_{n\in\mathbb{Z}^-}$ be a bounded backward trajectory of $A$, and let $\{x_n^*\}_{n\in\mathbb{Z}^+}$ be a forward trajectory of $A^*$. Then, for all $n \in \mathbb{Z}^+$,

$$\langle x_0, x_0^* \rangle = \langle A^n x_{-n}, x_0^* \rangle = \langle x_{-n}, (A^*)^n x_0^* \rangle = \langle x_{-n}, x_n^* \rangle. \tag{1}$$

Letting $n \to \infty$ and using the strong stability of $A^*$ and the boundedness of $x$, we find that $\langle x_0, x_0^* \rangle = 0$. This being true for all $x_0^* \in \mathcal{X}^*$, we must have $x_0 = 0$. Shifting $x$ $k$ steps to the right and repeating the same argument we find that $x_{-k} = 0$ for all $x \in \mathbb{Z}^+$. Thus, $x = 0$, and we have shown that $A$ does not have a nontrivial bounded backward trajectory.

Let us begin the proof of the converse part by observing that by the uniform boundedness principle, $\sup_{n\in\mathbb{Z}^+}\|A^n\| := M < \infty$, and hence also $\sup_{n\in\mathbb{Z}^+}\|(A^*)^n\| = M < \infty$. In particular $A^*$ is stable. Suppose that $A^*$ is not strongly stable. Choose some $x_0^* \in \mathcal{X}$ so that $x_n^* := (A^*)^n x_0 \not\to 0$ as $n \to \infty$. By the stability of $A^*$, this implies that $\inf_{n\in\mathbb{Z}^+}\|(A^*)^n x_0\| := \epsilon > 0$. We can therefore find some $x_{-n}^n \in \mathcal{X}$ with $\|x_{-n}^n\| \le 1/\epsilon$ such that $\langle x_{-n}^n, x_n^* \rangle = 1$. Let $x^n$ denote the sequence $\{x_k^n\}_{k\in\mathbb{Z}^-}$, where $x_k^n = A^{k-n} x_{-n}^n$ for $k \in [-n, 0]$ and $x_k^n = 0$ for $k < -n$. Then $\|x_k^n\| \le M/\epsilon$ for all $k \in \mathbb{Z}^-$. In particular, the sequence $\{x^n\}_{n\in\mathbb{Z}^+}$ is uniformly bounded in $\ell^\infty(\mathbb{Z}^+;\mathcal{X})$. Moreover, by construction, the elements of each sequence $x^n$ satisfy $x_k^n = A x_{k-1}^n$ for all $k \in [-n+1, 0]$. In particular, by (1),

$$\langle x_0^n, x_0^* \rangle = 1. \tag{2}$$

Since the unit ball in $\mathcal{X}$ is weakly sequentially compact, it is possible to find a subsequence $\{x^{n_{1,j}}\}_{j\in\mathbb{Z}^+}$ such that $x_0^{n_{1,j}}$ converges weakly to a limit $x_0$ in $\mathcal{X}$. It follows from (2) that $\langle x_0, x_0^* \rangle = 1$, hence $x_0 \ne 0$. By repeating the same argument with the original sequence $\{x^n\}_{n\in\mathbb{Z}^+}$ replaced by $\{x^{n_{1,j}}\}_{j\in\mathbb{Z}^+}$ we get another subsequence $\{x^{n_{2,j}}\}_{j\in\mathbb{Z}^+}$ such that both $x_0^{n_{2,j}}$ tends weakly to $x_0$ and $x_{-1}^{n_{2,j}}$ tends weakly to $x_{-1}$ for some $x_{-1} \in \mathcal{X}$. The operator $A$ is norm-continuous, hence weakly continuous, and therefore we must have $x_0 = A x_{-1}$. Continuing in the same way, with $\{x^{n_{1,j}}\}_{j\in\mathbb{Z}^+}$ replaced by $\{x^{n_{2,j}}\}_{j\in\mathbb{Z}^+}$ we get another subsequence $\{x^{n_{3,j}}\}_{j\in\mathbb{Z}^+}$ such that $x_{-1}^{n_{3,j}}$ tends weakly to $x_{-1}$ and $x_{-2}^{n_{3,j}}$ tends weakly to some vector $x_{-2}$ satisfying $x_{-1} = A x_{-2}$. The same process can be repeated indefinitely to produce a sequence $\{x_k\}_{k\in\mathbb{Z}^-}$, where $\|x_k^n\| \le M/\epsilon$ for all $k \in \mathbb{Z}^-$ and $x_k = A x_{k-1}$ for all $k \in \mathbb{Z}^-$. This proves the existence of a nontrivial bounded backward trajectory of $A$. $\square$

The same result is also valid in continuous time. In this case we replace $A$ by a $C_0$ semigroup $t \mapsto \mathfrak{A}^t$, $t \in \mathbb{R}^+ := [0, \infty)$. A *forward trajectory of* $\mathfrak{A}$ is defined on $\mathbb{R}^+$, and it is of the type $t \mapsto \mathfrak{A}^t x_0$ for some initial value $x_0$. The semigroup $\mathfrak{A}$ is *bounded* (or stable) if all forward trajectories are bounded, and it it *strongly stable* if all forward trajectories tend to zero at infinity. A *backward trajectory* is a continuous function $x$ defined on $\mathbb{R}^- = (-\infty, 0]$ satisfying $x(t) = \mathfrak{A}^{t-s} x(s)$ for all $s \le t \le 0$. It is nontrivial if it is does not vanish identically. The *adjoint* semigroup $t \mapsto \mathfrak{A}^{*t}$ is defined by $\mathfrak{A}^{*t} = (\mathfrak{A}^t)^*$, and it is also a $C_0$ semigroup.

**Theorem 2.** *Let $t \mapsto \mathfrak{A}^t$ be a bounded $C_0$ semigroup on a reflexive Banach space $\mathcal{X}$. Then the adjoint semigroup $t \mapsto \mathfrak{A}^{*t}$ is strongly stable if and only if $\mathfrak{A}$ does not have a nontrivial bounded backward trajectory.*

*Proof.* Define $A = \mathfrak{A}^1$. Then $\mathfrak{A}^*$ is strongly stable if and only if $A^*$ is strongly stable. By Theorem 1, this is true if and only if $A$ does not have a nontrivial bounded backward trajectory. However, there is a one-to-one correspondence between the bounded nontrivial backward trajectories of $\mathfrak{A}$ and those of $A$: Given a backward trajectory $t \mapsto x(t)$, $t \in \mathbb{R}^-$, of $\mathfrak{A}$ we get a backward trajectory $\{x_n\}_{n=-\infty}^0$ of $A$ by defining $x_n = x(n)$, and given a backward trajectory $\{x_n\}_{n=-\infty}^0$ of $A$ we can fill it in to get a backward trajectory of $\mathfrak{A}$ by defining $x(t) = \mathfrak{A}^{t-[t]}x_{[t]}$, $t \in \mathbb{R}^-$, where $[t]$ is the largest integer less than or equal to $t$. Thus, $\mathfrak{A}$ does not have a nontrivial backward trajectory if and only if $\mathfrak{A}^*$ is strongly stable. □

*Acknowledgment* The importance of the notion of a bounded backward trajectory became clear to us in a recent joint work with Mark Opmeer (not yet published) on an alternative interpretation of the discrete time dual forward cost minimization problem.