The Stationary State/Signal Systems Story

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Abstract. We give an introduction to the theory of linear stationary s/s (state/signal) systems in continuous time. A s/s system has a state space which plays the same role as the state space of an ordinary i/s/o (input/state/output) system, but it differs from an i/s/o systems in the sense that the interaction signal which connects the system to the outside world has not been divided a priori into one part which is called the "input" and another part which is called the "output". The class of s/s systems can be used to model, e.g., linear time-invariant circuits which may contain both lumped and distributed components. To each s/s system corresponds in general an infinite number of i/s/o systems which differ from each other by the choice of how the interaction signal has been divided into an input part and output part. Each such i/s/o system is called an i/s/o representation of the given s/s system.

We begin by giving an introduction to the time domain theory for i/s/o and s/s systems, then continue by taking a brief look at the frequency domain theory for i/s/o and s/s systems, and end with a short overview of the notions of passivity or conservativity of i/s/o and s/s systems. In all cases the s/s results that we present can be formulated in such a way that they do not depend on any particular i/s/o representation of the s/s system, but it is still true that there is a strong connection between the central properties of a s/s system and the corresponding properties of its i/s/o representations.

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1. Introduction to state/signal systems

1.1. Input/state/output systems in the time domain

A "well-posed" linear stationary discrete time i/s/o (input/state/output) system is of the form

$$\Sigma_{\mathrm{i/s/o}}:\begin{cases} x(n+1) = Ax(n) + Bu(n), \\ \quad y(n) = Cx(n) + Du(n), \end{cases} \qquad n \in \mathbb{Z}^+. \tag{1.1}$$

Here the *input* u, the *state* x, and the *output* y take their values in three Hilbert spaces, the input space $\mathcal{U}$, the state space $\mathcal{X}$, and the output space $\mathcal{Y}$, respectively, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, and A, B, C, and D, are bounded linear operators with the appropriate domain and range spaces. These operators are called as follows: A is the main operator, B is the control operator, C is the observation operator, and D is the feed-through operator. By a *future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a sequence $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ defined on $\mathbb{Z}^+$ with values in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ which satisfies (1.1) for all $n \in \mathbb{Z}^+$.

If we here replace the discrete time axis $\mathbb{Z}^+$ by the continuous time axis $\mathbb{R}^+ = [0, \infty)$ and at the same time replace the first equation in (1.1) by the corresponding differential equation, then we get a bounded linear stationary continuous time i/s/o system of the form

$$\Sigma_{\mathrm{i/s/o}}:\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad t \in \mathbb{R}^+. \tag{1.2}$$

The input u, the state x, and the output y still take their values in the Hilbert spaces $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, respectively, and the main operator A, the control operator B, the observation operator C, and the feed-through operator D are still bounded linear operators. By a *classical future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which satisfies (1.2) for all $t \in \mathbb{R}^+$, with x continuously differentiable with values in $\mathcal{X}$ and $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ continuous with values in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$.

Unfortunately, typical stationary i/s/o systems modelled by partial differential equations are *not bounded* in the sense that even if it might be possible to describe the dynamics of the system with an equation of the type (1.2), the operators A, B, C, and D need not be bounded. For this reason a more general version of (1.2) is needed. Clearly, equation (1.2) can be rewritten in the form

$$\Sigma_{\mathrm{i/s/o}}: \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \tag{1.3}$$

where S is the bounded block matrix operator $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. We get a much more general class of linear stationary continuous time i/s/o systems by simply allowing the operator S in (1.3) bo be unbounded (but still closed) and rewriting (1.3) in the form

$$\Sigma_{\mathrm{i/s/o}}\colon \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}(S), \\ \begin{bmatrix} \dot x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{1.4}$$

This class of systems covers "all" the standard models from mathematical physics. We call S *the generator* of $\Sigma_{\mathrm{i/s/o}}$. Usually the domain $\mathrm{dom}\,(S)$ of S is assumed to be dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$.

Definition 1.1.

(i) By a regular (continuous time stationary) *i/s/o (input/state/output)* node we mean a colligation $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are Hilbert spaces, and $S\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is a closed linear operator with dense domain.

(ii) The *main operator* A of $\Sigma_{\mathrm{i/s/o}}$ (or of S) is defined by

$$\begin{aligned} \mathrm{dom}\,(A) &:= \left\{ x \in \mathcal{X} \,\middle|\, \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S) \right\}, \\ Ax &:= \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \qquad x \in \mathrm{dom}\,(A). \end{aligned} \tag{1.5}$$

Here $\begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix}$ stands for the operator which maps $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ into x.

(iii) By a *classical future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which satisfies (1.4) for all $t \in \mathbb{R}^+$, with x continuously differentiable with values in $\mathcal{X}$ and $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ continuous with values in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$.

(iv) By a *generalized future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which is the limit of a sequence $\left[\begin{smallmatrix} x_n \\ u_n \\ y_n \end{smallmatrix}\right]$ of classical future trajectories of $\Sigma_{\mathrm{i/s/o}}$ in the sense that $x_n \to x$ in $C(\mathbb{R}^+; \mathcal{X})$ and $\left[\begin{smallmatrix} u_n \\ y_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$.

(v) By a regular (time domain) *i/s/o system* system we mean an i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ together with the sets of all classical and generalized future trajectories of Σ.

Above $C(\mathbb{R}^+; \mathcal{X})$ stands for the space of continuous function on $\mathbb{R}^+$ with values in $\mathcal{X}$, and convergence in $C(\mathbb{R}^+; \mathcal{X})$ means uniform convergence on each finite subinterval of $\mathbb{R}^+$. The space $L^2_{\mathrm{loc}}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$ consists of functions which belong locally to L^2 over $\mathbb{R}^+$ with values in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and convergence in $L^2_{\mathrm{loc}}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$ means convergence in L^2 on each finite subinterval of $\mathbb{R}^+$.

Note that if S is bounded, then S has a block matrix decomposition $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, and (1.4) is equivalent to (1.2).

1.2. State/signal systems in the time domain

The idea behind the definition of a s/s (state/signal) system is to remove the distinction between the "input" and the "output" of an i/s/o system. This can be done in several ways. One way is to define the *signal space* to be the product $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ of $\mathcal{X}$ and $\mathcal{Y}$, and to replace the input u and the output y by the combined i/o (input/output) *signal* $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$. After that one absorbs the "output" equation in (1.4) into the domain of a new operator F (whose domain will no longer be dense in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$), and rewrites (1.4) in the form

$$\Sigma\colon \begin{cases} \begin{bmatrix} x(t)\\ w(t)\end{bmatrix} \in \operatorname{dom}(F), \\ \dot x(t) = F\left(\begin{bmatrix} x(t)\\ w(t)\end{bmatrix}\right), \end{cases} \qquad t\in\mathbb{R}^+, \tag{1.6}$$

$$\operatorname{dom}(F) = \left\{ \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix} \in \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} x_0\\ u_0\end{smallmatrix}\right] \in \operatorname{dom}(S),\ y_0 = \begin{bmatrix} 0 & 1_{\mathcal{Y}}\end{bmatrix} S \left[\begin{smallmatrix} x_0\\ u_0\end{smallmatrix}\right]\right\},$$

$$F\begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0\end{bmatrix} S\left[\begin{smallmatrix} x_0\\ u_0\end{smallmatrix}\right],$$

where $\begin{bmatrix} 0 & 1_{\mathcal{Y}}\end{bmatrix}$ stands for the operator which maps $\left[\begin{smallmatrix} x\\ y\end{smallmatrix}\right] \in \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ into y. Note that (1.6) can be regarded as a special case of (1.4) with $\mathcal{U} = \mathcal{W}$ and $\mathcal{Y} = \{0\}$, apart from the fact that $\operatorname{dom}(F)$ need no longer be dense in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$.

We can also go one step further and replace the operator F in (1.6) by its graph $V = \operatorname{gph}(F)$. More precisely, we still take $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$, define the *node space* $\mathfrak{K}$ to be $\mathfrak{K} = \begin{bmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{bmatrix}$, and rewrite (1.6) in the form

$$\Sigma\colon \begin{bmatrix} \dot x(t)\\ x(t)\\ w(t)\end{bmatrix} \in V, \qquad t\in\mathbb{R}^+. \tag{1.7}$$

The *generating subspace* $V = \operatorname{gph}(F)$ of Σ can alternatively be interpreted as a reordered version of the graph of the original generator S in (1.4):

$$\begin{aligned} V &= \left\{ \begin{bmatrix} z_0\\ x_0 \\ \left[\begin{smallmatrix} u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix} \in \mathfrak{K} \,\middle|\, \left[\begin{smallmatrix} x_0\\ u_0\end{smallmatrix}\right] \in \operatorname{dom}(S),\ \left[\begin{smallmatrix} z_0\\ y_0\end{smallmatrix}\right] = S\left[\begin{smallmatrix} x_0\\ u_0\end{smallmatrix}\right]\right\} \\ &= \left\{ \begin{bmatrix} z_0\\ x_0 \\ \left[\begin{smallmatrix} u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix} \in \mathfrak{K} \,\middle|\, \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix} \in \operatorname{dom}(F),\ z_0 = F\begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0\\ y_0\end{smallmatrix}\right]\end{bmatrix}\right\}. \end{aligned} \tag{1.8}$$

Definition 1.2.

(i) By a *s/s (state/signal) node* we mean a colligation $\Sigma = (V;\mathcal{X},\mathcal{W})$, where $\mathcal{X}$ and $\mathcal{W}$ are Hilbert spaces and V is a closed subspace of the product space space $\mathfrak{K} = \begin{bmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{bmatrix}$.

(ii) By a *classical future trajectory* of Σ we mean a pair of functions $\left[\begin{smallmatrix} x\\ w\end{smallmatrix}\right]$ which satisfies (1.7) for all $t\in\mathbb{R}^+$, with x continuously differentiable with values in $\mathcal{X}$ and w continuous with values in $\mathcal{W}$.

(iii) By a *generalized future trajectory* of Σ we mean a pair of functions $\left[\begin{smallmatrix} x\\ w\end{smallmatrix}\right]$ which is the limit of a sequence $\left[\begin{smallmatrix} x_n\\ w_n\end{smallmatrix}\right]$ of classical future trajectories of Σ in the sense that $x_n \to x$ in $C(\mathbb{R}^+;\mathcal{X})$ and $w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W})$.

(iv) By a (time domain) *s/s system* system we mean an s/s node $\Sigma = (V;\mathcal{X},\mathcal{W})$ together with the sets of all classical and generalized future trajectories of Σ.

It is also possible to go in the opposite direction, i.e., to start with a state/signal system of the type (1.7), and to rewrite it into an i/s/o system of the type (1.4) under some additional "regularity" assumptions on the generating subspace V. In this case we start by decomposing the signal space $\mathcal{W}$ (which now is supposed to be an arbitrary Hilbert space with no particular structure) into a direct sum $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, and try to rewrite (1.7) into the form (1.4) with $\mathcal{U}$ as input space and $\mathcal{Y}$ as output space, for some closed operator S with dense domain. This will *not be possible for every possible decomposition* $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$. The closedness of S is not a problem (since the graph of S can be "identified" with V after the permutation of some of the components of V), but the *existence* of a (single-valued) *operator S with dense domain* is more problematic. This is equivalent to the following two conditions on V and on the decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$:

(i) if $\left[\begin{smallmatrix} z \\ 0 \\ y \end{smallmatrix}\right] \in V$ and $y \in \mathcal{Y}$, then $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = 0$,
(ii) the projection onto the second component of V and $\mathcal{U}$ along the first component of V and $\mathcal{Y}$ is dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$.

The first of these conditions means that the z-component and the y-component of a vector $\left[\begin{smallmatrix} z \\ x \\ u+y \end{smallmatrix}\right] \in V$ is determined uniquely by x and u, and the second conditions says that the map from $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ should have dense domain. If these two conditions hold, and if we denote the linear map from $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ by S, then S is the generator of a regular i/s/o node $\Sigma_{\mathrm{i/s/o}}$, and V has the graph representation

$$V := \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \in \operatorname{dom}(S) \text{ and } \begin{bmatrix} z \\ P_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} = S \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \right\}. \tag{1.9}$$

Here $P_{\mathcal{U}}^{\mathcal{Y}}$ is the projection onto $\mathcal{U}$ along $\mathcal{Y}$, and $P_{\mathcal{Y}}^{\mathcal{U}}$ is the complementary projection.

Definition 1.3. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. By a *regular i/s/o representation* of Σ we mean a regular i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $\mathcal{U} \dotplus \mathcal{Y}$ is a direct sum decomposition of $\mathcal{W}$ and V and S are connected to each other by (1.9).

Not every s/s node has a regular i/s/o representation. It is not difficult to see that if $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has a regular i/s/o representation, then Σ must be "regular" in the following sense:

Definition 1.4. A s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is *regular* if it satisfies the following two conditions:

(i) $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Rightarrow z = 0$;
(ii) The projection of V onto its middle component is dense in $\mathcal{X}$.

The two conditions (i) and (ii) above have the following interpretations: (i) means that $\dot{x}(t)$ in (1.7) is determined uniquely by $x(t)$ and $w(t)$, and (ii) permits the set of all initial states $x(0)$ of a classical future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ to be dense in the state space $\mathcal{X}$

Theorem 1.5. *Every regular i/s/o node has at least one (and usually infinitely many) regular i/s/o representations.*

The proof of this theorem is found in [AS14b, Chapter 2].

If a s/s node $\Sigma = (V;\mathcal{X},\mathcal{W})$ has a *bounded* i/s/o representation, then V must satisfy the stronger conditions (i)–(iii) listed below:

Definition 1.6. A s/s node $\Sigma = (V;\mathcal{X},\mathcal{W})$ is *bounded* if it satisfies the following conditions:

(i) $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Rightarrow z = 0$;

(ii) For every $x_0 \in \mathcal{X}$ there exists some $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$.

(iii) The projection of V onto its second and third components is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

The interpretation of condition (i) in Definition 1.6 is the same as in Definition 1.4. This condition is equivalent to the condition that V has a graph representation

$$V = \left\{ \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \middle| \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \operatorname{dom}(F) \text{ and } z = F\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \right\} \tag{1.10}$$

for some closed operator $F\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \to \mathcal{X}$. Condition (iii) says that $\operatorname{dom}(F)$ is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, and hence by the closed graph theorem, F is continuous. In other words, $\dot{x}(t)$ in (1.7) depends continuously on $\left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right]$. Finally, condition (ii) permits every $x_0 \in \mathcal{X}$ to be the initial state $x(0)$ of some classical future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ.

Theorem 1.7. *Every bounded s/s node has at least one (and usually infinitely many) bounded i/s/o representations.*

Also the proof of this theorem is found in [AS14b, Chapter 2].

As we noticed above, a s/s node Σ cannot have a regular i/s/o representation unless Σ is regular. From time to time it is useful to also study s/s nodes which are not regular. In that case it is still possible to obtain i/s/o representations, but these will no longer be regular. Instead they will be i/s/o nodes of the following type:

Definition 1.8.

(i) By a (continuous time stationary) *i/s/o (input/state/output)* node we mean a colligation $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$, where $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are Hilbert spaces, and $S\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is a closed multi-valued linear operator.

(ii) The (multi-valued) *main operator* A of $\Sigma_{\mathrm{i/s/o}}$ (or of S) is defined by

$$\begin{aligned} &\operatorname{dom}(A) := \{x \in \mathcal{X} \,|\, \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \operatorname{dom}(S)\}, \\ &z \in Ax \Leftrightarrow z \in \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \qquad x \in \operatorname{dom}(A). \end{aligned} \tag{1.11}$$

(iii) By a *classical future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which satisfies

$$\Sigma_{\mathrm{i/s/o}}\colon \begin{cases} \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+, \tag{1.12}$$

with x continuously differentiable with values in $\mathcal{X}$ and $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ continuous with values in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$.

(iv) By a *generalized future trajectory* of $\Sigma_{\mathrm{i/s/o}}$ we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which is the limit of a sequence $\left[\begin{smallmatrix} x_n \\ u_n \\ y_n \end{smallmatrix}\right]$ of classical future trajectories of $\Sigma_{\mathrm{i/s/o}}$ in the sense that $x_n \to x$ in $C(\mathbb{R}^+;\mathcal{X})$ and $\left[\begin{smallmatrix} u_n \\ y_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+;\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$.

(v) By a (time domain) *i/s/o system* system we mean an i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ together with the sets of classical and generalized future trajectories of $\Sigma_{\mathrm{i/s/o}}$.

See, e.g., [AS14b] for a short introduction to the notion of a multi-valued linear operator.

Definition 1.9. Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s node. By an *i/s/o representation* of Σ we mean an i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$, where $\mathcal{U} \dotplus \mathcal{Y}$ is a direct sum decomposition of $\mathcal{W}$ and V and S are connected to each other by

$$V := \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \in \operatorname{dom}(S) \text{ and } \begin{bmatrix} z \\ P_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} \in S \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \right\}. \tag{1.13}$$

See [AS14b, Chapter 2] for a more detailed description of this class of non-regular i/s/o nodes and i/s/o representations.

1.3. Various notions for state/signal systems

The definition of a (regular or non-regular) i/s/o representation of a (regular or non-regular) s/s node immediately implies the following results:

Lemma 1.10. *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s node, and let $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ be an i/s/o representation of Σ. Then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical or generalized future trajectory of Σ if and only if $\left[\begin{smallmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \\ P_{\mathcal{Y}}^{\mathcal{U}} w \end{smallmatrix}\right]$ is a classical or generalized future trajectory of $\Sigma_{\mathrm{i/s/o}}$.*

Thanks to Lemma 1.10, it is possible to extend all those notions for i/s/o systems that can be expressed in terms of properties of classical or generalized future trajectories of i/s/o systems to the s/s case. In this way it is possible to introduce and study, e.g., the following notions for s/s systems:

- driving variable and output nulling representations of s/s systems,
- existence and uniqueness of classical and generalized trajectories of s/s systems,
- well-posedness of s/s systems,
- s/s systems of boundary control type,
- controllability and observability of s/s systems,
- stability, stabilizability, and detectability of s/s systems,
- past, future, and two-sided time domain behaviors of s/s systems,
- frequency domain analysis of s/s systems,

- external equivalence of s/s systems,
- intertwinements of s/s systems.
- similarities and pseudo-similarities of s/s systems,
- restrictions, projections, compressions, and dilations of s/s systems,
- minimal s/s systems,
- the dual and the adjoint of a s/s system,
- passive past, future, and two-sided time domain behaviors,
- passive frequency domain behaviors,
- optimal and $*$-optimal s/s systems (available storage and required supply),
- passive balanced s/s systems,
- energy and co-energy preserving s/s systems,
- controllable energy-preserving and observable co-energy preserving realizations of passive signal bundles,
- quadratic optimal control and KYP-theory for s/s systems,
- s/s systems with extra symmetries (reality, reciprocity, real-reciprocity),
- relationships between the symmetries of a s/s system and the symmetries of its i/s/o representations,
- s/s versions of the de Branges complementary spaces of type $\mathcal{H}$ and $\mathcal{D}$.

Some of these notions are discussed in [AS14b], some of them are discussed in the other articles listed in the reference list, and some of them still remain to be properly developed.

In this article we shall still take a closer look at

- i/s/o and s/s systems in the frequency domain,
- The characteristic node and signal bundles of a s/s system,
- $\mathcal{J}$-passive and $\mathcal{J}$-conservative i/s/o systems,
- passive and conservative s/s systems,
- passive signal bundles,
- conservative realizations of passive signal bundles.

2. State/Signal Systems in the Frequency Domain

2.1. Input/state/output systems in the frequency domain

Let $\Sigma_{\rm i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node, and let $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ be a classical future trajectory of $\Sigma_{\rm i/s/o}$. If x, $\dot{x}$, u, and y in (1.4) are Laplace transformable, then it follows from (1.4) (since we assume S to be closed) that the Laplace transforms $\hat{x}$, $\hat{u}$, and $\hat{y}$ of x, u, and y satisfy the *i/s/o resolvent equation* (with $x^0 := x(0)$)

$$\widehat{\Sigma}_{\rm i/s/o}\colon \begin{cases} \left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{smallmatrix}\right] \in \operatorname{dom}(S), \\ \begin{bmatrix} \lambda\hat{x}(\lambda) - x^0 \\ \hat{y}(\lambda) \end{bmatrix} \in S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \end{cases} \tag{2.1}$$

for all those $\lambda \in \mathbb{C}$ for which the Laplace transforms converge (to see this it suffices to multiply by (1.4) by $e^{-\lambda t}$ and integrate by parts in the $\dot{x}$-component.) If $\Sigma_{i/s/o}$ is regular, or more generally, if S is single-valued, then we may replace the second inclusion "$\in$" in (2.1) by the equality "$=$"

Definition 2.1. Let $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node.

(i) $\lambda \in \mathbb{C}$ belongs to the *resolvent set* $\rho(\Sigma_{i/s/o})$ *of* $\Sigma_{i/s/o}$ if for every $x^0 \in \mathcal{X}$ and for every $\hat{u}(\lambda) \in \mathcal{U}$ there is a unique pair of vectors $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ satisfying the i/s/o resolvent equation (2.1). This set is alternatively called the *i/s/o resolvent set of* S and denoted by $\rho_{i/s/o}(S)$.

(ii) For each $\lambda \in \rho(\Sigma_{i/s/o})$ we define the *i/s/o resolvent matrix* $\widehat{\mathfrak{S}}(\lambda)$ *of* $\Sigma_{i/s/o}$ *at* λ to be the linear operator $\left[\begin{smallmatrix} x^0 \\ \hat{u}(\lambda) \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$.

Since S is assumed to be closed, also $\widehat{\mathfrak{S}}(\lambda)$ is closed (see [AS14b, Chapter 4] for details). Therefore by the closed graph theorem, for each $\lambda \in \rho(\Sigma_{i/s/o})$ the i/s/o resolvent matrix $\widehat{\mathfrak{S}}(\lambda)$ is a bounded linear operator. In particular, this implies that $\widehat{\mathfrak{S}}(\lambda)$ has a block matrix representation

$$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}, \quad \lambda \in \rho(\Sigma_{i/s/o}), \tag{2.2}$$

where each of the components $\widehat{\mathfrak{A}}(\lambda)$, $\widehat{\mathfrak{B}}(\lambda)$, $\widehat{\mathfrak{C}}(\lambda)$, and $\widehat{\mathfrak{D}}(\lambda)$ is a bounded linear operator with the appropriate domain and range space. Thus, if $\lambda \in \rho(\Sigma_{i/s/o})$, then (2.1) holds if and only if

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix}. \tag{2.3}$$

Conversely, if there exist four bounded linear operators $\widehat{\mathfrak{A}}(\lambda)$, $\widehat{\mathfrak{B}}(\lambda)$, $\widehat{\mathfrak{C}}(\lambda)$, and $\widehat{\mathfrak{D}}(\lambda)$ with the appropriate domain and ranges spaces such that (2.1) is equivalent to (2.3), then $\lambda \in \rho(\Sigma_{i/s/o})$, and the operator $\widehat{\mathfrak{S}}(\lambda)$ defined by (2.2) is the i/s/o resolvent matrix of $\Sigma_{i/s/o}$ at the point λ.

Definition 2.2. The components $\widehat{\mathfrak{A}}$, $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{C}}$, and $\widehat{\mathfrak{D}}$ of the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ are called as follows:

(i) $\widehat{\mathfrak{A}}$ is the *s/s (state/state) resolvent function* of $\Sigma_{i/s/o}$,
(ii) $\widehat{\mathfrak{B}}$ is the *i/s (input/state) resolvent function* of $\Sigma_{i/s/o}$,
(iii) $\widehat{\mathfrak{C}}$ is the *s/o (state/output) resolvent function* of $\Sigma_{i/s/o}$,
(iv) $\widehat{\mathfrak{D}}$ is the *i/o (input/output) resolvent function* of $\Sigma_{i/s/o}$,

The state/state resolvent function $\widehat{\mathfrak{A}}$ is the usual resolvent of the main operator A of $\Sigma_{i/s/o}$. Here the resolvent set of A and the resolvent of A is defined in the same way as in Definition 2.1 with $\mathcal{U} = \mathcal{Y} = \{0\}$, i.e., λ belongs to the resolvent set $\rho(A)$ of A if it is true for every $x^0 \in \mathcal{X}$ that there exists a unique $z_\lambda \in \mathcal{X}$ such that $\lambda z_\lambda - x^0 \in A z_\lambda$, in which case the the bounded linear operator which maps x^0 into z_λ is called the the resolvent of A (evaluated at λ). This operator is usually denoted by $(\lambda - A)^{-1}$ since it is the (single-valued) inverse of the (possibly multi-valued) operator $\lambda - A$.

The i/o resolvent function $\widehat{\mathfrak{D}}$ is known in the literature under different names, such as "the transfer function", or "the characteristic function", or "the Weyl function". In operator theory the i/s resolvent function $\widehat{\mathfrak{B}}$ is sometimes called the Γ-field.

The fact that (2.1) and (2.3) are equivalent to each other leads to the following graph representations of S and $S-\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right]$ which will be needed later:

Lemma 2.3. *Let $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ be an i/s/o node with $\rho(\Sigma_{\mathrm{i/s/o}}) \neq \emptyset$. Then for each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ the graph of $(S-\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right])$ has the representation*

$$\operatorname{gph}\left(S-\begin{bmatrix}\lambda & 0\\ 0 & 0\end{bmatrix}\right) = \operatorname{rng}\left(\begin{bmatrix}-1_{\mathcal{X}} & 0\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\\ \hline \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ 0 & 1_{\mathcal{U}}\end{bmatrix}\right), \tag{2.4}$$

where $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$ is the i/s/o resolvent matrix of $\Sigma_{\mathrm{i/s/o}}$, and the graph of S has the representation

$$\operatorname{gph}(S) = \operatorname{rng}\left(\begin{bmatrix}\lambda\widehat{\mathfrak{A}}(\lambda) - 1_{\mathcal{X}} & \lambda\widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\\ \hline \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ 0 & 1_{\mathcal{U}}\end{bmatrix}\right). \tag{2.5}$$

Definition 2.1 above is both natural and simple, and it may be surprising that in the case where S is single-valued and densely defined the above definition is equivalent to the condition that S is a so called "operator node" in the sense of [Sta05].

Definition 2.4 ([Sta05, Definition 4.7.2]). By an *operator node* (in the sense of [Sta05]) on a triple of Hilbert spaces $(\mathcal{X},\mathcal{U},\mathcal{Y})$ we mean a linear operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ with the following properties:

(i) S is closed.
(ii) The main operator A of S has dense domain and nonempty resolvent set.
(iii) $\begin{bmatrix}1_{\mathcal{X}} & 0\end{bmatrix} S$ can be extended to a bounded linear operator $\begin{bmatrix}A_{-1} & B\end{bmatrix} : \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \mathcal{X}_{-1}$, where $\mathcal{X}_{-1}$ is the so called *extrapolation space* induced by A (i.e., the completion of $\mathcal{X}$ with respect to the norm $\|x\|_{\mathcal{X}_{-1}} = \|(\alpha - A)^{-1}x\|_{\mathcal{X}}$ where α is some fixed point in $\rho(A)$).
(iv) $\operatorname{dom}(S) = \left\{\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right] \,\middle|\, A_{-1}x + Bu \in \mathcal{X}\right\}$.

Theorem 2.5. *An operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ is an operator node in the sense of Definition 2.4 if and only if $\operatorname{dom}(S)$ is dense in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ and $\rho_{\mathrm{i/s/o}}(S) \neq \emptyset$. Moreover, if $\rho_{\mathrm{i/s/o}}(S) \neq \emptyset$, then $\rho_{\mathrm{i/s/o}}(S) = \rho(A)$ where A is the main operator of S.*

The proof of this theorem is given in [AS14b, Chapter 4].

As the following lemma shows, it is possible to use the s/s resolvent function $\widehat{\mathfrak{A}}$ to check the regularity of an i/s/o system $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ with nonempty resolvent set.

Lemma 2.6. *Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a i/s/o node with $\rho(\Sigma_{\mathrm{i/s/o}}) \neq \emptyset$, with main operator A, and with and s/s resolvent function $\widehat{\mathfrak{A}}$. Then*

(i) *The following conditions are equivalent:*
  (a) *S is single-valued;*
  (b) *A is single-valued;*
  (c) *$\widehat{\mathfrak{A}}(\lambda)$ is injective for some $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ (or equivalently, for all $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$).*

(ii) *Also the following conditions are equivalent:*
  (a) $\mathrm{dom}\,(S)$ *is dense in* $\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$*;*
  (b) $\mathrm{dom}\,(A)$ *is dense in $\mathcal{X}$;*
  (c) *$\widehat{\mathfrak{A}}(\lambda)$ has dense range for some $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ (or equivalently, for all $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$).*

In particular, $\Sigma_{\mathrm{i/s/o}}$ is a regular i/s/o system if and only if A is single-valued and has dense domain, or equivalently, if and only if $\widehat{\mathfrak{A}}(\lambda)$ is injective and has dense range for some $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ (or equivalently, for all $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$).

The proof of this lemma is given in [AS14b, Chapter 4].

The i/s/o resolvent matrix has the following properties:

Lemma 2.7. *Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node with $\rho(\Sigma_{\mathrm{i/s/o}}) \neq \emptyset$. Then the resolvent set $\rho(\Sigma_{\mathrm{i/s/o}})$ of $\Sigma_{\mathrm{i/s/o}}$ is open, the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ of $\Sigma_{\mathrm{i/s/o}}$ is analytic on $\rho(\Sigma_{\mathrm{i/s/o}})$, and it satisfies the* i/s/o resolvent identity

$$\widehat{\mathfrak{S}}(\lambda) - \widehat{\mathfrak{S}}(\mu) = \widehat{\mathfrak{S}}(\mu) \begin{bmatrix} (\mu-\lambda) & 0 \\ 0 & 0 \end{bmatrix} \widehat{\mathfrak{S}}(\lambda) = \widehat{\mathfrak{S}}(\lambda) \begin{bmatrix} (\mu-\lambda) & 0 \\ 0 & 0 \end{bmatrix} \widehat{\mathfrak{S}}(\mu) \tag{2.6}$$

for all μ, $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$. In terms of the components of the i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$ the above identity can be rewritten into the equivalent form

$$\begin{aligned}
\widehat{\mathfrak{A}}(\lambda) - \widehat{\mathfrak{A}}(\mu) &= (\mu-\lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{A}}(\lambda) = (\mu-\lambda)\widehat{\mathfrak{A}}(\lambda)\widehat{\mathfrak{A}}(\mu), \\
\widehat{\mathfrak{B}}(\lambda) - \widehat{\mathfrak{B}}(\mu) &= (\mu-\lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{B}}(\lambda) = (\mu-\lambda)\widehat{\mathfrak{A}}(\lambda)\widehat{\mathfrak{B}}(\mu), \\
\widehat{\mathfrak{C}}(\lambda) - \widehat{\mathfrak{C}}(\mu) &= (\mu-\lambda)\widehat{\mathfrak{C}}(\mu)\widehat{\mathfrak{A}}(\lambda) = (\mu-\lambda)\widehat{\mathfrak{C}}(\lambda)\widehat{\mathfrak{A}}(\mu), \\
\widehat{\mathfrak{D}}(\lambda) - \widehat{\mathfrak{D}}(\mu) &= (\mu-\lambda)\widehat{\mathfrak{C}}(\mu)\widehat{\mathfrak{B}}(\lambda) = (\mu-\lambda)\widehat{\mathfrak{C}}(\lambda)\widehat{\mathfrak{B}}(\mu).
\end{aligned} \tag{2.7}$$

The proof of this lemma is given in [AS14b, Chapter 4].

Motivated by Lemma 2.7 we make the following definition.

Definition 2.8. Let Ω be an open subset of the complex plane $\mathbb{C}$. An analytic $\mathcal{B}(\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]; \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right])$-valued function $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$ defined in Ω is called an *i/s/o pseudo-resolvent in* $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega)$ if it satisfies the identity (2.6) for all μ, $\lambda \in \Omega$.

Thus, the i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$ of an i/s/o node $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with $\rho(\Sigma_{\mathrm{i/s/o}}) \neq \emptyset$ is an i/s/o pseudo-resolvent in $\rho(\Sigma_{\mathrm{i/s/o}})$.

In [Opm05] Mark Opmeer makes systematic use of the notion of an i/s/o pseudo-resolvent, but instead of calling $\widehat{\mathfrak{S}}$ an i/s/o pseudo-resolvent he calls $\widehat{\mathfrak{S}}$ a "resolvent linear system", and calls $\widehat{\mathfrak{A}}$ the "pseudo-resolvent", $\widehat{\mathfrak{B}}$

the "incoming wave function", $\widehat{\mathfrak{C}}$ the "outgoing wave function", and $\mathfrak{D}$ the "characteristic function" of the resolvent linear system $\widehat{\mathfrak{S}}$. In the same article he also investigates what can be said about time domain trajectories (in the distribution sense) of resolvent linear systems satisfying some additional conditions. One of these additional set of conditions is that Ω should contain some right-half plane and that $\widehat{\mathfrak{S}}$ should satisfy a polynomial growth bound in this right-half plane.

The converse of Lemma 2.7 is also true in the following form.

Theorem 2.9. *Let Ω be an open subset of the complex plane $\mathbb{C}$. Then every i/s/o pseudo-resolvent $\widehat{\mathfrak{S}}$ in $(\mathcal{X},\mathcal{U},\mathcal{Y};\Omega)$ is the restriction to Ω of the i/s/o resolvent of some i/s/o node $\Sigma_{\rm i/s/o} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ satisfying $\rho(\Sigma_{\rm i/s/o}) \supset \Omega$. The i/s/o node $\Sigma_{\rm i/s/o}$ is determined uniquely by $\widehat{\mathfrak{S}}(\lambda)$ where λ is some arbitrary point in Ω, and $\widehat{\mathfrak{S}}$ has a unique extension to $\rho(\Sigma_{\rm i/s/o})$. This extension is maximal in the sense that $\widehat{\mathfrak{S}}$ cannot be extended to an i/s/o pseudo-resolvent on any larger open subset of $\mathbb{C}$.*

See [AS14b, Chapter 4] for the proof.

Theorem 2.9 is well-known in the case where the system has no input and no output (so that S is equal to its main operator A), and where $\widehat{\mathfrak{A}}(\lambda)$ is injective and has dense range for some $\lambda \in \Omega$; see, e.g., [Paz83, Theorem 9.3, p. 36]. A multi-valued version of this theorem, still with no input and output, is found in [DdS87, Remark, pp. 148–149].

2.2. State/signal systems in the frequency domain

Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s node, and let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a classical future trajectory of Σ. If x, $\dot{x}$, and w in (1.7) are Laplace transformable, then it follows from (1.7) (since we assume V to be closed) that the Laplace transforms $\hat{x}$, and $\hat{w}$ x and w satisfy (with $x^0 := x(0)$)

$$\widehat{\Sigma}\colon \begin{bmatrix} \lambda\hat{x}(\lambda) - x^0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V \tag{2.8}$$

for all those $\lambda \in \mathbb{C}$ for which the Laplace transforms converge (to see this it suffices to multiply by (1.4) by $\mathrm{e}^{-\lambda t}$ and integrate by parts in the $\dot{x}$-component.) This formula can be rewritten in the form

$$\begin{bmatrix} x^0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) := \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V. \tag{2.9}$$

Definition 2.10. The family of subspaces $\widehat{\mathfrak{E}} : \{\widehat{\mathfrak{E}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ defined in (2.8) is called the *characteristic node bundle* of the s/s node $\Sigma = (V;\mathcal{X},\mathcal{W})$.

The characteristic node bundle is a special case of a vector bundle:

Definition 2.11. Let $\mathcal{Z}$ be a Hilbert vector space.

(i) By a *vector bundle in $\mathcal{Z}$* we mean a family of subspaces $\mathfrak{G} = \{\mathfrak{G}(\lambda)\}_{\lambda\in \mathrm{dom}(\mathfrak{G})}$ of $\mathcal{Z}$ parameterized by a complex parameter $\lambda \in \mathrm{dom}(\mathfrak{G}) \subset \mathbb{C}$.

(ii) For each $\lambda \in \mathrm{dom}(\mathfrak{G})$, the subspace $\mathfrak{G}(\lambda)$ of $\mathcal{Z}$ is called the *fiber* of $\mathfrak{G}$ at λ.

(iii) The vector bundle $\mathfrak{G}$ is *analytic* at a point $\lambda_0 \in \mathrm{dom}(\mathfrak{G})$ if there exists a neighborhood $\mathcal{O}(\lambda_0)$ of λ_0 and some direct sum decomposition $\mathcal{Z} = \mathcal{U} \dotplus \mathcal{Y}$ of $\mathcal{Z}$ such that the restriction of $\mathfrak{G}$ to $\mathcal{O}(\lambda_0)$ is the graph of an analytic $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued function in $\mathcal{O}(\lambda_0)$.

(iv) The vector bundle $\mathfrak{G}$ is *analytic* if $\mathrm{dom}(\mathfrak{G})$ is open and $\mathfrak{G}$ is analytic at every point in $\mathrm{dom}(\mathfrak{G})$.

(v) The vector bundle $\mathfrak{G}$ is *entire* if $\mathfrak{G}$ is analytic in the full complex plane $\mathbb{C}$.

Lemma 2.12. *The characteristic node bundle $\widehat{\mathfrak{E}}$ of a s/s node $\Sigma = (V;\mathcal{X},\mathcal{W})$ is an entire vector bundle in the node space $\mathfrak{K} = \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$.*

This is easy to see (and proved in [AS14b, Chapter 1]).

Lemma 2.13. *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s node with the i/s/o representation $\Sigma_{\mathrm{i/s/o}} = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$, suppose that $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$. Denote the characteristic node bundle of Σ by $\widehat{\mathfrak{E}}$ and the i/s/o resolvent matrix of $\Sigma_{\mathrm{i/s/o}}$ by $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$. Then V and $\widehat{\mathfrak{E}}(\lambda)$ have the representations*

$$V = \mathrm{rng}\left(\begin{bmatrix}1_{\mathcal{X}} - \lambda\widehat{\mathfrak{A}}(\lambda) & -\lambda\widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \mathcal{I}_{\mathcal{Y}}\widehat{\mathfrak{C}}(\lambda) & \mathcal{I}_{\mathcal{U}} + \mathcal{I}_{\mathcal{Y}}\widehat{\mathfrak{D}}(\lambda)\end{bmatrix}\right), \tag{2.10}$$

$$\widehat{\mathfrak{E}}(\lambda) = \mathrm{rng}\left(\begin{bmatrix}1_{\mathcal{X}} & 0\\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \mathcal{I}_{\mathcal{Y}}\widehat{\mathfrak{C}}(\lambda) & \mathcal{I}_{\mathcal{U}} + \mathcal{I}_{\mathcal{Y}}\widehat{\mathfrak{D}}(\lambda)\end{bmatrix}\right), \tag{2.11}$$

where $\mathcal{I}_{\mathcal{U}}$ and $\mathcal{I}_{\mathcal{Y}}$ are the injection operators $\mathcal{I}_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{W}$ and $\mathcal{I}_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{W}$.

This follows from (1.13), Lemma 2.3, and (2.8) (see [AS14b, Chapter 4] for details).

Note that (2.10) can be interpreted as a graph representation of $\widehat{\mathfrak{E}}(\lambda)$ over the first copy of $\mathcal{X}$ and the input space $\mathcal{U}$. It follows from Lemma 2.13 that V (and $\widehat{\mathfrak{E}}(\lambda)$) are determined uniquely by the decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ and the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ of $\Sigma_{\mathrm{i/s/o}}$ evaluated at some arbitrary point $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$.

In i/s/o systems theory one is often interested in the "pure i/o behavior", which one gets by "ignoring the state". More precisely, one takes the initial state $x^0 = 0$, and looks at the relationship between the input u and the output y, ignoring the state x. If we in the frequency domain setting take $x^0 = 0$ and ignore $\hat{x}$, then the full frequency domain identity (2.3) simplifies into $\hat{y}(\lambda) = \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda)$, where $\widehat{\mathfrak{D}}(\lambda)$ is the i/o resolvent function of $\Sigma_{\mathrm{i/s/o}}$.

The same procedure can be carried out in the case of a s/s system: We take $x^0 = 0$ and ignore the values of $\hat{x}(\lambda)$ in (2.7). Then it follows from (2.8) that $\hat{w}(\lambda) \in \widehat{\mathfrak{F}}(\lambda)$, where

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \;\middle|\; \begin{bmatrix} 0 \\ z \\ w \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \text{ for some } z \in \mathcal{X} \right\}. \tag{2.12}$$

Definition 2.14. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. The family of subspaces $\widehat{\mathfrak{F}} : \{\widehat{\mathfrak{F}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathcal{W}$ defined by (2.11) is called the *characteristic signal bundle* of Σ.

Whereas the characteristic node bundle $\widehat{\mathfrak{E}}$ of Σ is an entire vector bundle, the same is not true for the signal bundle $\widehat{\mathfrak{F}}$ of Σ. Even the dimension of the fibers $\widehat{\mathfrak{F}}(\lambda)$ may change from one point to another. However, the following result is true:

Lemma 2.15. *If $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is an i/s/o representation of the s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with $\rho(\Sigma_{\mathrm{i/s/o}}) \neq \emptyset$, then for each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ the fibers of the characteristic signal bundle $\widehat{\mathfrak{F}}$ of Σ have the graph representation*

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \,\middle|\, P_{\mathcal{Y}}^{\mathcal{U}} w = \widehat{\mathfrak{D}}(\lambda) P_{\mathcal{U}}^{\mathcal{Y}} \right\}, \qquad \lambda \in \rho(\Sigma_{\mathrm{i/s/o}}). \tag{2.13}$$

This follows from Lemma 2.13.

Lemma 2.16. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node with the i/s/o representation $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, suppose that $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$. Denote the characteristic node bundle of Σ by $\widehat{\mathfrak{E}}$. Then, for each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ the fiber $\widehat{\mathfrak{E}}(\lambda)$ of $\widehat{\mathfrak{E}}$ is a closed subspace of $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, and it has the following properties:*

(i) $\left[\begin{smallmatrix} 0 \\ x \\ 0 \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda) \Rightarrow x = 0$;

(ii) *For every $z \in \mathcal{X}$ there exists some $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda)$.*

(iii) *The projection of $\widehat{\mathfrak{E}}(\lambda)$ onto its first and third components is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.*

This follows from Lemma 2.13.

Another equivalent way of formulating Lemma 2.16 is to say that for each $\lambda \in \rho(\Sigma_{\mathrm{i/s/o}})$ the fiber $\widehat{\mathfrak{E}}(\lambda)$ becomes a bounded s/s node after we interchange the first and the second component of $\widehat{\mathfrak{E}}(\lambda)$.

Definition 2.17. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node with node bundle $\widehat{\mathfrak{E}}$. Then the *resolvent set* $\rho(\Sigma)$ of Σ consists of all those points $\lambda \in \mathbb{C}$ for which conditions (i)–(iii) in Lemma 2.16 hold.

Theorem 2.18. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. Then $\rho(\Sigma)$ is the union of the resolvent sets of all i/s/o representations of Σ.*

See [AS14b, Chapter 4] for the proof.

Lemma 2.19. *The characteristic signal bundle $\widehat{\mathfrak{F}}$ of a s/s node Σ is analytic in $\rho(\Sigma)$.*

This follows from Definition 2.11, Lemma 2.15, and Theorem 2.18.

3. Passive and Conservative i/s/o and s/s systems

In this section we have, for simplicity, restricted the discussion to the regular case, i.e., the case where both the s/s system and its i/s/o representations are regular. As shown in [AS14b], the extension to the non-regular case is straightforward.

3.1. $\mathcal{J}$-passive and $\mathcal{J}$-conservative i/s/o systems

Definition 3.1. Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular i/s/o node.

(i) $\Sigma_{\mathrm{i/s/o}}$ is (forward) *solvable* if it is true that for every $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \operatorname{dom}(S)$ there exists at least one classical future trajectory $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ of $\Sigma_{\mathrm{i/s/o}}$ with $\left[\begin{smallmatrix} x(0) \\ u(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]$.

(ii) The *adjoint* of $\Sigma_{\mathrm{i/s/o}}$ is the i/s/o node $\Sigma^*_{\mathrm{i/s/o}} = (S^*; \mathcal{X}, \mathcal{Y}, \mathcal{U})$, where S^* is the adjoint of S.

Definition 3.2. Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular i/s/o node with adjoint $\Sigma^*_{\mathrm{i/s/o}} = (S^*; \mathcal{X}, \mathcal{Y}, \mathcal{U})$, and suppose that both $\Sigma_{\mathrm{i/s/o}}$ and $\Sigma^*_{\mathrm{i/s/o}}$ are solvable.

(i) $\Sigma_{\mathrm{i/s/o}}$ is *scattering conservative* if all its classical future trajectories $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ satisfy the balance equation

$$\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|y(s)\|^2_{\mathcal{Y}} \,\mathrm{d}s = \|x(0)\|^2_{\mathcal{X}} + \int_0^t \|u(s)\|^2_{\mathcal{U}} \,\mathrm{d}s, \quad t \in \mathbb{R}^+, \tag{3.1}$$

and the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ has the same property. If the above conditions hold with the equality sign in (3.1) by "$\le$" then $\Sigma_{\mathrm{i/s/o}}$ is *scattering passive.*

(ii) Let $\Psi\colon \mathcal{Y} \to \mathcal{U}$ be a unitary operator. Then $\Sigma_{\mathrm{i/s/o}}$ is Ψ*-impedance conservative* if all its classical future trajectories (u, x, y) satisfy the balance equation

$$\|x(t)\|^2_{\mathcal{X}} = \|x(0)\|^2_{\mathcal{X}} + 2\Re \int_0^t \langle u(s), \Psi y(s)\rangle_{\mathcal{U}} \,\mathrm{d}s, \quad t \in \mathbb{R}^+, \tag{3.2}$$

and the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ has the same property with Ψ replaced by Ψ^*. If the above conditions hold with the equality sign in (3.1) by "$\le$" then $\Sigma_{\mathrm{i/s/o}}$ is Ψ*-impedance passive.*

(iii) Let $\mathcal{J}_{\mathcal{U}}$ and $\mathcal{J}_{\mathcal{Y}}$ be signature operators in $\mathcal{U}$ respectively $\mathcal{Y}$ (i.e., $\mathcal{J}_{\mathcal{U}} = \mathcal{J}_{\mathcal{U}}^* = \mathcal{J}_{\mathcal{U}}^{-1}$ and $\mathcal{J}_{\mathcal{Y}} = \mathcal{J}_{\mathcal{Y}}^* = \mathcal{J}_{\mathcal{Y}}^{-1}$). Then $\Sigma_{\mathrm{i/s/o}}$ is $(J_{\mathcal{U}}, J_{\mathcal{Y}})$*-transmission conservative* if all its classical future trajectories (u, x, y) satisfy the balance equation

$$\begin{aligned} \|x(t)\|^2_{\mathcal{X}} &+ \int_0^t \langle y(s), J_{\mathcal{Y}} y(s)\rangle_{\mathcal{Y}} \,\mathrm{d}s \\ &= \|x(0)\|^2_{\mathcal{X}} + \int_0^t \langle u(s), J_{\mathcal{U}} u(s)\rangle_{\mathcal{U}} \,\mathrm{d}s, \quad t \in \mathbb{R}^+, \end{aligned} \tag{3.3}$$

and the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ has the same property with $(J_{\mathcal{U}}, J_{\mathcal{Y}})$ replaced by $(J_{\mathcal{Y}}, J_{\mathcal{U}})$. If the above conditions hold with the equality sign in (3.1) by "$\le$" then $\Sigma_{\mathrm{i/s/o}}$ is $(J_{\mathcal{U}}, J_{\mathcal{Y}})$*-transmission passive.*

The three different balance equations in Lemma 3.3 can all be written in the common form

$$\|x(t)\|^2_{\mathcal{X}} = \|x(0)\|^2_{\mathcal{X}} + \int_0^t \left\langle \begin{bmatrix} u(s) \\ y(s) \end{bmatrix}, \mathcal{J} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}} \mathrm{d}s, \quad t \in \mathbb{R}^+, \tag{3.4}$$

where $\mathcal{J}$ *is a signature operator* in the product space $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$:

(i) $\mathcal{J} = \mathcal{J}_{\mathrm{scat}} = \left[\begin{smallmatrix} 1_{\mathcal{U}} & 0 \\ 0 & -1_{\mathcal{Y}} \end{smallmatrix}\right]$ in the *scattering* case,
(ii) $\mathcal{J} = \mathcal{J}_{\mathrm{imp}} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$ in the Ψ*-impedance* case,
(iii) $\mathcal{J} = \mathcal{J}_{\mathrm{tra}} = \left[\begin{smallmatrix} J_{\mathcal{U}} & 0 \\ 0 & -J_{\mathcal{Y}} \end{smallmatrix}\right]$ in the $(\mathcal{J}_{\mathcal{U}}, \mathcal{J}_{\mathcal{Y}})$*-transmission* case.

It is also possible of combine the three different parts of Definition 3.2 into one general definition. In that definition we need two different signature operators, one in the space $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and the other in the space $\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$. The connection between these two operators is the following: If $\mathcal{J}$ is a signature operator in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, then we define the operator $\mathcal{J}_*$ by

$$\mathcal{J}_* = \begin{bmatrix} 0 & -1_{\mathcal{Y}} \\ 1_{\mathcal{U}} & 0 \end{bmatrix} \mathcal{J} \begin{bmatrix} 0 & 1_{\mathcal{U}} \\ -1_{\mathcal{Y}} & 0 \end{bmatrix} \tag{3.5}$$

It is easy to see that $\mathcal{J}_*$ is a signature operator in $\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ whenever $\mathcal{J}$ is a signature operator in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ and that $(\mathcal{J}_*)_* = \mathcal{J}$.

Definition 3.3. Let $\Sigma_{\mathrm{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular i/s/o node with adjoint $\Sigma^*_{\mathrm{i/s/o}} = (S^*; \mathcal{X}, \mathcal{Y}, \mathcal{U})$, and suppose that both $\Sigma_{\mathrm{i/s/o}}$ and $\Sigma^*_{\mathrm{i/s/o}}$ are solvable. Let $\mathcal{J}$ be a signature operator in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and define $\mathcal{J}_*$ by (3.5). Then $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-conservative if all its classical future trajectories $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ satisfy the balance equation (3.4), and the adjoint system $\Sigma^*_{\mathrm{i/s/o}}$ has the same property with $\mathcal{J}$ replaced by $\mathcal{J}_*$. If the above conditions hold with the equality sign in (3.4) by "$\le$" then $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$*-passive.*

The reader is invited to check that Definition 3.2 can indeed be interpreted as a special case of Definition 3.3 (with the appropriate choice of $\mathcal{J} = \mathcal{J}_{\mathrm{scat}}$, $\mathcal{J} = \mathcal{J}_{\mathrm{imp}}$, or $\mathcal{J} = \mathcal{J}_{\mathrm{tra}}$).

Formula (3.4) treats the *input u and the output y in an equal way*: the operator $\mathcal{J}$ *is simply a signature operator in the signal space* $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and it defines a Kreĭn space inner product in $\mathcal{W}$. From the point of view of (3.4) it does not matter if u is the input and y the output, or the other way around, or if neither u nor y is the input or output.

It is well-known that one can pass from a Ψ-impedance or $(J_{\mathcal{U}}, J_{\mathcal{Y}})$-transmission passive or conservative i/s/o system to a scattering passive or conservative i/s/o system by simply *reinterpreting which part of the combined i/o signal* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ *is the input, and which part is the input.*

(i) If $\Sigma_{i/s/o}$ is Ψ-impedance conservative, and if we take the new input and output to be

$$\begin{bmatrix} u_{\rm scat} \\ y_{\rm scat} \end{bmatrix} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1_{\mathcal{U}} & \Psi \\ \Psi^* & -1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} u_{\rm imp} \\ y_{\rm imp} \end{bmatrix},$$

then the resulting i/s/o system is scattering conservative.

(ii) If $\Sigma_{i/s/o}$ is $(J_{\mathcal{U}}, J_{\mathcal{Y}})$-transmission conservative, and if we take the new input and output to be

$$\begin{bmatrix} u_{\rm scat} \\ y_{\rm scat} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{U}^+} & P_{\mathcal{Y}^-} \\ P_{\mathcal{U}^-} & P_{\mathcal{Y}^+} \end{bmatrix} \begin{bmatrix} u_{\rm tra} \\ y_{\rm tra} \end{bmatrix},$$

where $(P_{\mathcal{U}^+}, P_{\mathcal{U}^-})$ and $(P_{\mathcal{Y}^+}, P_{\mathcal{Y}^-})$ are complementary projections onto the positive and negative subspaces of $J_{\mathcal{U}}$ and $J_{\mathcal{Y}}$, respectively, then the resulting i/s/o system is again scattering conservative.

The two transforms described above have the following common interpretation: We decompose the Kreĭn space $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ with the $\mathcal{J}$-inner product into a positive part and an orthogonal negative part (= a fundamental decomposition), and choose the input to be the positive part of $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ and the output to be the negative part of w. Of course, these transformations lead to new dynamic equations with new generators $S_{\rm scat}$, which can be explicitly derived from the original generators $S_{\rm imp}$ and $S_{\rm tra}$, but the formulas for $S_{\rm scat}$ tend to be complicated, especially when $S_{\rm imp}$ and $S_{\rm tra}$ are unbounded. For this reason it makes sense to reformulate the $\mathcal{J}$-passivity and $\mathcal{J}$-conservativity conditions described above into a state/signal setting.[1]

3.2. Passive and conservative state/signal systems

Let $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular i/s/o system, and let $\Sigma = (V; \mathcal{X}, \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right])$ be the s/s system induced by $\Sigma_{i/s/o}$, i.e., the generating subspace V of Σ is given by (1.8). If $\Sigma_{i/s/o}$ is $\mathcal{J}$-passive or $\mathcal{J}$-conservative for some signature operator $\mathcal{J}$ in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, then what does this tell us about the s/s system Σ?

First of all, the solvability condition of $\Sigma_{i/s/o}$ implies an analogous condition for Σ:

Definition 3.4. A s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is (forward) *solvable* if it is true that for every $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ there exists at least one classical future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ satisfying $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]$.

It follows from Definitions 3.1 and 3.4 and Lemma 1.10 that if $\Sigma_{i/s/o}$ is a regular i/s/o representation of a s/s node Σ, then Σ is solvable if and only if $\Sigma_{i/s/o}$ is solvable.

By Lemma 1.10, $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is a classical future trajectory of $\Sigma_{i/s/o}$ if and only if $\left[\begin{smallmatrix} x \\ x \end{smallmatrix}\right]$ is a classical future trajectory Σ, where $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$. Thus, if $\Sigma_{i/s/o}$ is $\mathcal{J}$-conservative, then every classical future trajectory $\left[\begin{smallmatrix} x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right]$ of Σ satisfies (3.4).

[1] This was the primary motivation for the development of the s/s systems theory in the first place.

If instead $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-passive, then every classical future trajectory $\left[\begin{smallmatrix} x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right]$ of Σ satisfies (3.4) with "=" replace by "≤".

Up to this point we have throughout assumed that the signal space $\mathcal{W}$ of a s/s node is a *Hilbert space*, but it follows from (3.4) that in the study of passive and conservative systems it more natural to allow $\mathcal{W}$ to be a *Kreĭn space*, i.e., to allow the inner product in $\mathcal{W}$ to be indefinite. More precisely, we let $\mathcal{W}$ be the product space $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ equipped with the Kreĭn space inner product

$$\left[\begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}\right]_{\mathcal{W}} = \left\langle \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \mathcal{J} \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}}, \qquad \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \in \begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}. \tag{3.6}$$

With this notation (3.4) becomes

$$\|x(t)\|^2_{\mathcal{X}} = \|x(0)\|^2_{\mathcal{X}} + \int_0^t [w(s), w(s)]_{\mathcal{W}}\, \mathrm{d}s, \quad t \in \mathbb{R}^+. \tag{3.7}$$

Differentiating (3.7) with respect to t we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} = [w(t), w(t)]_{\mathcal{W}}, \qquad t \in \mathbb{R}^+,$$

or equivalently,

$$-\langle \dot{x}(t), x(t)\rangle_{\mathcal{X}} - \langle x(t), \dot{x}(t)\rangle_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} = 0, \qquad t \in \mathbb{R}^+. \tag{3.8}$$

In particular, this equation is true for $t = 0$. If we assume that Σ is solvable (or equivalently, $\Sigma_{\mathrm{i/s/o}}$ is solvable), then it follows from (3.8) that

$$-\langle z_0, x_0\rangle_{\mathcal{X}} - \langle x_0, z_0\rangle_{\mathcal{X}} + [w_0, w_0]_{\mathcal{W}} = 0, \qquad \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V. \tag{3.9}$$

We can make also the *node space* $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ into a Kreĭn space by introducing the following *node inner product* in $\mathfrak{K}$:

$$\left[\left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ x_2 \\ w_2 \end{smallmatrix}\right]\right]_{\mathfrak{K}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}, \quad \left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ x_2 \\ w_2 \end{smallmatrix}\right] \in \mathfrak{K}. \tag{3.10}$$

Clearly (3.9) says that $V \subset V^{[\perp]}$, where

$$V^{[\perp]} := \left\{ \left[\begin{smallmatrix} z_* \\ x_* \\ w_* \end{smallmatrix}\right] \in \mathfrak{K} \,\middle|\, \left[\left[\begin{smallmatrix} z_* \\ x_* \\ w_* \end{smallmatrix}\right], \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]\right]_{\mathfrak{K}} = 0 \text{ for all } \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V \right\}. \tag{3.11}$$

In other words, V is a *neutral subspace of* $\mathfrak{K}$. If $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-passive instead of $\mathcal{J}$-conservative, then the same argument shows that

$$\left[\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]\right]_{\mathfrak{K}} \geq 0 \text{ for all } \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V,$$

i.e., V is a *nonnegative subspace of* $\mathfrak{K}$.

Above we have used only one half of Definition 3.3, namely the half with refers to the the i/s/o representation $\Sigma_{\mathrm{i/s/o}}$ itself, and not the half which refers to the adjoint i/s/o node $\Sigma^*_{\mathrm{i/s/o}}$. By adding the conditions imposed on $\Sigma^*_{\mathrm{i/s/o}}$ to the above argument it is possible to show that

(i) $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-conservative if and only if Σ satisfies $V = V^{[\perp]}$ (i.e., V is a *Lagrangian* subspace of $\mathfrak{K}$), and

(ii) $\Sigma_{\mathrm{i/s/o}}$ is $\mathcal{J}$-passive if and only if V is a maximal nonnegative subspace of $\mathfrak{K}$ (i.e., V is nonnegative, and it is not strictly contained in any other nonnegative subspace of $\mathfrak{K}$).

This motivates the following definition:

Definition 3.5.

(i) By a *conservative s/s system* Σ we mean a regular s/s system whose signal space $\mathcal{W}$ is a Kreĭn space, and whose generating subspace V *is a Lagrangian subspace* of the node space $\mathfrak{K}$ (with respect to the inner product (3.10)).

(ii) By a *passive s/s system* Σ we mean a regular s/s system whose signal space $\mathcal{W}$ is a Kreĭn space, and whose generating subspace V *is a maximal nonnegative subspace* of the node space $\mathfrak{K}$ (with respect to the inner product (3.10)).

Thus, in particular, every conservative s/s system is also passive.

Note that Definition 3.5 does *not explicitly require that* Σ *must be solvable*, which was assumed in the derivation of (3.9). However, it turns out that this condition is redundant in Definitions 3.5, i.e., the regularity of Σ combined with either the condition $V = V^{[\perp]}$ or the assumption that V is maximal nonnegative implies that Σ is solvable.

3.3. Passive and conservative realizations

In i/s/o systems theory one is often interested in the "converse problem" of finding a "realization" of a given analytic "transfer function" φ with some "additional properties". By a realization we mean an i/s/o system whose i/o resolvent function coincides with φ is some specified open subset Ω of $\mathbb{C}$. For example,

(i) φ is a *"Schur function"* over $\mathbb{C}^+$, and one wants to construct a *scattering conservative realization* $\Sigma_{\mathrm{i/s/o}}$ of φ,

(ii) φ is a *"positive real function"* over $\mathbb{C}^+$, and one wants to construct an *impedance conservative realization* $\Sigma_{\mathrm{i/s/o}}$ of φ.

(iii) φ is a *"Potapov function"* over $\mathbb{C}^+$, and one wants to construct a *transmission conservative realization* $\Sigma_{\mathrm{i/s/o}}$ of φ.

In the state/signal setting all *these three problems collapse into one and the same problem*: Given a *passive signal bundle* over $\mathbb{C}^+$ (this notion will be defined in Definition 3.7 below), we want to construct a *conservative s/s realization* of this signal bundle, i.e., a conservative s/s system Σ with $\mathbb{C}^+ \subset \rho(\Sigma)$ such that the given passive signal bundle coincides with the characteristic signal bundle $\widehat{\mathfrak{F}}$ of Σ in $\mathbb{C}^+$.

Theorem 3.6. *Let Σ be a passive s/s system with signal space space $\mathcal{W}$ and characteristic signal bundle $\widehat{\mathfrak{F}}$. Then*

(i) $\mathbb{C}^+ \subset \rho(\Sigma)$ *(and hence $\widehat{\mathfrak{F}}$ is analytic in $\mathbb{C}^+$),*

(ii) *for each $\lambda \in \mathbb{C}^+$ the fiber $\widehat{\mathfrak{F}}(\lambda)$ of $\widehat{\mathfrak{F}}$ is a maximal nonnegative subspace of $\mathcal{W}$.*

See [AS14b] for the proof of this theorem.

Definition 3.7. By a *passive signal bundle* in a Kreĭn (signal) space $\mathcal{W}$ we mean an analytic signal bundle Ψ in $\mathbb{C}^+$ with the property that for each $\lambda \in \mathbb{C}^+$ the fiber $\Psi(\lambda)$ is a maximal nonnegative subspace of $\mathcal{W}$.

This leads us to the following problem:

Problem 3.8 (Conservative State/Signal Realization Problem). *Given a passive signal bundle Ψ, find a conservative s/s system Σ such that the characteristic signal bundle of Σ coincides with Ψ in $\mathbb{C}^+$.*

One such construction is carried out in [AKS11]. The setting in [AKS11] is different from the one described here, but it follows from [AKS11], e.g., that every passive signal bundle Ψ has a "simple" conservative s/s realization, and that such a realization is unique up to a unitary similarity transformation in the state space. Here "simplicity" means that the system is minimal within the class of conservative s/s systems, i.e., a conservative s/s system is simple if and only if it does not have any nontrivial conservative compression.

4. A short history

I first met Dima (Prof. Damir Arov) at the MTNS conference 1998 in Padova where he gave a plenary talk on "Passive Linear Systems and Scattering Theory". Five years later, in the fall of 2003, Dima came to work with me in Åbo for one month, and that was the beginning of our joint stationary state/signal systems story. We decided to "join forces" to study the relationship between the (external) reciprocal symmetry of a conservative linear system and the (internal) symmetry structure of the system in three different settings, namely the scattering, the impedance, and the transmission setting. Instead of writing three separate papers with three separate sets of results and proofs we wanted to rationalize and to find some "general setting" that would cover the "common part" of the theory. The basic plan was to first develop the theory in such a "general setting" as far as far as possible, before discussing the three related symmetry problems mentioned above in detail.

After a couple of days we realized that the "behavioral approach" of [BS06] seemed to provide a suitable "general setting". This setting gave us a natural mathematical model for a "linear time-invariant circuit" which may contain both lumped and distributed components.

To make the work more tractable from a technical point of view we decided to begin by studying the discrete time case. As time went by the borderline between the "general theory" and the application to the original symmetry problem kept moving forward. Our first paper had to be split in two because it became too long. Then the second part had to be split in two because it became too long, then the third part had to be split in to, and so on. Every time the paper was split into two the original symmetry problem was postponed to the second unfinished half, and our "general solution" to the symmetry problem was not submitted until 2011. By that time we had published more than 500 pages on the s/s systems theory in 13 papers (in addition to numerous conference papers). The specific applications of our symmetry paper to the scattering, impedance, and transmission settings is still "work in progress".

In 2006 Mikael Kurula joined the s/s team, and together with him we begun to also study the continuous time problem. See the reference list for details.

Since 2009 Dima and I have spent most of our common research time on writing a book on linear stationary systems in continuous time. It started out as a manuscript about s/s systems in discrete time. In 2012 we shifted the focus to s/s systems in continuous time. After one more year the manuscript was becoming too long to be published as a single volume, so we decided to split the book into two volumes. A partial preliminary draft of the first volume of this book is available as [AS14b].

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