# State/Signal Linear Time-Invariant Systems Theory, Part IV: Affine Representations of Discrete Time Systems

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*Dedicated to the memory of Mark Grigorevich Kreĭn.*

**Abstract**

In the paper we continue to develop the linear discrete time invariant state/signal systems theory that was initiated in a sequence of earlier papers (Parts I—III). The trajectories of a state/signal system $\Sigma$ with a Hilbert state space $\mathcal{X}$ and a Hilbert or Krein signal space $\mathcal{W}$ consists of a pair of sequences $(x(\cdot), w(\cdot))$ that after an admissible input/output decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of the signal space can be

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obtained from the set of trajectories $(x(\cdot), u(\cdot), y(\cdot))$ of a standard input/state/output system by taking $w(\cdot) = y(\cdot) + u(\cdot)$. In Part I we studied the families of all admissible decompositions of $\mathcal{W}$ for a given state/signal system $\Sigma$ and the corresponding input/state/output representations and their transfer functions, together with two other types of representations, namely driving variable and output nulling representations. By combining the two representations with input/output decompositions of the signal space we here obtain right and left affine input/state/output representations of $\Sigma$, and we also get left and right affine transfer functions and generalized input/output transfer functions. As opposed to standard transfer functions, a generalized transfer function not need to be holomorphic at the origin. This makes it possible to realize every rational matrix-valued function (even those that has a pole at the origin) as the generalized transfer function of a minimal state/signal system whose state space has a finite dimension equal to the McMillan degree of the given function. We finally apply the theory to the case where $\Sigma$ is passive to obtain right and left affine transmission and impedance representations, left and right affine transmission matrices and impedances, and generalized input/output transmission matrices of a passive system. In particular, it is shown that any meromorphic $J$-contractive matrix-valued function defined in the unit disk, including those that have a pole at the origin, is the generalized transmission matrix of a minimal passive state/signal system as well as of a simple conservative state/signal system. Similar operator-valued results presented here, too.

**Keywords**

System, state, signal, input, output, passive, conservative, scattering, impedance, transmission, realization, Krein space.

# 1 Introduction

This article is a continuation of the articles [AS05], [AS06a], and [AS06b], which we in the sequel refer to as "Part I", "Part II", and "Part III". In Part I we developed a linear discrete time-invariant s/s (state/signal) systems theory in a general setting. This theory differs from the standard i/s/o (input/state/output) systems theory in the sense that we do not distinguish between input and output signals, but only between an "internal" state $x \in \mathcal{X}$ and an "external" signal $w \in \mathcal{W}$, where the state space $\mathcal{X}$ and signal space $\mathcal{W}$ are vector spaces. In Part I both of these were assumed to be Hilbert spaces, but no use was made of the specific inner product (in particular, we made no use of orthogonality), so that all results in Part I remain valid if we replace the inner product by another inner product that induces an equivalent norm. This makes it possible to apply the results from Part I also in the case where $\mathcal{X}$ and $\mathcal{W}$ are Kreĭn spaces. Here, as in Parts II and III, we still take the state space $\mathcal{X}$ to be a Hilbert space, but we allow the signal space $\mathcal{W}$ will be a Kreĭn space (this additional generality will be important only in the last sections of this paper where we discuss passivity). When we cite a particular result from one of Parts I–III we shall simply add a roman number "I", "II", or "III" to the corresponding number appearing there. Thus, for example, Theorem III.3.6 stands for Theorem 3.6 in Part III, and (I.6.16) stands for formula (6.16) in Part I.

A *trajectory* $(x(\cdot), w(\cdot))$ of a linear time-invariant s/s system $\Sigma$ in discrete time consists of a state sequence $x(n) \in \mathcal{X}$ and a signal sequence $w(n) \in \mathcal{W}$, $n \in \mathbb{Z}^+ := 0, 1, 2, \ldots$, that satisfy the system of equations

$$\begin{aligned} x(n+1) &= F \begin{bmatrix} x(n) \\ w(n) \end{bmatrix}, \quad n \in \mathbb{Z}^+, \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where $F$ is a bounded linear operator with closed domain $\mathcal{D}(F)$ in the product space $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ and range $\mathcal{R}(F) \subset \mathcal{X}$. We assume throughout that $\mathcal{D}(F)$ has the property that for every $x \in \mathcal{X}$ there is at least one $w \in \mathcal{W}$ such that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F)$. This property guarantees that for every $x_0 \in \mathcal{X}$ there exists at least one trajectory $(x(\cdot), w(\cdot))$ of the system with initial state $x(0) = x_0$. We remark that $x_0$ and the sequence $w(\cdot)$ together determine the trajectory $(x(\cdot), w(\cdot))$ uniquely.

For any i/o (input/output) decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of the signal space $\mathcal{W}$ of a s/s system $\Sigma$ as the direct sum of two spaces $\mathcal{U}$ (the input space) and $\mathcal{Y}$ (the output space) we get corresponding set of i/s/o trajectories $(x(\cdot), u(\cdot), y(\cdot))$ of $\Sigma$, where the sequences $u(n) \in \mathcal{U}$ and $y(n) \in \mathcal{Y}$, $n \in \mathbb{Z}^+$, are sequences obtained from the trajectories $(x(\cdot), w(\cdot))$ of $\Sigma$ by writing

$w(n) = u(n) + y(n)$, $n \in \mathbb{Z}^+$, i.e.,

$$u(n) = P_{\mathcal{U}}^{\mathcal{Y}} w(n), \quad y(n) = P_{\mathcal{Y}}^{\mathcal{U}} w(n), \quad n \in \mathbb{Z}^+;$$

here $P_{\mathcal{U}}^{\mathcal{Y}}$ is the projection onto $\mathcal{U}$ along $\mathcal{Y}$, and $P_{\mathcal{Y}}^{\mathcal{U}}$ is the complementary projection. As shown in Part I, there exists at least one decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ such that the set of i/s/o trajectories $(x(\cdot), u(\cdot), y(\cdot))$ of $\Sigma$ constructed above coincide with the set of trajectories of a standard i/s/o system $\Sigma_{i/s/o} = \left( \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ defined by the system of equations

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \qquad n \in \mathbb{Z}^+, \\ x(0) &= x_0, \end{aligned} \tag{1.2}$$

where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}\left(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]\right)$. Such a decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is called an *admissible i/o decomposition of* $\mathcal{W}$ *for* $\Sigma$, and the corresponding i/s/o system $\Sigma_{i/s/o}$ is called an *i/s/o representation of* $\Sigma$.

Every i/s/o system $\Sigma_{i/s/o} = \left( \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ has an *is/so (input-state/state-output) transfer function*[1]

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} (1_{\mathcal{X}} - zA)^{-1} & z(1_{\mathcal{X}} - zA)^{-1}B \\ C(1_{\mathcal{X}} - zA)^{-1} & zC(1_{\mathcal{X}} - zA)^{-1}B + D \end{bmatrix}, \quad z \in \Lambda_A, \tag{1.3}$$

where the set $\Lambda_A$ consists of those $z \in \mathbb{C}$ for which $1_{\mathcal{X}} - zA$ has a bounded inverse plus the point at infinity if $A$ has a bounded inverse in $\mathcal{X}$. Here the block $\mathfrak{D}(z)$ is the standard i/o transfer function

$$\mathfrak{D}(z) = zC(1_{\mathcal{X}} - zA)^{-1}B + D, \quad z \in \Lambda_A, \tag{1.4}$$

In terms of this is/so transfer function the connection between the formal series $\hat{u}(z) = \sum_{n=0}^{\infty} u(n)z^n$, $\hat{x}(z) = \sum_{n=0}^{\infty} x(n)z^n$, $\hat{y}(z) = \sum_{n=0}^{\infty} y(n)z^n$, and $x_0$, where $(x(\cdot), u(\cdot), y(\cdot))$ is a trajectory of (1.2), is given by

$$\begin{bmatrix} \hat{x}(z) \\ \hat{y}(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(z) \end{bmatrix}. \tag{1.5}$$

In particular,

$$\hat{y}(z) = \mathfrak{D}(z)\hat{u}(z) \text{ when } x_0 = 0. \tag{1.6}$$

[1] In Parts I–II we used the name *four block transfer function* instead of is/so transfer function.

Not every direct sum decomposition of $\mathcal{W}$ is admissible. To be able to treat also the nonadmissible case we here introduce right and left affine generalizations of the notions of i/s/o representations and their transfer functions. These are defined for arbitrary i/o decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. By a *right affine i/s/o representation of* $\Sigma$ we mean an i/s/o system

$$\Sigma^r_{i/s/o} = \left( \left[ \begin{array}{c|c} A' & B' \\ \hline C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{array} \right]; \mathcal{X}, \mathcal{L}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right] \right) \tag{1.7}$$

(where the new input space $\mathcal{L}$ is an auxiliary Hilbert space) with the property that the correspondence $\left(x(\cdot), \ell(\cdot), \left[\begin{smallmatrix} y(\cdot) \\ u(\cdot) \end{smallmatrix}\right]\right) \to (x(\cdot), u(\cdot), y(\cdot))$ is a bijective map from the set of trajectories of the of i/s/o system $\Sigma^r_{i/s/o}$ described by the set of equations

$$\begin{aligned} x(n+1) &= A'x(n) + B'\ell(n), \\ y(n) &= C'_{\mathcal{Y}}x(n) + D'_{\mathcal{Y}}\ell(n), \\ u(n) &= C'_{\mathcal{U}}x(n) + D'_{\mathcal{U}}\ell(n), \qquad n \in \mathbb{Z}^+,\ \ell(n) \in \mathcal{L} \end{aligned} \tag{1.8}$$

onto the set of i/s/o trajectories of the s/s system $\Sigma$ corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. It is easily seen that $\Sigma^r_{i/s/o}$ has this property if and only if

$$\begin{bmatrix} C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} P^{\mathcal{U}}_{\mathcal{Y}} C' & P^{\mathcal{U}}_{\mathcal{Y}} D' \\ P^{\mathcal{Y}}_{\mathcal{U}} C' & P^{\mathcal{Y}}_{\mathcal{U}} D' \end{bmatrix} \tag{1.9}$$

and

$$\Sigma_{dv/s/s} = \left( \left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$$

is a driving variable representation of the s/s system $\Sigma$ of the type introduced and studied in Part I (see Section 2 for details). Explicitly, this means that $\Sigma_{dv/s/s}$ is an i/s/o system with the same state space $\mathcal{X}$ as $\Sigma$, the auxiliary (Hilbert) input space $\mathcal{L}$, and output space $\mathcal{W}$, that $\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right] \in \mathcal{B}\left(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]\right)$, that $D'$ has a bounded left inverse (i.e., $D'$ is injective and has a closed range), and that the operator $F$ in (1.1) and its domain $\mathcal{D}(F)$ are given by

$$\begin{aligned} &\mathcal{D}(F) = \left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \,\middle|\, w = C'x + D'\ell,\ \ell \in \mathcal{L}, x \in \mathcal{X} \right\}, \\ &F\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = A'x + B'\ell \text{ if } w = C'x + D'\ell,\ \ell \in \mathcal{L},\ x \in \mathcal{X}. \end{aligned}$$

The is/so transfer function of the i/s/o system $\Sigma^r_{i/s/o}$ is called a right affine is/so transfer function of the s/s system $\Sigma$ corresponding to the i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of the signal space $\mathcal{W}$. Likewise, the i/o transfer function $\mathfrak{D}' = \begin{bmatrix} \mathfrak{D}'_{\mathcal{Y}} \\ \mathfrak{D}'_{\mathcal{U}} \end{bmatrix}$ of $\Sigma^r_{i/s/o}$ is called a right affine i/o transfer function of $\Sigma$, corresponding to this decomposition.

To each fixed i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of $\mathcal{W}$ corresponds infinitely many right affine i/s/o representations of the s/s system $\Sigma$, but there exist simple connections between them that follow from the corresponding connections between different driving variable representations of $\Sigma$, obtained in Part I. This connection will be presented in Section 2 together with the corresponding connections between the different right affine is/so transfer functions of $\Sigma$. These connections permit us to introduce the notion of a generalized is/so transfer function $\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix}$ of $\Sigma$ with domain $\Omega(\Sigma;\mathcal{U},\mathcal{Y})$, corresponding to a given i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of $\mathcal{W}$. As we will show in Theorem 5.2, this decomposition is admissible if and only if $0 \in \Omega(\Sigma;\mathcal{U},\mathcal{Y})$. In this case the generalized is/so transfer function of $\Sigma$ coincides with the is/so transfer function of the corresponding i/s/o representation $\Sigma_{i/s/o}$, and in particular, $\Omega(\Sigma;\mathcal{U},\mathcal{Y}) = \Lambda_A$, where $A$ is the main operator of $\Sigma_{i/s/o}$. One way to define the generalized is/so transfer function is to start with an arbitrary right affine i/s/o represenation $\Sigma^r_{i/s/o}$ given by (1.7), to define

$$\Omega(\Sigma;\mathcal{Y},\mathcal{U}) = \left\{ z \in \mathbb{C} \;\middle|\; \text{the operator } \begin{bmatrix} 1_\mathcal{X} - zA' & -zB' \\ C'_\mathcal{U} & D'_\mathcal{U} \end{bmatrix} \text{ has a bounded inverse} \right\}, \tag{1.10}$$

and to define

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} 1_\mathcal{X} & 0 \\ C'_\mathcal{Y} & D'_\mathcal{Y} \end{bmatrix} \begin{bmatrix} 1_\mathcal{X} - zA' & -zB' \\ C'_\mathcal{U} & D'_\mathcal{U} \end{bmatrix}^{-1}, \quad z \in \Omega(\Sigma;\mathcal{U},\mathcal{Y}). \tag{1.11}$$

As we show in Section 5, the is/so generalized transfer function and its domain $\Omega(\Sigma;\mathcal{U},\mathcal{Y})$ *are independent of the particular right affine i/s/o representation that we use in the above definition.* The block $\mathfrak{D}(z)$ is called the generalized i/o transfer function, and the block $\mathfrak{A}(z)$ is called the generalized resolvent of $\Sigma$, corresponding to the i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$.

The corresponding *left* notions are defined in an analogous way, by replacing the driving variable representations used above by the output nulling representations defined in Part I. A *left affine i/s/o representation* of the s/s system $\Sigma$ is an i/s/o system

$$\Sigma^l_{i/s/o} = \left( \left[ \begin{array}{c|cc} A'' & B''_\mathcal{Y} & B''_\mathcal{U} \\ \hline C'' & D''_\mathcal{Y} & D''_\mathcal{U} \end{array} \right]; \mathcal{X}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \mathcal{K} \right) \tag{1.12}$$

(where the new output space $\mathcal{K}$ is an auxiliary Hilbert space) with the property that the correspondence $(x(\cdot), u(\cdot), y(\cdot)) \to \left( x(\cdot), \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix}, 0 \right)$ is a bijective map from the set of i/s/o trajectories of the s/s system $\Sigma$ corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ onto the subset of trajectories

$$\begin{aligned} x(n+1) &= A''x(n) + B''_\mathcal{Y} y(n) + B''_\mathcal{U} u(n), \\ k(n) &= C''x(n) + D''_\mathcal{Y} y(n) + D''_\mathcal{U} u(n), \qquad n \in \mathbb{Z}^+, \end{aligned} \tag{1.13}$$

of the of i/s/o system $\Sigma^l_{i/s/o}$ for which the error signal $k(\cdot)$ vanishes identically. As we show in Theorem 5.8, given any i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ and any left affine i/s/o representation $\Sigma^l_{i/s/o}$ given by (1.12), the domain $\Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ in (1.10) of the generalized is/so transfer function may also be defined by

$$\Omega(\Sigma; \mathcal{Y}, \mathcal{U}) = \Big\{ z \in \mathbb{C} \;\Big|\; \text{the operator } \left[\begin{smallmatrix} 1_{\mathcal{X}} - zA'' & -zB''_{\mathcal{Y}} \\ -C'' & -D''_{\mathcal{Y}} \end{smallmatrix}\right] \text{ has a bounded inverse} \Big\}, \tag{1.14}$$

and the generalized is/so transfer fuction in (1.11) may also be defined by the formula

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} - zA'' & -zB''_{\mathcal{Y}} \\ -C'' & -D''_{\mathcal{Y}} \end{bmatrix}^{-1} \begin{bmatrix} 1_{\mathcal{X}} & zB''_{\mathcal{U}} \\ 0 & D''_{\mathcal{U}} \end{bmatrix}, \qquad z \in \Omega(\Sigma; \mathcal{Y}, \mathcal{U}). \tag{1.15}$$

At the end of Sections 3 and 4 we also apply the results described above to the case where the s/s system $\Sigma$ is stabilizable, detectable, or LFT-stabilizable (LFT stands for Linear Fractional Transformation). We recall that an i/s/o system $\Sigma_{i/s/o} = (\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is called *stable* if the trajectories $(x(\cdot), u(\cdot), y(\cdot))$ of this system have the property that $x(\cdot) \in \ell^\infty(\mathcal{X})$ and $y(\cdot) \in \ell^2(\mathcal{Y})$ whenever $u(\cdot) \in \ell^2(\mathcal{U})$. A right or left affine i/s/o representation is stable if it is stable when regarded as an i/s/o system. A s/s system $\Sigma$ is called (a) *stabilizable*, or (b) *detectable*, or (c) *LFT-stabilizable* if it has a stable (a) driving varialbe representation, or (b) output nulling representation, or (c) i/s/o representation, respectively. According to Lemma I.9.2, the i/o transfer function of a stable i/s/o system is bounded and analytic in the unit disk, i.e., it belongs to $H^\infty$ over $\mathbb{D}$. It follows in the LFT-stabilizable case that for every i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ there exist corresponding right and left affine i/o transfer functions which are bounded and analytic in $\mathbb{D}$, and such that these affine i/o transfer functions (each one consisting of one pair of functions) are even right and left coprime in $H^\infty$ over $D$, respectively.

In Section 8 our general results on affine representations, affine transfer functions, and generalized transfer functions are applied to *passive* s/s systems, whose theory was developed in Parts II–III. A s/s system $\Sigma$ is passive if the trajectories $(x(\cdot), w(\cdot))$ of $\Sigma$ have the forward passivity property

$$\|x(n+1)\|^2_{\mathcal{X}} - \|x(n)\|^2_{\mathcal{X}} \le [w(n), w(n)]_{\mathcal{W}}, \qquad n \in \mathbb{Z}^+, \tag{1.16}$$

where $[\cdot,\cdot]_{\mathcal{W}}$ is the inner product in the Kreĭn signal space $\mathcal{W}$, and if also the trajectories of the adjoint system $\Sigma_*$ of $\Sigma$ have the same property. The latter system has the same state space $\mathcal{X}$ as $\Sigma$, its signal space is $\mathcal{W}_* = -\mathcal{W}$, and the sets of trajectories $(x(\cdot), w(\cdot))$ of $\Sigma$ and $(x_*(\cdot), w_*(\cdot))$ of $\Sigma_*$ are determined from each other through the "orthogonality relation" (see Section II.4)

$$\begin{aligned} -(x(n+1), x_*(0))_{\mathcal{X}} + (x(0), x_*(n+1))_{\mathcal{X}} \\ + \sum_{k=0}^{n} \langle w(k), w_*(n-k)\rangle_{\langle \mathcal{W}, \mathcal{W}_*\rangle} = 0, \end{aligned} \tag{1.17}$$

where $\langle \cdot, \cdot\rangle_{\langle \mathcal{W}, \mathcal{W}_*\rangle}$ is the duality pairing introduced in Section II.4 (see Subsection 2.3 for a short description of this duality pairing).

In the development of passive s/s systems theory in Parts II–III we made essential use of known results on the geometry of Kreĭn spaces. These spaces appear naturally since the forward passivity property (1.16) holds if and only if the graph $V$ (defined in (2.2) below) of the operator $F$ in (1.1) is nonnegative in the Kreĭn space $\mathfrak{K} = \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ with respect to the indefinite inner product

$$\left[\begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix}, \begin{bmatrix} \dot{x}' \\ x' \\ w' \end{bmatrix}\right]_{\mathfrak{K}} = -(\dot{x}, \dot{x}')_{\mathcal{X}} + (x, x')_{\mathcal{X}} + [w, w']_{\mathcal{W}}, \quad \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix}, \begin{bmatrix} \dot{x}' \\ x' \\ w' \end{bmatrix} \in \mathfrak{K}. \tag{1.18}$$

Moreover, a system is passive if and only if $V$ is a maximal nonnegative subspace of $\mathfrak{K}$, and it is conservative if and only if $V$ is a Lagrangean subspace of $\mathfrak{K}$, i.e., $V$ is its own orthogonal companion.

Passive systems are always LFT-stabilizable, since every fundamental decomposition $\mathcal{W} = -\mathcal{Y} \,[\dotplus]\, \mathcal{U}$ (where $-\mathcal{Y}$ is an anti-Hilbert space and $\mathcal{U}$ is a Hilbert space) is admissible, and the corresponding i/s/o system, which we call a *scattering representation*, is stable. The results obtained here for LFT-stabilizable systems therefor applies, and it gives us $H^\infty$ coprime right and left affine i/o transfer functions corresponding to other types of direct sum decompositions of the signal space. The two most important cases (in addition to the scattering case mentioned above), are the *Lagrangean decompositions* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ (where both $\mathcal{Y}$ and $\mathcal{U}$ are Lagrangean subspaces of $\mathcal{W}$), and the *orthogonal decompositions* $\mathcal{W} = -\mathcal{Y} \,[\dotplus]\, \mathcal{U}$ (where $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal in $\mathcal{W}$, but they are not necessarily Hilbert spaces).

We use the word "impedance" in connection with an i/s/o representation, or a right or left affine representation, or a right or left affine transfer function, or a generalized transfer function, if the underlying decomposition of the signal space is Lagrangean. The word "scattering" is used if the decomposition is fundamental, and the word "transmission" if the decomposition is orthogonal. Scattering representations of passive s/s systems were studied in Part II, and admissible i/s/o impedance and transmission representations and transfer functions of passive systems were studied in Part III. Essential differences exist between the scattering, impedance, and transmission cases.

As we mentioned above, every fundamental decomposition of the signal space of a passive s/s system is admissible. However, not every orthogonal decomposition is admissible for a passive s/s system, and there exist s/s system whose signal space do not have any admissible Lagrangean decomposition (even if we require these decompositions to have the correct input and output dimensions). This provides us with a motivation to develop the theory of right and left affine i/s/o representations of the s/s system and their is/so transfer functions. As we shall see, the notion of a generalized i/o transfer function of the type introduced here is important in the case of orthogonal decompositions of the signal space of a passive s/s system, but not in the case of Lagrangean decompositions.

In Part III we among others proved the existence of simple conservative and minimal passive i/s/o realizations of a given function $\mathfrak{D}$ of the Potapov class $P(\Omega;\mathcal{U},\mathcal{Y})$ under the additional assumption that $\Omega$ contains the point zero (by a realization of $\mathfrak{D}$ we mean an i/s/o system whose transmission matrix coincides with $\mathfrak{D}$ on $\Omega$).[2] In Section 8 we extend that result to the case where $0 \notin \Omega$, replacing the earlier i/s/o realizations by s/s realizations, and show how to realize $\mathfrak{D} \in P(\Omega;\mathcal{U},\mathcal{Y})$ in the general case as the generalized i/o transfer function of a simple conservative or minimal passive s/s system.

**Notations**. The following standard notations are used below. $\mathbb{C}$ is the complex plane, $\mathbb{D}$ is the open unit disk in $\mathbb{C}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

The space of bounded linear operators from one Kreĭn space $\mathcal{X}$ to another Kreĭn space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X};\mathcal{Y})$, and we abbreviate $\mathcal{B}(\mathcal{X};\mathcal{X})$ to $\mathcal{B}(\mathcal{X})$. The domain, range, and kernel of a linear operator $A$ are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$, respectively. The restriction of $A$ to some subspace $\mathcal{Z} \subset \mathcal{D}(A)$ is denoted by $A|_{\mathcal{Z}}$. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$. For each $A \in \mathcal{B}(\mathcal{X})$ we let $\Lambda_A$ be the set of points $z \in \mathbb{C}$ for which $(1_{\mathcal{X}} - zA)$ has a bounded inverse, plus the point at infinity if $A$ is boundedly invertible. We denote the projection onto a closed subspace $\mathcal{Y}$ of a space $\mathcal{X}$ along some complementary subspace $\mathcal{U}$ by $P_{\mathcal{Y}}^{\mathcal{U}}$, and by $P_{\mathcal{Y}}$ if $\mathcal{Y}$ is orthogonal to $\mathcal{U}$ with respect to a Hilbert or Kreĭn space inner product in $\mathcal{X}$. The closed linear span of a set of subsets $\mathfrak{R}_\alpha \subset \mathcal{X}$ where $\alpha$ runs over some index set $\Lambda$ is denoted by $\vee_{\alpha\in\Lambda}\mathfrak{R}_\alpha$.

For a Hilbert space $\mathcal{U}$ the Hilbert space $\ell^2(\mathbb{Z}^+;\mathcal{U})$ contain those $\mathcal{U}$-valued sequences $u(\cdot)$ on $\mathbb{Z}^+$ which satisfy $\sum_{n\in\mathbb{Z}^+}\|u(n)\|^2 < \infty$, and the Hilbert space $H^2(\mathbb{D};\mathcal{U})$ consists of $\mathcal{U}$-valued analytic functions $\varphi$ on $\mathbb{D}$ for which $\sup_{0<r<1}\int_{|\zeta|=1}\|\varphi(r\zeta)\|_{\mathcal{U}}^2 d|\zeta| < \infty$. For a Banach spaces $\mathcal{U}$ and $\mathcal{Y}$ the Banach

[2]This is related to known results about reproducing kernel Hilbert spaces, as presented in, e.g., [ADRdS97].

space $\ell^\infty(\mathbb{Z}^+;\mathcal{U})$ consists of all bounded $\mathcal{U}$-valued sequences on $\mathbb{Z}^+$, and the Banach space $H^\infty(\mathbb{D};\mathcal{U},\mathcal{Y})$ consists of all bounded $\mathcal{B}(\mathcal{U},\mathcal{Y})$-valued analytic functions on $\mathbb{D}$.

We denote the ordered product of the two locally convex topological vector spaces $\mathcal{Y}$ and $\mathcal{U}$ by $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. In particular, although $\mathcal{Y}$ and $\mathcal{U}$ may be Hilbert or Kreĭn spaces (in which case the product topology on $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ is induced by an inner product), we shall not require that $\left[\begin{smallmatrix}\mathcal{Y}\\ 0\end{smallmatrix}\right] \perp \left[\begin{smallmatrix}0\\ \mathcal{U}\end{smallmatrix}\right]$ in $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. We identify a vector $\left[\begin{smallmatrix}y\\ 0\end{smallmatrix}\right] \in \left[\begin{smallmatrix}\mathcal{Y}\\ 0\end{smallmatrix}\right]$ with $y \in \mathcal{Y}$ and a vector $\left[\begin{smallmatrix}0\\ u\end{smallmatrix}\right] \in \left[\begin{smallmatrix}0\\ \mathcal{U}\end{smallmatrix}\right]$ with $u \in \mathcal{U}$, and then we sometimes write $\mathcal{Y} \dotplus \mathcal{U}$ instead of $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$, interpreting $\mathcal{Y} \dotplus \mathcal{U}$ as an *ordered direct sum.*

We denote the inner product in a Hilbert space $\mathcal{X}$ by $(\cdot,\cdot)_{\mathcal{X}}$, the inner product in a Kreĭn space $\mathcal{W}$ by $[\cdot,\cdot]_{\mathcal{W}}$. The set of all vectors that are orthogonal to a set $\mathcal{G}$ is denoted by $G^{[\perp]}$ in the case of a Kreĭn space and by $\mathcal{G}^\perp$ in the case of a Hilbert space. The orthogonal sum of two Hilbert spaces $\mathcal{Y}$ and $\mathcal{U}$ is denoted by $\mathcal{Y} \oplus \mathcal{U}$, and the orthogonal sum of two Kreĭn spaces $\mathcal{Y}$ and $\mathcal{U}$ is denoted by $\mathcal{Y}\,[\dotplus]\,\mathcal{U}$.

The set $\Omega(\Sigma;\mathcal{U},\mathcal{Y})$ is defined in (1.10). The notations $\Omega(\Sigma)$ and $\Omega_0(\Sigma)$ are introduced in Notations 6.5 and 6.11, respectively. The notation $\mathfrak{V}(z)$ is introduced in (5.9), and the notations $V(z)$, $\mathfrak{S}(z)$, and $\Sigma(z)$ in Lemma 6.1.

# 2 Preliminaries

## 2.1 Three kinds of representations

Instead of using the representation (1.1) for the trajectories $(x(\cdot),w(\cdot))$ of a s/s system $\Sigma$ with the Hilbert or Kreĭn state space $\mathcal{X}$ and signal space $\mathcal{W}$ we shall often use a *graph representation*

$$\begin{bmatrix} x(n+1)\\ x(n)\\ w(n)\end{bmatrix} \in V, \quad n \in \mathbb{Z}^+, \quad x(0) \ = x_0, \tag{2.1}$$

where $V$ is the graph of the operator $F$, i.e.,

$$V = \left\{ \left[\begin{smallmatrix}\dot{x}\\ x\\ w\end{smallmatrix}\right] \;\middle|\; \dot{x} = F(x,w),\ \left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \mathcal{D}(F)\right\}. \tag{2.2}$$

As shown in Section I.2, the properties that we required in the introduction from the operator $F$ in (1.1) and its domain are equivalent to the following properties of the graph $V$ of $F$:

(i) $V$ is closed in the product $\mathfrak{K} = \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$;

(ii) For every $x \in \mathcal{X}$ there is some $\left[\begin{smallmatrix} \dot{x} \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} \dot{x} \\ x \\ w \end{smallmatrix}\right] \in V$;

(iii) If $\left[\begin{smallmatrix} \dot{x} \\ 0 \\ 0 \end{smallmatrix}\right] \in V$, then $\dot{x} = 0$;

(iv) The set $\left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} \dot{x} \\ x \\ w \end{smallmatrix}\right] \in V \text{ for some } \dot{x} \in \mathcal{X} \right\}$ is closed in the product $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

A colligation $\Sigma := (V; \mathcal{X}, \mathcal{W})$ where the state space $\mathcal{X}$ and the signal space $\mathcal{W}$ are Hilbert or Kreĭn spaces and $V$ is a subspace of the node space $\mathfrak{K} := \left[\begin{smallmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ with properties (i)–(iv) is called a *s/s node*. A pair of sequences $(x(\cdot), w(\cdot))$ of vectors $x(n) \in \mathcal{X}$ and $w(n) \in \mathcal{W}$, $n \in \mathbb{Z}^+$, satisfying (2.1) is called a trajectory generated by $V$ with initial state $x_0$, and $V$ is called the generating subspace. By a linear discrete time invariant s/s system we understand a s/s node $\Sigma$ together with the set of all trajectories generated by $V$ via (2.1). We still use the same notation $\Sigma := (V; \mathcal{X}, \mathcal{W})$ for the s/s system as for the system node.

An i/s/o system $\Sigma_{dv/s/s} := \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$, where $\mathcal{L}$ is an auxiliar Hilbert space (whose elements are called driving variables) and $\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right])$ is called a driving variable representation of a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ if 1) $D'$ has a bounded left inverse and $V$ is given by

$$V = \mathcal{R}\left(\begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix}\right) = \left\{ \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in \mathfrak{K} \;\middle|\; \begin{matrix} \dot{x} = A'x + B'\ell, \\ w = C'x + D'\ell, \end{matrix} \; \ell \in \mathcal{L} \right\}. \tag{2.3}$$

This is equivalent to the following property: the correspondence $\big(x(\cdot), \ell(\cdot), w(\cdot)\big) \to (x(\cdot), w(\cdot), )$ is a bijective map from the set of trajectories of the of i/s/o system $\Sigma_{dv/s/s}$ onto the set of trajectories of the s/s system $\Sigma$. It was proved in Section I.3 that there exist infinite many such representations for a s/s system, that can be parameterized by means of one fixed driving variable representation in the following way.

**Theorem 2.1.** *[Theorem I.3.3.] Let* $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ *be a driving variable representation of a state signal system* $\Sigma$*, and let*

$$\begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ K' & M' \end{bmatrix} \tag{2.4}$$

*where*

$$K' \in \mathcal{B}(\mathcal{X}; \mathcal{L}),\ M' \in \mathcal{B}(\mathcal{L}_1; \mathcal{L}),\ \text{and } M' \text{ has a bounded inverse,} \tag{2.5}$$

*for some Hilbert space* $\mathcal{L}_1$*. Then* $\Sigma^1_{dv/s/s} = \left(\left[\begin{smallmatrix} A'_1 & B'_1 \\ C'_1 & D'_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}_1, \mathcal{W}\right)$ *is a driving variable representation of* $\Sigma$*. Conversely, every driving variable representation* $\Sigma^1_{dv/s/s}$ *of* $\Sigma$ *may be obtained from formula* (2.4) *for some operators* $K'$ *and* $M'$ *satisfying* (2.5)*. The operators* $K'$ *and* $M'$ *are uniquely defined by* $\Sigma_{dv/s/s}$ *and* $\Sigma^1_{dv/s/s}$ *via*

$$D'K' = C'_1 - C' \text{ and } D'M' = D'_1. \tag{2.6}$$

An i/s/o system $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}\right)$, where $\mathcal{K}$ is an auxiliary Hilbert space (whose elements are called error variables) and $\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{K} \end{smallmatrix}\right])$ is called an output nulling representation of a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ if 1) $D''$ is surjective and 2)

$$V = \mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & A'' & B'' \\ 0 & C'' & D'' \end{bmatrix}\right) = \left\{ \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in \mathfrak{K} \;\middle|\; \begin{matrix} \dot{x} = A''x + B''w \\ 0 = C''x + D''w \end{matrix} \right\} \tag{2.7}$$

This is equivalent to the following property: the correspondence $\big(x(\cdot), w(\cdot), 0\big) \to (x(\cdot), w(\cdot), )$ is a bijective map from the set of trajectories $\big(x(\cdot), w(\cdot), k(\cdot)\big)$ of $\Sigma_{s/s/on}$ for which $k(n) = 0$ for all $n \in \mathbb{Z}^+$ onto the set of trajectories of the s/s system $\Sigma$. It was proved in Section I.4 that there exist infinite many such representations for a s/s system, that can be parameterized by means of one fixed output nulling representation in the following way:

**Theorem 2.2** (Theorem I.4.3)**.** *Let* $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}\right)$ *be an output nulling representation of a s/s system* $\Sigma$*, and let*

$$\begin{bmatrix} A''_1 & B''_1 \\ C''_1 & D''_1 \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & K'' \\ 0 & M'' \end{bmatrix} \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}, \tag{2.8}$$

*where*

$$K'' \in \mathcal{B}(\mathcal{K}, \mathcal{X}),\ M'' \in \mathcal{B}(\mathcal{K}, \mathcal{K}_1),\ \text{and } M'' \text{ has a bounded inverse}, \tag{2.9}$$

*for some Hilbert space* $\mathcal{K}_1$*. Then* $\Sigma^1_{s/s/on} = \left(\left[\begin{smallmatrix} A''_1 & B''_1 \\ C''_1 & D''_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}_1\right)$ *is an output nulling representation of* $\Sigma$*. Conversely, every output nulling representation* $\Sigma^1_{s/s/on}$ *of* $\Sigma$ *may be obtained from the formula* (2.8) *for some operators* $M''$ *and* $K''$ *satisfying* (2.9)*. The operators* $M''$ *and* $K''$ *are uniquely defined by* $\Sigma_{s/s/on}$ *and* $\Sigma^1_{s/s/on}$ *via*

$$M''D'' = D''_1 \text{ and } K''D'' = B''_1 - B''. \tag{2.10}$$

An i/s/o system $\Sigma_{i/s/o} = ([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are taken from a direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of the signal space of the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called an i/s/o representation of $\Sigma$ if $[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in \mathcal{B}([\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}]; [\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}])$ and $V$ is given by

$$\begin{aligned} V = \mathcal{R}\left(\begin{bmatrix} A & B \\ 1_{\mathcal{X}} & 0 \\ C & D \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) &= \mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & A & 0 & B \\ 0 & C & -1_{\mathcal{Y}} & D \end{bmatrix}\right) \\ &= \left\{ \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in \mathfrak{K} \,\middle|\, \begin{aligned} \dot{x} &= Ax + Bu, \\ w &= Cx + Du + u, \end{aligned} \right\} \end{aligned} \tag{2.11}$$

This is equivalent to the following property: the correspondence $\big(x(\cdot), u(\cdot), y(\cdot)\big) \to (x(\cdot), w(\cdot))$ is a bijective map from the set of i/s/o trajectories of $\Sigma_{i/s/o}$ onto the set of trajectories of $\Sigma$. A decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ for which such an i/s/o representation exists is called *admissible*. The existence of infinitely many admissible decompositions of $\mathcal{W}$ was established in Section I.5. Consequently, there also exist infinitely many i/s/o representations. A description of the connections between these representations is given in Section I.5. The relationships between the coefficients and the is/so transfer functions of driving variable, output nulling, and i/s/o representations of a given s/s system $\Sigma$ are described in Sections I.4–6.

## 2.2 Passive Systems

In Parts II and III we developed the theory of passive s/s systems. The state space of a passive system is a Hilbert space, but the signal space $\mathcal{W}$ is a Kreĭn space with an indefinite inner product $[\cdot,\cdot]_{\mathcal{W}}$. The results from Part I (where we took $\mathcal{W}$ to be a Hilbert space) are still applicable if we replace the inner product in $\mathcal{W}$ by a positive inner product which induces a norm topology on $\mathcal{W}$ that is equivalent to the its strong topology.

A s/s system $\Sigma$ with a Hilbert state space $\mathcal{X}$ and a Kreĭn space $\mathcal{W}$ is *forward passive* if all its trajectories $(x(\cdot), w(\cdot))$ satisfy the forward passivity inequality (1.16). This can be interpreted as a positivity condition of the subspace $V$ in (2.1) with respect to the Kreĭn space inner product given by (1.18) in the *node space* $\mathfrak{K} := -\mathcal{X} [\dotplus] \mathcal{X} [\dotplus] \mathcal{W} = \left[\begin{smallmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$. Thus, $\Sigma$ is forward passive if and only if $[k,k]_{\mathfrak{K}} \geq 0$ for all $k \in \mathfrak{K}$., i.e., if and only if $V$ is a nonnegative subspace of the Kreĭn node space $\mathfrak{K}$.

The notions of backward passivity and passivity use the notion of the adjoint of a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$. This is another s/s system $\Sigma_* =$

$(V_*;\mathcal{X},\mathcal{W}_*)$ with the same state space $\mathcal{X}$, with signal space is $\mathcal{W}_* = -\mathcal{W}$, and with generating subspace

$$V_* = \left\{ k_* = \begin{bmatrix} \dot{x}_* \\ x_* \\ w_* \end{bmatrix} \in \mathfrak{K}_* \;\middle|\; \left[\begin{bmatrix} \dot{x}_* \\ x_* \\ w_* \end{bmatrix}, \begin{bmatrix} x \\ \dot{x} \\ w \end{bmatrix}\right]_{\mathfrak{K}} = 0 \text{ for all } \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in V \right\}. \qquad (2.12)$$

A s/s system $\Sigma$ is *backward passive* if its adjoint $\Sigma_*$ is forward passive, and $\Sigma$ is *passive* if it is both forward and backward passive. As shown in Theorem II.5.6, $\Sigma$ is passive if and only if its generating subspace $V$ is a maximal nonnegative subspace of the node space $\mathfrak{K}$.

By a *fundamental decomposition* of the Kreĭn space $\mathcal{W}$ of a s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ we mean an orthogonal decomposition $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$, where $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal Hilbert spaces with the induced inner products

$$\begin{aligned} (y,y')_{\mathcal{Y}} &= -[y,y']_{\mathcal{W}}, \qquad y,\ y' \in \mathcal{Y}, \\ (u,u')_{\mathcal{U}} &= [u,u']_{\mathcal{W}}, \qquad u,\ u' \in \mathcal{U}. \end{aligned}$$

If such a decomposition is admissible, then we call the corresponding i/s/o representation a *scattering represenation* of $\Sigma$, and its i/o transfer function is called a *scattering matrix*. As shown in Theorem II.5.7, a forward passive s/s system $\Sigma$ is passive if and only if its signal space has an admissible fundamental decomposition, in which case every fundamental decomposition is admissible.

Every scattering representation $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ of a passive s/s system is stable. Indeed, because of the passivity, every trajectory $(x(\cdot),u(\cdot),y(\cdot))$ of $\Sigma_{i/s/o}$ satisfies

$$\|x(k+1)\|_{\mathcal{X}}^2 + \|y(k)\|_{\mathcal{Y}}^2 \le \|x(k)\|_{\mathcal{X}}^2 + \|u(k)\|_{\mathcal{U}}^2, \qquad k \in \mathbb{Z}^+, \qquad (2.13)$$

and by summing over $k = 0,1,\dots,n$ we get

$$\|x(n+1)\|_{\mathcal{X}}^2 + \sum_{k=0}^{n} \|y(k)\|_{\mathcal{Y}}^2 \le \|x(0)\|_{\mathcal{X}}^2 + \sum_{k=0}^{n} \|u(k)\|_{\mathcal{U}}^2, \qquad n \in \mathbb{Z}^+. \quad (2.14)$$

Consequently, $x \in \ell^\infty(\mathcal{X})$ and $y \in \ell^2(\mathcal{Y})$ whenever $u \in \ell^2(\mathcal{U})$, which shows that $\Sigma_{i/s/o}$ is stable. In particular, conditions 1)–4) in Lemma I.9.2 hold. Thus, every passive system is LFT-stabilizable, and hence both stabilizable and detectable.

In Parts II and III we studied, in addition to the fundamental decompositions of the signal space mentioned above, also two other types of decompositions, namely *orthogonal* and *Lagrangean decompositions*.

In an *orthogonal decomposition* the signal space $\mathcal{W}$ is split into $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$, where $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal in $\mathcal{W}$ and $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn spaces with the inner products

$$[y, y']_{\mathcal{Y}} = -[y, y']_{\mathcal{W}}, \qquad [u, u']_{\mathcal{U}} = [u, u']\mathcal{W}$$

inherited from $\mathcal{W}$. Thus, if we decompose $w$, $w' \in \mathcal{W}$ into $w = y + u$, $w' = y' + u'$, where $y$, $y' \in \mathcal{Y}$ and $u$, $u' \in \mathcal{U}$, then

$$[w, w']_{\mathcal{W}} = -[y, y']_{\mathcal{Y}} + [u, u']_{\mathcal{U}}, \tag{2.15}$$

If an orthogonal decomposition is admissible for a given s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, then we call the corresponding i/s/o representation a *transmission representation* of $\Sigma$, and its i/o transfer function is called a *transmission matrix*.

A *Lagrangean decompositions* $\mathcal{W} = \mathcal{F} \dot{+} \mathcal{E}$ is a direct sum decomposition where $\mathcal{F}$ and $\mathcal{E}$ are no longer orthogonal, but both $\mathcal{F}$ and $\mathcal{E}$ are *Lagrangean* subspaces of $\mathcal{W}$, meaning that they coincide with their own orthogonal companion. In particular, they are *neutral*, i.e., $[f, f]_{\mathcal{W}} = 0$ and $[e, e]_{\mathcal{W}} = 0$ for all $f \in \mathcal{F}$ and $e \in \mathcal{E}$.[3] As shown in Lemma III.2.3, it is possible to choose Hilbert space inner products in $\mathcal{F}$ and $\mathcal{E}$ which are compatible with the strong topology inherited from $\mathcal{W}$ in such a way that there exists a unitary operator $\Psi \colon \mathcal{E} \to \mathcal{F}$ so that if if we decompose $w$, $w' \in \mathcal{W}$ into $w = f + e$, $w' = f' + e'$, where $f$, $f' \in \mathcal{F}$ and $e$, $e' \in \mathcal{E}$, then

$$[w, w']_{\mathcal{W}} = (f, \Psi e')_{\mathcal{F}} + (e, \Psi^* f')_{\mathcal{F}}, \tag{2.16}$$

To indicate that the inner products in $\mathcal{F}$ and $\mathcal{E}$ have been chosen in this way we write $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dot{+}} \mathcal{E}$. If a Lagrangean decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dot{+}} \mathcal{E}$ is admissible for a given s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, then we call the corresponding i/s/o representation an *impedance representation* of $\Sigma$, and its i/o transfer function is called an *impedance matrix*. Transmission and impedance representations of passive s/s systems were studied in Part III. In this paper we will study their affine counterparts.

## 2.3 Direct Sum Decompositions of Kreĭn Spaces

In the definition of the adjoint s/s system $\Sigma_*$ we replaced the original signal space $\mathcal{W}$ by the dual signal space $\mathcal{W}_* = -\mathcal{W}$. In the sequel we shall identify

[3] A necessary condition for the existence of a Lagrangean decomposition is that the positive and negative indeces $\mathrm{ind}_+\mathcal{W}$ and $\mathrm{ind}_-\mathcal{W}$ are equal. Here $\mathrm{ind}_-\mathcal{W} = \dim \mathcal{W}_-$ and $\mathrm{ind}_+\mathcal{W} = \dim \mathcal{W}_+$, where $\mathcal{W} = -\mathcal{W}_- \, [\dot{+}] \, \mathcal{W}_+$ is an arbitrary fundamental decomposition of $\mathcal{W}$.

the dual of $\mathcal{W}$ with $\mathcal{W}_*$ through the duality pairing $\langle\cdot,\cdot\rangle_{\langle\mathcal{W},\mathcal{W}_*\rangle}$ defined by

$$\langle w, w_*\rangle_{\langle\mathcal{W},\mathcal{W}_*\rangle} = [w, \mathcal{I}w_*]_{\mathcal{W}} = [\mathcal{I}^*w, w_*]_{\mathcal{W}_*},\ w \in \mathcal{W},\ w_* \in \mathcal{W}_*, \tag{2.17}$$

where $\mathcal{I}$ is the identity operator from $\mathcal{W}_*$ to $\mathcal{W}$. This operator is anti-unitary, i.e., $\mathcal{I}^* = -\mathcal{I}^{-1}$ (see Section II.4 for details).

In the following we shall have to compute adjoints of operators which are defined on $\mathcal{W}$ or take their values in $\mathcal{W}$. This can be done in two ways: whenever we compute the adjoint with respect to the duality pairing $\langle\cdot,\cdot\rangle_{\langle\mathcal{W},\mathcal{W}_*\rangle}$ we denote the adjoint with the superscript $^\dagger$, and whenever we compute the adjoints with respect to the inner product of $\mathcal{W}$ we use the superscript $^*$. Thus, if, for example, $C' \in \mathcal{B}(X;\mathcal{W})$ and $B'' \in \mathcal{B}(\mathcal{W};\mathcal{X})$, where $\mathcal{X}$ is a Hilbert space whose dual we identify with $\mathcal{X}$ itself in the standard way, then for all $x \in \mathcal{X}$, $w \in \mathcal{W}$, and $w_* \in \mathcal{W}_*$,

$$\begin{aligned}
[C'x, w]_{\mathcal{W}} &= (x, (C')^*w)_{\mathcal{X}}, & (B''w, x)_{\mathcal{X}} &= [w, (B'')^*x]_{\mathcal{W}},\\
\langle C'x, w_*\rangle_{\langle\mathcal{W},\mathcal{W}_*\rangle} &= (x, (C')^\dagger w_*)_{\mathcal{X}}, & (B''w, x)_{\mathcal{X}} &= \langle w, (B'')^\dagger x\rangle_{\langle\mathcal{W},\mathcal{W}_*\rangle}.
\end{aligned}$$

Combining this with (2.17) we find that

$$(C')^\dagger = C^*\mathcal{I}, \qquad \mathcal{I}(B'')^\dagger = (B'')^*. \tag{2.18}$$

In the same way one can show (see Section II.4 for details) that if $D \in \mathcal{B}(\mathcal{W})$, then

$$\mathcal{I}(D)^\dagger = D^*\mathcal{I}. \tag{2.19}$$

In the affine i/s/o representations which will be presented below we frequently decompose operators $C' \in \mathcal{B}(\mathcal{X},\mathcal{W})$ and $B'' \in \mathcal{B}(\mathcal{W},\mathcal{X})$ into blocks of the type

$$\begin{aligned}
C' &= \begin{bmatrix} C'_{\mathcal{Y}} \\ C'_{\mathcal{U}} \end{bmatrix} := \begin{bmatrix} P^{\mathcal{U}}_{\mathcal{Y}}C' \\ P^{\mathcal{Y}}_{\mathcal{U}}C' \end{bmatrix} \in \mathcal{B}(\mathcal{X}; \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]),\\
B'' &= \begin{bmatrix} B''_{\mathcal{Y}} & B''_{\mathcal{U}} \end{bmatrix} := \begin{bmatrix} B''|_{\mathcal{Y}} & B''|_{\mathcal{U}} \end{bmatrix} \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]; \mathcal{X}),
\end{aligned} \tag{2.20}$$

where $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is a direct sum decomposition of $\mathcal{W}$. The corresponding block decompositions of the operators $(C')^\dagger$ and $(B'')^\dagger$ depend on how we interpret the duals of the spaces $\mathcal{Y}$ and $\mathcal{U}$. Out of the different choices available we prefer to identify the duals of $\mathcal{Y}$ and $\mathcal{U}$ by $\mathcal{Y}_*$ and $\mathcal{U}_*$, respectively, given by

$$\mathcal{Y}_* = \mathcal{U}^{\langle\perp\rangle}, \qquad \mathcal{U}_* = \mathcal{Y}^{\langle\perp\rangle}, \tag{2.21}$$

where the orthogonal complements are computed with respect to the duality pairing $\langle\cdot,\cdot\rangle_{\langle\mathcal{W},\mathcal{W}_*\rangle}$ between $\mathcal{W}$ and $\mathcal{W}_* = -\mathcal{W}$ introduced earlier, i.e.,

$$\begin{aligned}
\mathcal{Y}_* &= \big\{y_* \in \mathcal{W}_* \bigm| \langle u, y_*\rangle_{\langle\mathcal{W},\mathcal{W}_*\rangle} = 0 \text{ for all } u \in \mathcal{U}\big\},\\
\mathcal{U}_* &= \big\{u_* \in \mathcal{W}_* \bigm| \langle y, u_*\rangle_{\langle\mathcal{W},\mathcal{W}_*\rangle} = 0 \text{ for all } y \in \mathcal{Y}\big\}.
\end{aligned} \tag{2.22}$$

The method by which we identify the duals of $\mathcal{Y}$ and $\mathcal{U}$ with $\mathcal{Y}_*$ and $\mathcal{U}_*$ will be explained after the following lemma.

**Lemma 2.3.** *Let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a direct sum decomposition of the Kreĭn space $\mathcal{W}$, and define $\mathcal{U}_*$ and $\mathcal{Y}_*$ by* (2.21)*, where the orthogonal complements are computed with respect to the duality pairing* (2.17) *between $\mathcal{W}$ and $\mathcal{W}_* = -\mathcal{W}$. Then*

1) $\mathcal{W}_* = \mathcal{U}_* \dotplus \mathcal{Y}_*$,
2) $(P_{\mathcal{Y}}^{\mathcal{U}})^\dagger = P_{\mathcal{Y}_*}^{\mathcal{U}_*}$ *and* $(P_{\mathcal{U}}^{\mathcal{Y}})^\dagger = P_{\mathcal{U}_*}^{\mathcal{Y}_*}$*, where* $(P_{\mathcal{Y}}^{\mathcal{U}})^\dagger$ *and* $(P_{\mathcal{U}}^{\mathcal{Y}})^\dagger$ *are the adjoints of* $P_{\mathcal{Y}}^{\mathcal{U}} \in \mathcal{B}(\mathcal{W})$ *and* $P_{\mathcal{U}}^{\mathcal{Y}} \in \mathcal{B}(\mathcal{W})$*, respectively, computed with respect to the duality pairing* (2.17).

*Proof.* The two operators $P_{\mathcal{U}}^{\mathcal{Y}}$ and $P_{\mathcal{Y}}^{\mathcal{U}}$ are complementary projections in $\mathcal{W}$, i.e., $(P_{\mathcal{U}}^{\mathcal{Y}})^2 = P_{\mathcal{U}}^{\mathcal{Y}}$, $(P_{\mathcal{Y}}^{\mathcal{U}})^2 = P_{\mathcal{Y}}^{\mathcal{U}}$, and $P_{\mathcal{U}}^{\mathcal{Y}} + P_{\mathcal{Y}}^{\mathcal{U}} = 1_{\mathcal{W}}$. Taking the adjoints of these three equations we find that $(P_{\mathcal{Y}}^{\mathcal{U}})^\dagger$ and $(P_{\mathcal{U}}^{\mathcal{Y}})^\dagger$ are complementary projections in $\mathcal{W}_*$. Moreover, $\mathcal{N}\left((P_{\mathcal{U}}^{\mathcal{Y}})^\dagger\right) = (\mathcal{R}\left(P_{\mathcal{U}}^{\mathcal{Y}}\right))^{\langle\perp\rangle} = \mathcal{U}^{\langle\perp\rangle} = \mathcal{Y}_*$, and in the same way $\mathcal{N}\left((P_{\mathcal{Y}}^{\mathcal{U}})^\dagger\right) = \mathcal{U}_*$. Thus, $\mathcal{W}_* = \mathcal{U}_* \dotplus \mathcal{Y}_*$, $(P_{\mathcal{Y}}^{\mathcal{U}})^\dagger = P_{\mathcal{Y}_*}^{\mathcal{U}_*}$ and $(P_{\mathcal{U}}^{\mathcal{Y}})^\dagger = P_{\mathcal{U}_*}^{\mathcal{Y}_*}$. □

Every bounded linear functional on $\mathcal{U}$ can be extended to a bounded linear functional on $\mathcal{W}$, so it has a representation of the type $u \mapsto \langle u, w_* \rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle}$ for some $w_* \in \mathcal{W}_*$. But

$$\langle u, w_* \rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle} = \langle P_{\mathcal{U}}^{\mathcal{Y}} u, w_* \rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle} = \langle u, (P_{\mathcal{U}}^{\mathcal{Y}})^\dagger w_* \rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle} = \langle u, P_{\mathcal{U}_*}^{\mathcal{Y}_*} w_* \rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle},$$

so it is possible to replace the vector $w_* \in \mathcal{W}$ in this representation by the vector $u_* = P_{\mathcal{U}_*}^{\mathcal{Y}_*} w_* \in \mathcal{U}_*$. Every nonzero $u_* \in \mathcal{U}_*$ induces a nonzero functional on $\mathcal{U}$ by the above formula, because if $u_* \in \mathcal{U}_*$ and $\langle u, u_* \rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle} = 0$ for all $u \in \mathcal{U}$, then $u_* \in \mathcal{U}^{\langle\perp\rangle} \cap \mathcal{U}_* = \mathcal{Y}_* \cap \mathcal{U}_* = \{0\}$, and hence $u_* = 0$. This shows that we can identify the dual of $\mathcal{U}$ with $\mathcal{U}_*$ by the duality pairing

$$\langle u, u_* \rangle_{\langle \mathcal{U},\mathcal{U}_*\rangle} = \langle u, u_* \rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle}, \;\; u \in \mathcal{U}, \;\; u_* \in \mathcal{U}_*. \tag{2.23}$$

The same computation can be repeated with $\mathcal{U}$ replaced by $\mathcal{Y}$, so that we identify the dual of $\mathcal{Y}$ with $\mathcal{Y}_* = \mathcal{U}^{\langle\perp\rangle}$ by the duality pairing

$$\langle y, y_* \rangle_{\langle \mathcal{Y},\mathcal{Y}_*\rangle} = \langle y, y_* \rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle}, \;\; y \in \mathcal{Y}, \;\; y_* \in \mathcal{Y}_*. \tag{2.24}$$

Note that these duality pairings do not depend in any way on any particular inner product that we may choose to use in $\mathcal{Y}_*$ and $\mathcal{U}_*$. Also note that if $w \in \mathcal{W}$, $w_* \in \mathcal{W}_*$ and if we denote $u = P_{\mathcal{U}}^{\mathcal{Y}} w$, $y = P_{\mathcal{Y}}^{\mathcal{U}} w$, $u_* = P_{\mathcal{U}_*}^{\mathcal{Y}_*} w_*$ and $y_* = P_{\mathcal{Y}_*}^{\mathcal{U}_*} w_*$, then

$$\langle w, w_* \rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle} = \langle y, y_* \rangle_{\langle \mathcal{Y},\mathcal{Y}_*\rangle} + \langle u, u_* \rangle_{\langle \mathcal{U},\mathcal{U}_*\rangle}. \tag{2.25}$$

**Lemma 2.4.** *Let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a direct sum decomposition of the Kreĭn space $\mathcal{W}$, and let $\mathcal{W}_* = \mathcal{U}_* \dotplus \mathcal{Y}_*$ be the dual decomposition of $\mathcal{W}_*$ in Lemma 2.3, with $\mathcal{Y}_*$ and $\mathcal{U}_*$. Let $C' \in \mathcal{B}(\mathcal{X};\mathcal{W})$ and $B'' \in \mathcal{B}(\mathcal{W};\mathcal{X})$, decompose these operators as in* (2.20)*, and decompose the adjoint operators $(C')^\dagger$ and $(B'')^\dagger$ in the same way with respect to the dual decomposition $\mathcal{W}_* = \mathcal{U}_* \dotplus \mathcal{Y}_*$ of $\mathcal{W}_*$ into*

$$\begin{aligned} (B'')^\dagger &= \begin{bmatrix} (B'')^\dagger_{\mathcal{U}_*} \\ (B'')^\dagger_{\mathcal{Y}_*} \end{bmatrix} := \begin{bmatrix} P^{\mathcal{Y}_*}_{\mathcal{U}_*}(B'')^\dagger \\ P^{\mathcal{U}_*}_{\mathcal{Y}_*}(B'')^\dagger \end{bmatrix} \in \mathcal{B}(\mathcal{X}; \left[\begin{smallmatrix} \mathcal{U}_* \\ \mathcal{Y}_* \end{smallmatrix}\right]), \\ (C')^\dagger &= \begin{bmatrix} (C')^\dagger_{\mathcal{U}_*} & (C')^\dagger_{\mathcal{Y}_*} \end{bmatrix} := \begin{bmatrix} (C')^\dagger|_{\mathcal{U}_*} & (C')^\dagger|_{\mathcal{Y}_*} \end{bmatrix} \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{U}_* \\ \mathcal{Y}_* \end{smallmatrix}\right]; \mathcal{X}), \end{aligned} \tag{2.26}$$

*Then*

$$\begin{aligned} (C'_{\mathcal{Y}})^\dagger &= (C')^\dagger_{\mathcal{Y}_*}, \qquad (C'_{\mathcal{U}})^\dagger = (C')^\dagger_{\mathcal{U}_*} \\ (B''_{\mathcal{Y}})^\dagger &= (B'')^\dagger_{\mathcal{Y}_*}, \qquad (B''_{\mathcal{U}})^\dagger = (B'')^\dagger_{\mathcal{U}_*}, \end{aligned} \tag{2.27}$$

*where the adjoints on the left-hande side of the equality signs are computed with respect to the duality pairings between $\mathcal{Y}$ and $\mathcal{Y}_*$ and between $\mathcal{U}$ and $\mathcal{U}_*$, and the adjoints on the right-hand side are computed with respect to the duality pairing between $\mathcal{W}$ and $\mathcal{W}_*$.*

*Proof.* For each $x \in \mathcal{X}$, $y_* \in \mathcal{Y}$, and $u_* \in \mathcal{U}$ we have by (2.25)

$$\begin{aligned} (x, (C'_{\mathcal{Y}})^\dagger y_*)_{\mathcal{X}} + (x, (C'_{\mathcal{U}})^\dagger u_*)_{\mathcal{X}} &= \langle C'_{\mathcal{Y}} x, y_* \rangle_{\langle \mathcal{Y}, \mathcal{Y}_* \rangle} + \langle C'_{\mathcal{U}} x, u_* \rangle_{\langle \mathcal{U}, \mathcal{U}_* \rangle} \\ &= \langle C' x, y_* + u_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = (x, (C'_{\mathcal{Y}})^\dagger (y_* + u_*))_{\mathcal{X}} \\ &= (x, (C')^\dagger|_{\mathcal{Y}_*} y_*)_{\mathcal{X}} + (x, (C')^\dagger|_{\mathcal{U}_*} u_*)_{\mathcal{X}}. \end{aligned}$$

This proves the first two identites in (2.27). In the same way we have for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $u \in \mathcal{U}$,

$$\begin{aligned} (y, (B''_{\mathcal{Y}})^\dagger x)_{\langle \mathcal{Y}, \mathcal{Y}_* \rangle} + (u, (B''_{\mathcal{U}})^\dagger x)_{\langle \mathcal{U}, \mathcal{U}_* \rangle} &= (B'' y, x)_{\mathcal{X}} + (B''_{\mathcal{U}} u, x)_{\mathcal{X}} \\ &= (B''(y+u), x)_{\mathcal{X}} = (y+u, (B'')^\dagger x)_{\langle \mathcal{W}, \mathcal{W}_* \rangle} \\ &= (y, P^{\mathcal{U}_*}_{\mathcal{Y}_*}(B'')^\dagger x)_{\langle \mathcal{Y}, \mathcal{Y}_* \rangle} + (u, P^{\mathcal{Y}_*}_{\mathcal{U}_*}(B'')^\dagger x)_{\langle \mathcal{U}, \mathcal{U}_* \rangle}. \end{aligned}$$

This proves the last two identities in (2.27). □

# 3 Right Affine Representations

As we have shown in Part III, not every orthogonal decomposition $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ of the Kreĭn signal space is admissible for a passive s/s system $\Sigma$, i.e., not every such decomposition gives rise to an i/s/o (transmission) representation of $\Sigma$. For the Lagrangean decompositions $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$ the situation is even more problematic, since none of the Lagrangean decompositions may be admissible, as shown in Example III.5.12. In this connection it is natural to introduce and study right and left affine representations of the system.

In the introduction we defined what we mean by a right affine i/s/o representation of a s/s system $\Sigma$, and we mentioned in passing that they can be obtained from the corresponding driving variable representations by splitting the signal component of a trajectory into an input and an output component. As explained in the introduction, for any i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of the signal space $\mathcal{W}$ of a s/s system $\Sigma$ the trajectories $(x(\cdot), w(\cdot))$ can be written in the i/s/o form $(x(\cdot), u(\cdot), y(\cdot))$ where

$$u(n) = P_{\mathcal{U}}^{\mathcal{Y}} w(n), \quad y(n) = P_{\mathcal{Y}}^{\mathcal{U}} w(n), \quad n \in \mathbb{Z}^+.$$

A sequence $(x(\cdot), u(\cdot), y(\cdot))$ obtained in this way will be called an *i/s/o trajectory of* $\Sigma$, corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. By first fixing a particular i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ and then applying this decompositions to the class of all driving variable representations of a given s/s system we get a bijective correspondence

$$\left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right) \to \left(\left[\begin{smallmatrix} A' & B' \\ \hline C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]\right)$$

from the set of all driving variable representations $\Sigma_{dv/s/s}$ of $\Sigma$ to the set of all right affine i/s/o representations $\Sigma_{i/s/o}^{r}$ of $\Sigma$, corresponding to this particular decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$, by formula (1.9).

Clearly, a right affine i/s/o representation of a s/s system $\Sigma$ corresponding to some given decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is not unique, since the driving variable representation on which it is based is not unique. However, if we have two such representations, the one given above, and another one

$$\Sigma_{i/s/o}^{r,1} := \left(\left[\begin{smallmatrix} A'_1 & B'_1 \\ \left[\begin{smallmatrix} (C'_1)_{\mathcal{Y}} \\ (C'_1)_{\mathcal{U}} \end{smallmatrix}\right] & \left[\begin{smallmatrix} (D'_1)_{\mathcal{Y}} \\ (D'_1)'_{\mathcal{U}} \end{smallmatrix}\right] \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}_1, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]\right), \tag{3.1}$$

then by Theorem 2.1, there exist a bounded invertible operator $M' \in \mathcal{B}(\mathcal{L}_1; \mathcal{L})$ and an operator $K' \in \mathcal{B}(\mathcal{X}, \mathcal{L})$ such that

$$\begin{bmatrix} A'_1 & B'_1 \\ (C'_1)_{\mathcal{Y}} & (D'_1)_{\mathcal{Y}} \\ 1_{\mathcal{X}} & 0 \\ (C'_1)_{\mathcal{U}} & (D'_1)_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} A' & B' \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \\ 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ K' & M' \end{bmatrix}. \tag{3.2}$$

The converse is also true.

In a standard i/s/o representation $\Sigma_{i/s/o}$ of a s/s system node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ we can interpret the generating subspace $V$ as the graph of the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ by reordering the components into the form

$$V = \left\{ \begin{bmatrix} \left[\begin{smallmatrix} \dot{x} \\ y \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \end{bmatrix} \;\middle|\; \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\}. \tag{3.3}$$

In the right affine i/s/o representation $\Sigma^r_{i/s/o}$ the subspace $V$ has the image representation (with the same reordering of the components)

$$V = \left\{ \begin{bmatrix} \begin{bmatrix} \dot{x} \\ y \end{bmatrix} \\ \begin{bmatrix} x \\ u \end{bmatrix} \end{bmatrix} \;\middle|\; \begin{matrix} \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A' & B' \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} x \\ \ell \end{bmatrix}, \\ \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} x \\ \ell \end{bmatrix}, \end{matrix} \quad \begin{bmatrix} x \\ \ell \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{L} \end{bmatrix} \right\}. \tag{3.4}$$

This coincides with the standard i/s/o representation (2.11) if $\mathcal{L} = \mathcal{U}$, $C'_{\mathcal{U}} = 0$, and $D'_{\mathcal{U}} = 1_{\mathcal{U}}$. If $D'_{\mathcal{U}}$ has a bounded inverse, then $\ell(n)$ may be solved in terms of $x(n)$ and $u(n)$ from the last equation, and we will arrive at a standard i/s/o representation of $\Sigma$. The condition that $D'_{\mathcal{U}}$ has a bounded inverse is actually *equivalent* to the admissibility of the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$; see Lemma I.5.9.

The operators $\left[\begin{smallmatrix} A' & B' \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{smallmatrix}\right]$ in the representation (3.4) are right coprime in the following (Bezout) sense: there exist two operators $P' \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right])$ and $Q' \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right])$ (called Bezout factors) such that the following identity (often called the right Bezout identity) holds:

$$P' \begin{bmatrix} A' & B' \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{bmatrix} + Q' \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix} = 1_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right]}. \tag{3.5}$$

Indeed, the operator

$$\begin{bmatrix} \left[\begin{smallmatrix} A' & B' \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{smallmatrix}\right] \end{bmatrix} \in \mathcal{B}\left( \begin{bmatrix} \mathcal{X} \\ \mathcal{L} \end{bmatrix}; \begin{bmatrix} \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \end{bmatrix} \right)$$

is injective since $D' = \left[\begin{smallmatrix} D'_{\mathcal{Y}} \\ D'_{\mathcal{U}} \end{smallmatrix}\right] \in \mathcal{B}(\mathcal{L}; \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$ is injective, and its range (equal to $V$) is closed. Consequently, it has a bounded left inverse

$$\begin{bmatrix} P' & Q' \end{bmatrix} \in \mathcal{B}\left( \begin{bmatrix} \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{L} \end{bmatrix} \right),$$

and (3.5) holds for the pair $(P', Q')$.

Let $(x(\cdot), u(\cdot), y(\cdot))$ be a trajectory of the right affine i/s/o representation $\Sigma^r_{i/s/o}$ given by (1.7) of the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$. Let $\hat{x}(\cdot)$, $\hat{u}(\cdot)$, and $\hat{y}(\cdot)$ be the formal power series of $x(\cdot)$, $u(\cdot)$, and $y(\cdot)$, respectively, and let $\hat{\ell}(\cdot)$ be the formal power series of the corresponding driving variable sequence $\ell(\cdot)$ (by Proposition I.3.2, $\ell(\cdot)$ is determined uniquely by ($x(\cdot)$, $u(\cdot)$, and $y(\cdot)$)). Then it follows from (1.8) that

$$\begin{aligned} \hat{x}(z) &= \mathfrak{A}'(z)x_0 + \mathfrak{B}'(z)\hat{\ell}(z), \\ \hat{y}(z) &= \mathfrak{C}'_{\mathcal{Y}}(z)x_0 + \mathfrak{D}'_{\mathcal{Y}}(z)\hat{\ell}(z), \\ \hat{u}(z) &= \mathfrak{C}'_{\mathcal{U}}(z)x_0 + \mathfrak{D}'_{\mathcal{U}}(z)\hat{\ell}(z), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} &\left[\begin{array}{c|c} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \hline \mathfrak{C}'_{\mathcal{Y}}(z) & \mathfrak{D}'_{\mathcal{Y}}(z) \\ \mathfrak{C}'_{\mathcal{U}}(z) & \mathfrak{D}'_{\mathcal{U}}(z) \end{array}\right] \\ &= \left[\begin{array}{c|c} (1_{\mathcal{X}} - zA')^{-1} & z(1_{\mathcal{X}} - zA')^{-1}B' \\ \hline C'_{\mathcal{Y}}(1_{\mathcal{X}} - zA')^{-1} & C'_{\mathcal{Y}}z(1_{\mathcal{X}} - zA')^{-1}B' + D'_{\mathcal{Y}} \\ C'_{\mathcal{U}}(1_{\mathcal{X}} - zA')^{-1} & C'_{\mathcal{U}}z(1_{\mathcal{X}} - zA')^{-1}B' + D'_{\mathcal{U}} \end{array}\right], \quad z \in \Lambda_{A'}. \end{aligned} \tag{3.7}$$

The above four block function is called the *right affine is/so transfer function* of the right affine i/s/o representation $\Sigma^r_{i/s/o}$ of $\Sigma$, corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. In particular, as a part of this is/so affine transfer function we find the *right affine i/o transfer function* $\mathfrak{D}' = \left[\begin{smallmatrix} \mathfrak{D}'_{\mathcal{Y}} \\ \mathfrak{D}'_{\mathcal{U}} \end{smallmatrix}\right]$ of $\Sigma$, corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. A right affine is/so transfer function of a given s/s system $\Sigma$ corresponding to an i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is not unique, since the right affine i/s/o representation is not unique. The connection between two different right affine is/so transfer functions can be obtained from the corresponding connection between two driving variable representations presented in (I.6.12), and it leads to a formula of the type

$$\begin{bmatrix} \mathfrak{D}'_{1,\mathcal{Y}}(z) \\ \mathfrak{D}_{1,\mathcal{U}'}(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{D}'_{\mathcal{Y}}(z) \\ \mathfrak{D}'_{\mathcal{U}}(z) \end{bmatrix} m'(z), \qquad z \in \Lambda_{A'_1} \cap \Lambda_{A'}, \tag{3.8}$$

for the connection between the right affine i/o transfer functions. Here $m'(z)$ is a holomorphic $\mathcal{B}(\mathcal{L}_1; \mathcal{L})$-valued function on $\Lambda_{A'_1} \cap \Lambda_{A'}$ with a (locally) bounded inverse (this function is given by $m'(z) = (1_{\mathcal{L}} - K'\mathfrak{B}'(z))^{-1}M'$; see formula (I.6.13)). In particular, if $\mathbb{D} \subset \Lambda_{A'_1} \cap \Lambda_{A'}$ (as is the case if both representations are stable), then (3.8) holds for all $z \in \mathbb{D}$.

Up to now the discussion has focused on right affine representations based on some arbitrary driving variable representation. Instead of using an arbitrary driving variable representation we may also use an i/s/o representation, interpreted as a driving variable representation, in which case some of our earlier formulas can be written in a more specific way.

Let $\Sigma^1_{i/s/o} = \left(\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1\right)$ be an i/s/o representation of $\Sigma = (V; \mathcal{X}, \mathcal{W})$ corresponding to some admissible decomposition $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ with is/so transfer function $\left[\begin{smallmatrix} \mathfrak{A}_1(z) & \mathfrak{B}_1(z) \\ \mathfrak{C}_1(z) & \mathfrak{D}_1(z) \end{smallmatrix}\right]$, and let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be another direct sum decomposition of $\mathcal{W}$ (not necessarily admissible). We define the is/so decompositions $\Theta$ and $\widetilde{\Theta}$ of the identity with respect to these two decompositions of $\mathcal{W}$ in the usual way, i.e.,

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} P^{\mathcal{U}}_{\mathcal{Y}}|_{\mathcal{Y}_1} & P^{\mathcal{U}}_{\mathcal{Y}}|_{\mathcal{U}_1} \\ P^{\mathcal{Y}}_{\mathcal{U}}|_{\mathcal{Y}_1} & P^{\mathcal{Y}}_{\mathcal{U}}|_{\mathcal{U}_1} \end{bmatrix}, \tag{3.9}$$

$$\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} = \begin{bmatrix} P^{\mathcal{U}_1}_{\mathcal{Y}_1}|_{\mathcal{Y}} & P^{\mathcal{U}_1}_{\mathcal{Y}_1}|_{\mathcal{U}} \\ P^{\mathcal{Y}_1}_{\mathcal{U}_1}|_{\mathcal{Y}} & P^{\mathcal{Y}_1}_{\mathcal{U}_1}|_{\mathcal{U}} \end{bmatrix}. \tag{3.10}$$

By interpreting $\Sigma^1_{i/s/o}$ as a driving variable representation of $\Sigma$ with driving variable space $\mathcal{L} = \mathcal{U}_1$ we can make the following substitution in the right affine formulas:

$$\begin{aligned} \begin{bmatrix} A' & B' \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{bmatrix} &= \begin{bmatrix} A_1 & B_1 \\ \Theta_{11} C_1 & \Theta_{11} D_1 + \Theta_{12} \end{bmatrix}, \\ \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21} C_1 & \Theta_{21} D_1 + \Theta_{22} \end{bmatrix}. \end{aligned} \tag{3.11}$$

One particular choice of the operators $P'$ and $Q'$ in (3.5) leads to the (right Bezout) identity

$$\begin{aligned} &\begin{bmatrix} 0 & 0 \\ 0 & \widetilde{\Theta}_{21} \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ \Theta_{11} C_1 & \Theta_{11} D_1 + \Theta_{12} \end{bmatrix} \\ &\qquad + \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & \widetilde{\Theta}_{22} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21} C_1 & \Theta_{21} D_1 + \Theta_{22} \end{bmatrix} = 1_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U}_1 \end{smallmatrix}\right]}, \end{aligned} \tag{3.12}$$

The right affine is/so transfer function (3.7) is now given by

$$\left[\begin{array}{c|c} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \hline \mathfrak{C}'_{\mathcal{Y}}(z) & \mathfrak{D}'_{\mathcal{Y}}(z) \\ \mathfrak{C}'_{\mathcal{U}}(z) & \mathfrak{D}'_{\mathcal{U}}(z) \end{array}\right] = \left[\begin{array}{c|c} \mathfrak{A}_1(z) & \mathfrak{B}_1(z) \\ \hline \Theta_{11}\mathfrak{C}_1(z) & \Theta_{11}\mathfrak{D}_1(z) + \Theta_{12} \\ \Theta_{21}\mathfrak{C}_1(z) & \Theta_{21}\mathfrak{D}_1(z) + \Theta_{22} \end{array}\right], \quad z \in \Lambda_{A_1}. \tag{3.13}$$

An identity similar to (3.12) is also valid for the right affine is/so transfer function (with the same Bezout factors), namely

$$\begin{bmatrix} 0 & 0 \\ 0 & \widetilde{\Theta}_{21} \end{bmatrix} \begin{bmatrix} \mathfrak{A}_1(z) & \mathfrak{B}_1(z) \\ \Theta_{11}\mathfrak{C}_1(z) & \Theta_{11}\mathfrak{D}_1(z) + \Theta_{12} \end{bmatrix} + \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & \widetilde{\Theta}_{22} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21}\mathfrak{C}_1(z) & \Theta_{21}\mathfrak{D}_1(z) + \Theta_{22} \end{bmatrix} = 1_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U}_1 \end{smallmatrix}\right]}, \quad z \in \Lambda_{A_1}. \tag{3.14}$$

From this identity we can extract the corresponding (right Bezout) identity

$$\widetilde{\Theta}_{21}\mathfrak{D}'_{\mathcal{Y}}(z) + \widetilde{\Theta}_{22}\mathfrak{D}'_{\mathcal{U}}(z) = 1_{\mathcal{U}}, \qquad z \in \Lambda_{A_1}, \tag{3.15}$$

for the right affine i/o transfer function

$$\begin{bmatrix} \mathfrak{D}'_{\mathcal{Y}}(z) \\ \mathfrak{D}'_{\mathcal{U}}(z) \end{bmatrix} = \begin{bmatrix} \Theta_{11}\mathfrak{D}_1(z) + \Theta_{12} \\ \Theta_{21}\mathfrak{D}_1(z) + \Theta_{22} \end{bmatrix}, \qquad z \in \Lambda_{A_1}. \tag{3.16}$$

Two holomorphic functions $a$ and $b$ on an open set $\Omega$ with values in $\mathcal{B}(\mathcal{L};\mathcal{U})$ and $\mathcal{B}(\mathcal{L};\mathcal{Y})$, respectively, are called *right (Bezout) coprime* (in the space of analytic operator-valued functions on $\Omega$) if there exists a pair of holomorphic functions $p$ and $q$ on $\Omega$ with values in $\mathcal{B}(\mathcal{U};\mathcal{L})$ and $\mathcal{B}(\mathcal{Y};\mathcal{L})$, respectively (called the Bezout factors) such that

$$p(z)a(z) + q(z)b(z) = 1_{\mathcal{L}}, \qquad z \in \Omega. \tag{3.17}$$

Thus, by (3.15) and (3.16), the holomorphic functions $\mathfrak{D}'_{\mathcal{Y}}$ and $\mathfrak{D}'_{\mathcal{U}}$ on $\Lambda_{A_1}$ are right coprime (and it is even possible to choose the Bezout factors to be constants).

We remark that the Bezout factors $p(z)$ and $q(z)$ in (3.17) determine the pair $\left[\begin{smallmatrix} a(z) \\ b(z) \end{smallmatrix}\right]$ uniquely on $\Omega$ within the class of all pairs of the type $\left[\begin{smallmatrix} a(z) \\ b(z) \end{smallmatrix}\right] m(z)$, where $m(z)$ is a holomorphic function on $\Omega$ with values in $\mathcal{B}(\mathcal{L})$. This is true because (3.17) gives

$$p(z)[a(z)m(z)] + q(z)[b(z)m(z)] = m(z), \qquad z \in \Omega,$$

and here the right-hand side is the identity only if $m(z) = 1_{\mathcal{L}}$, $z \in \mathbb{D}$. In the same sense the Bezout factors $\widetilde{\Theta}_{21}$ and $\widetilde{\Theta}_{22}$ in (3.15) determine $\mathfrak{D}'_{\mathcal{Y}}(z)$ and $\mathfrak{D}'_{\mathcal{U}}(z)$ uniquely on $\Lambda_{A_1}$, and the Bezout factors $\left[\begin{smallmatrix} 0 & 0 \\ 0 & \widetilde{\Theta}_{21} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \widetilde{\Theta}_{22} \end{smallmatrix}\right]$ in (3.14) determine the corresponding pair of right affine is/so transfer functions on $\Lambda_{A_1}$.

In the case where the s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ is stabilizable we can say more about the right affine i/s/o representation of $\Sigma$. Indeed, stabilizability is (by definition) equivalent to the existence of a stable driving variable representation of $\Sigma$. For each decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ we get from this particular driving variable representation a stable right affine representation $\Sigma^r_{i/s/o}$ of $\Sigma$ given in the form (1.7). According to Lemma I.9.2, its main operator and the is/so transfer function has the following properties:

1) $\|(A')^n\| \le C$ for some constant $C > 0$ all $n \in \mathbb{Z}^+$; in particular, $\mathbb{D} \in \Lambda_{A'}$.

2) $(\mathfrak{B}')^{\tilde{}} \in H^2(\mathbb{D}; \mathcal{X}, \mathcal{U})$,[4] where $(\mathfrak{B}')^{\tilde{}}(z) = \mathfrak{B}'(\overline{z})^*$.

3) $\begin{bmatrix} \mathfrak{C}'_{\mathcal{Y}} \\ \mathfrak{C}'_{\mathcal{Y}} \end{bmatrix} \in H^2(\mathbb{D}; \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$.

4) $\begin{bmatrix} \mathfrak{D}'_{\mathcal{Y}} \\ \mathfrak{D}'_{\mathcal{Y}} \end{bmatrix} \in H^\infty(\mathbb{D}; \mathcal{U}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$.

In particular, according to condition 4), the right affine i/o transfer function of $\Sigma$ corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is holomorphic and bounded in $\mathbb{D}$.

If we impose the even stronger condition of LFT-stabilizability on the system, then we can say more (recall, in particular, that every passive system is LFT-stabilizable). In this case $\Sigma$ has a stable i/s/o representation $\Sigma^1_{i/s/o} := \left(\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1\right)$. This representation can be interpreted as a stable driving variable representation of $\Sigma$, and from this representation we get the right affine i/s/o representation described in (3.9)–(3.16). In particular, we now see from (3.15) that the pair $(\mathfrak{D}'_{\mathcal{Y}}, \mathfrak{D}'_{\mathcal{U}})$ is right coprime in $H^\infty(\mathbb{D})$ (even with constant Bezout factors).

We will end this section by reformulating Theorem III.3.6 on the positivity properties of the i/o transfer function of a driving variable represenation of a forward passive s/s system by reinterpreting this transfer function as a right affine i/o transfer function of $\Sigma$. We treat two cases, namely the transmission case where the decomposition $\mathcal{W} = -\mathcal{Y} \, [\dotplus] \, \mathcal{U}$ is orthogonal, and the impedance case where the decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ is Lagrangean.

**Proposition 3.1.** *Let $\Sigma^r_{i/s/o}$ be a right affine i/s/o representation with main operator $A'$ and i/o transfer function $\begin{bmatrix} \mathfrak{D}'_{\mathcal{Y}}(z) \\ \mathfrak{D}'_{\mathcal{U}}(z) \end{bmatrix}$ of a forward passive s/s system*

[4] Here we denote by $H^2(\mathbb{D}; \mathcal{L}_1, \mathcal{L}_2)$ the Hardy class of holomorphic $\mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$-valued functions $f(\cdot)$ in $\mathbb{D}$ such that $f(\cdot)\ell \in H^2(\mathcal{L}_2)$ for every $\ell \in \mathcal{L}_1$). By the uniform boundedness principle, there exists a finite constant $C$ such that $\|f(\cdot)\ell\|_{H^2(\mathbb{D};\mathcal{L}_2)} \le C\|\ell\|_{\mathcal{L}_1}$ for all $\ell \in \mathcal{L}_1$.

$\Sigma = (V; \mathcal{X}, \mathcal{W})$, *corresponding to an orthogonal decomposition* $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$ *of the signal space.*[5] *Then the kernel*

$$K_{\mathfrak{D}'_\mathcal{Y},\mathfrak{D}'_\mathcal{U}}(z,\zeta) = \frac{\mathfrak{D}'_\mathcal{U}(z)^*\mathfrak{D}'_\mathcal{U}(\zeta) - \mathfrak{D}'_\mathcal{Y}(z)^*\mathfrak{D}'_\mathcal{Y}(\zeta)}{1-\overline{z}\zeta}, \qquad z,\zeta \in \Omega'_+ \tag{3.18}$$

*is positive definite on the set* $\Omega'_+ \times \Omega'_+$, *where* $\Omega'_+ = \Lambda_{A'} \cap \mathbb{D}$. *If* $\Sigma$ *is passive, then we may choose* $\Sigma^r_{i/s/o}$ *to be stable, in which case* $\mathbb{D} \subset \Lambda_{A'}$, $\Omega'_+ = \mathbb{D}$, $\mathfrak{D}'_\mathcal{Y} \in H^\infty(\mathbb{D}; \mathcal{L}, \mathcal{Y})$ *and* $\mathfrak{D}'_\mathcal{U} \in H^\infty(\mathbb{D}; \mathcal{L}, \mathcal{U})$. *If* $\Sigma^r_{i/s/o}$ *arises from a scattering i/s/o representation, then the functions* $\mathfrak{D}'_\mathcal{Y}$ *and* $\mathfrak{D}'_\mathcal{U}$ *are even right coprime in* $H^\infty(\mathbb{D})$.

*Proof.* By Theorem III.3.6, the kernel

$$K_{\mathfrak{D}'}(z,\zeta) = \frac{\mathfrak{D}'(z)^*\mathfrak{D}'(\zeta)}{1-\overline{z}\zeta}, \qquad z,\zeta, \in \Omega'_+ \tag{3.19}$$

is positive definite on $\Omega'_+ \times \Omega'_+$, where $\Omega'_+ = \Lambda_{A'} \cap \mathbb{D}$. For every $\ell$, $\ell' \in \mathcal{L}$ (where $\mathcal{L}$ is the driving variable space of $\Sigma^r_{i/s/o}$) we have

$$\begin{aligned}(\mathfrak{D}'(z)^*\mathfrak{D}'(\zeta)\ell, \ell')_\mathcal{L} &= [\mathfrak{D}'(\zeta)\ell, \mathfrak{D}'(z)\ell']_\mathcal{W} \\ &= [\mathfrak{D}'_\mathcal{U}(\zeta)\ell, \mathfrak{D}'_\mathcal{U}(z)\ell']_\mathcal{U} - [\mathfrak{D}'_\mathcal{Y}(\zeta)\ell, \mathfrak{D}'_\mathcal{Y}(z)\ell']_\mathcal{Y} \\ &= ([\mathfrak{D}'_\mathcal{U}(z)^*\mathfrak{D}'_\mathcal{U}(\zeta) - \mathfrak{D}'_\mathcal{Y}(z)^*\mathfrak{D}'_\mathcal{Y}(\zeta)]\ell, \ell')_\mathcal{L}.\end{aligned}$$

Thus,

$$\mathfrak{D}'(z)^*\mathfrak{D}'(\zeta) = \mathfrak{D}'_\mathcal{U}(z)^*\mathfrak{D}'_\mathcal{U}(\zeta) - \mathfrak{D}'_\mathcal{Y}(z)^*\mathfrak{D}'_\mathcal{Y}(\zeta), \qquad z,\zeta \in \Omega'_+, \tag{3.20}$$

and so $K_{\mathfrak{D}'_\mathcal{Y},\mathfrak{D}'_\mathcal{U}}(z,\zeta)$ is positive definite since $K_{\mathfrak{D}'}(z,\zeta)$ is positive definite. The two final claims (about the stable cases) follow from the stability discussion earlier in this section, and from the fact that every scattering representation of a passive s/s system is stable. □

**Proposition 3.2.** *Let* $\Sigma^r_{i/s/o}$ *be a right affine i/s/o representation with main operator* $A'$ *and i/o transfer function* $\begin{bmatrix} \mathfrak{D}'_\mathcal{F}(z) \\ \mathfrak{D}'_\mathcal{E}(z) \end{bmatrix}$ *of a forward passive system* $\Sigma = (V; \mathcal{X}, \mathcal{W})$, *corresponding to a Lagrangean decomposition* $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dot{+}} \mathcal{E}$ *of the signal space. Then the kernel*

$$K_{\mathfrak{D}'_\mathcal{F},\mathfrak{D}'_\mathcal{E}}(z,\zeta) = \frac{\mathfrak{D}'_\mathcal{E}(z)^*\Psi^*\mathfrak{D}'_\mathcal{F}(\zeta) + \mathfrak{D}'_\mathcal{F}(z)^*\Psi\mathfrak{D}'_\mathcal{E}(\zeta)}{1-\overline{z}\zeta}, \qquad z,\zeta \in \Omega'_+ \tag{3.21}$$

[5] Recall that this transfer function is defined and holomorphic on $\Lambda_{A'}$.

*is positive definite on the set* $\Omega'_+ \times \Omega'_+$*, where* $\Omega'_+ = \Lambda_{A'} \cap \mathbb{D}$*. If* $\Sigma$ *is passive, then we may choose* $\Sigma^r_{i/s/o}$ *to be stable, in which case* $\mathbb{D} \subset \Lambda_{A'}$*,* $\Omega'_+ = \mathbb{D}$*,* $\mathfrak{D}'_{\mathcal{F}} \in H^\infty(\mathbb{D}; \mathcal{L}, \mathcal{F})$ *and* $\mathfrak{D}'_{\mathcal{E}} \in H^\infty(\mathbb{D}; \mathcal{L}, \mathcal{E})$*. If* $\Sigma^r_{i/s/o}$ *arises from a scattering i/s/o representation, then the functions* $\mathfrak{D}'_{\mathcal{F}}$ *and* $\mathfrak{D}'_{\mathcal{E}}$ *are even right coprime in* $H^\infty(\mathbb{D})$*.*

*Proof.* As in the proof of Proposition 3.1 we observe that the kernel $K_{\mathfrak{D}'}(z,\zeta)$ in (3.19) is positive definite, where this time $\mathfrak{D}'(z) = \left[\begin{smallmatrix} \mathfrak{D}'_{\mathcal{F}}(z) \\ \mathfrak{D}'_{\mathcal{E}}(z) \end{smallmatrix}\right]$. For all $\ell, \ell' \in \mathcal{L}$ we have by (2.16)

$$\begin{aligned}(\mathfrak{D}'(z)^*\mathfrak{D}'(\zeta)\ell, \ell')_{\mathcal{L}} &= [\mathfrak{D}'(\zeta)\ell, \mathfrak{D}'(z)\ell']_{\mathcal{W}} \\ &= [\mathfrak{D}'_{\mathcal{F}}(\zeta)\ell, \psi\mathfrak{D}'_{\mathcal{E}}(z)\ell']_{\mathcal{F}} + [\mathfrak{D}'_{\mathcal{E}}(\zeta)\ell, \psi^*\mathfrak{D}'_{\mathcal{F}}(z)\ell']_{\mathcal{E}} \\ &= ([\mathfrak{D}'_{\mathcal{E}}(z)^*\psi^*\mathfrak{D}'_{\mathcal{F}}(\zeta) + \mathfrak{D}'_{\mathcal{F}}(z)^*\psi\mathfrak{D}'_{\mathcal{E}}(\zeta)]\ell, \ell')_{\mathcal{L}}.\end{aligned}$$

Thus,

$$\mathfrak{D}'(z)^*\mathfrak{D}'(\zeta) = \mathfrak{D}'_{\mathcal{E}}(z)^*\Psi^*\mathfrak{D}'_{\mathcal{F}}(\zeta) + \mathfrak{D}'_{\mathcal{F}}(z)^*\Psi\mathfrak{D}'_{\mathcal{E}}(\zeta), \quad z, \zeta \in \Omega'_+, \tag{3.22}$$

which means that the kernel $K_{\mathfrak{D}'}(z,\zeta)$ now becomes the kernel given in (3.21). □

# 4 Left Affine Representations

The notions and results presented above for right affine i/s/o representations of a s/s system $\Sigma$ have a natural left counterpart, where the driving variable representations used in the right representations are replaced by output nulling representations. We have already defined what we mean by an output nulling representation in Section 2, and what we mean by a left affine i/s/o repersentation in Section 1. The correspondence between these is the one induced by the i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$, so that $(x(\cdot), w(\cdot), 0)$ is a trajectory of the output nulling system $\Sigma_{s/s/on}$ (with vanishing error vector) if and only if $(x(\cdot), u(\cdot), y(\cdot))$ is a trajectory of the left affine i/s/o representation $\Sigma^l_{i/s/o}$, where

$$u(n) = P^{\mathcal{Y}}_{\mathcal{U}} w(n), \quad y(n) = P^{\mathcal{U}}_{\mathcal{Y}} w(n), \quad n \in \mathbb{Z}^+.$$

Just as in the right affine case, a left affine i/s/o representation of a s/s system $\Sigma$ corresponding to a given i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is not unique. By Theorem 2.2, given the representation $\Sigma^l_{i/s/o}$ in (1.12) and another left affine i/s/o representation

$$\Sigma^{l,1}_{i/s/o} := \left( \begin{bmatrix} A''_1 & (B''_1)_{\mathcal{Y}} & (B''_1)_{\mathcal{U}} \\ C''_1 & (D_1)''_{\mathcal{Y}} & (D_1)''_{\mathcal{U}} \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \mathcal{K}_1 \right), \tag{4.1}$$

corresponding to the i/o decomposition $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$, the latter can be obtained from the former through the formula

$$\begin{bmatrix} -1_{\mathcal{X}} & (B_1'')_{\mathcal{Y}} & A_1'' & (B_1'')_{\mathcal{U}} \\ 0 & (D_1)_{\mathcal{Y}}'' & C_1'' & (D_1)_{\mathcal{U}}'' \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & K'' \\ 0 & M'' \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & B_{\mathcal{Y}}'' & A'' & B_{\mathcal{U}}'' \\ 0 & D_{\mathcal{Y}}'' & C'' & D_{\mathcal{U}}'' \end{bmatrix}. \tag{4.2}$$

where $M''\in\mathcal{B}(\mathcal{K};\mathcal{K}_1)$ is boundedly invertible and $K''\in\mathcal{B}(\mathcal{K},\mathcal{X})$. This formula gives a complete parameterization of all left affine i/s/o representations of $\Sigma$ corresponding to the decomposition $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$.

In the left affine i/s/o representation $\Sigma_{i/s/o}^l$ the subspace $V$ has the kernel representation (with the same reordering of the components as in (3.4))

$$V=\left\{\begin{bmatrix}\left[\begin{smallmatrix}\dot x\\ y\end{smallmatrix}\right]\\ \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]\end{bmatrix} \;\middle|\; \begin{bmatrix} -1_{\mathcal{X}} & B_{\mathcal{Y}}'' \\ 0 & D_{\mathcal{Y}}'' \end{bmatrix}\begin{bmatrix}\dot x\\ y\end{bmatrix} + \begin{bmatrix} A'' & B_{\mathcal{U}}'' \\ C'' & D_{\mathcal{U}}'' \end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}\right\}. \tag{4.3}$$

In this representation the operators $\left[\begin{smallmatrix} -1_{\mathcal{X}} & B_{\mathcal{Y}}'' \\ 0 & D_{\mathcal{Y}}'' \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A'' & B_{\mathcal{U}}'' \\ C'' & D_{\mathcal{U}}'' \end{smallmatrix}\right]$ are left coprime in (Bezout) sense that

$$\begin{bmatrix} -1_{\mathcal{X}} & B_{\mathcal{Y}}'' \\ 0 & D_{\mathcal{Y}}'' \end{bmatrix} P'' + \begin{bmatrix} A'' & B_{\mathcal{U}}'' \\ C'' & D_{\mathcal{U}}'' \end{bmatrix} Q'' = 1_{\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{K}\end{smallmatrix}\right]}. \tag{4.4}$$

for some operators $P''\in\mathcal{B}(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{K}\end{smallmatrix}\right];\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right])$ and $Q''\in\mathcal{B}(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{K}\end{smallmatrix}\right];\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right])$. Indeed, the operator

$$\left[\begin{array}{cc|cc} -1_{\mathcal{X}} & B_{\mathcal{Y}}'' & A'' & B_{\mathcal{U}}'' \\ 0 & D_{\mathcal{Y}}'' & C'' & D_{\mathcal{U}}'' \end{array}\right] \in \mathcal{B}\left(\begin{bmatrix}\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]\\ \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]\end{bmatrix};\begin{bmatrix}\mathcal{X}\\ \mathcal{K}\end{bmatrix}\right)$$

is surjective and hence has a bounded right inverse

$$\begin{bmatrix}P''\\ Q''\end{bmatrix}\in\mathcal{B}\left(\begin{bmatrix}\mathcal{X}\\ \mathcal{K}\end{bmatrix};\begin{bmatrix}\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]\\ \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]\end{bmatrix}\right),$$

and (4.4) holds for this pair $(P'',Q'')$.

If $(x(\cdot),u(\cdot),y(\cdot))$ is a trajectory of the left affine i/s/o representation $\Sigma_{i/s/o}^l$ given in (1.12) of the s/s system $\Sigma=(V;\mathcal{X},\mathcal{W})$, and if $\hat x(\cdot)$, $\hat u(\cdot)$, and $\hat y(\cdot)$ be the formal power series of ($x(\cdot)$, $u(\cdot)$, and $y(\cdot)$), respectively, then it follows from (1.13) that

$$\begin{aligned} \hat x(z) &= \mathfrak{A}''(z)x_0+\mathfrak{B}_{\mathcal{Y}}''(z)\hat y(z)+\mathfrak{B}_{\mathcal{U}}''(z)\hat u(z),\\ 0 &= \mathfrak{C}''(z)x_0+\mathfrak{D}_{\mathcal{Y}}''(z)\hat y(z)+\mathfrak{D}_{\mathcal{U}}''(z)\hat u(z), \end{aligned} \tag{4.5}$$

where

$$\left[\begin{array}{c|cc} \mathfrak{A}''(z) & \mathfrak{B}_{\mathcal{Y}}''(z) & \mathfrak{B}_{\mathcal{U}}''(z) \\ \hline \mathfrak{C}''(z) & \mathfrak{D}_{\mathcal{Y}}''(z) & \mathfrak{D}_{\mathcal{U}}''(z) \end{array}\right] = \left[\begin{array}{c|cc} (1_{\mathcal{X}}-zA'')^{-1} & z(1_{\mathcal{X}}-zA'')^{-1}B_{\mathcal{Y}}'' & z(1_{\mathcal{X}}-zA'')^{-1}B_{\mathcal{U}}'' \\ \hline C''(1_{\mathcal{X}}-zA'')^{-1} & C''z(1_{\mathcal{X}}-zA'')^{-1}B_{\mathcal{Y}}''+D_{\mathcal{Y}}'' & C''z(1_{\mathcal{X}}-zA'')^{-1}B_{\mathcal{U}}''+D_{\mathcal{U}}'' \end{array}\right],\quad z\in\Lambda_{A''}. \tag{4.6}$$

The above four block function is called the *left affine is/so transfer function* of the left affine i/s/o representation of $\Sigma$ corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$. In particular, as a part of this left affine is/so transfer function we find the *left affine i/o transfer function* $\mathcal{D}'' = \begin{bmatrix} \mathfrak{D}''_{\mathcal{Y}} & \mathfrak{D}''_{\mathcal{U}} \end{bmatrix}$ corresponding to this decomposition of $\mathcal{W}$. Any two left affine i/o transfer functions differ from each other by an invertible factor to the left, i.e.,

$$\begin{bmatrix} \mathfrak{D}''_{1,\mathcal{Y}}(z) & \mathfrak{D}''_{1,\mathcal{U}}(z) \end{bmatrix} = m''(z) \begin{bmatrix} \mathfrak{D}''_{\mathcal{Y}}(z) & \mathfrak{D}''_{\mathcal{U}}(z) \end{bmatrix}, \qquad z \in \Lambda_{A''_1} \cap \Lambda_{A''}, \tag{4.7}$$

where $m''(z)$ is a holomorphic $\mathcal{B}(\mathcal{K};\mathcal{K}_1)$-valued function on $\Lambda_{A''_1} \cap \Lambda_{A''}$ with a (locally) bounded inverse; see formula (I.6.16). In particular, if both representations are stable, then (4.7) holds for all $z \in \mathbb{D}$.

By interpreting an i/s/o representation $\Sigma^1_{i/s/o} = \left( \left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1 \right)$ of $\Sigma$ as an output nulling representation of $\Sigma$ we can make a substitution analogous to the one in (3.11) in the left affine formulas, namely

$$\begin{aligned} \begin{bmatrix} -1_{\mathcal{X}} & B''_{\mathcal{Y}} \\ 0 & D''_{\mathcal{Y}} \end{bmatrix} &= \begin{bmatrix} -1_{\mathcal{X}} & B_1\widetilde{\Theta}_{21} \\ 0 & -\widetilde{\Theta}_{11} + D_1\widetilde{\Theta}_{21} \end{bmatrix}, \\ \begin{bmatrix} A'' & B''_{\mathcal{U}} \\ C'' & D''_{\mathcal{U}} \end{bmatrix} &= \begin{bmatrix} A_1 & B_1\widetilde{\Theta}_{22} \\ C_1 & -\widetilde{\Theta}_{12} + D_1\widetilde{\Theta}_{22} \end{bmatrix}. \end{aligned} \tag{4.8}$$

One particular choice of the operators $P''$ and $Q''$ in (4.4) leads to the (left Bezout) identity, analogous to (3.12),

$$\begin{aligned} &\begin{bmatrix} -1_{\mathcal{X}} & B_1\widetilde{\Theta}_{21} \\ 0 & -\widetilde{\Theta}_{11} + D_1\widetilde{\Theta}_{21} \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & 0 \\ 0 & -\Theta_{11} \end{bmatrix} \\ &\qquad + \begin{bmatrix} A_1 & B_1\widetilde{\Theta}_{22} \\ C_1 & -\widetilde{\Theta}_{12} + D_1\widetilde{\Theta}_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -\Theta_{21} \end{bmatrix} = 1_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y}_1 \end{smallmatrix}\right]}, \end{aligned} \tag{4.9}$$

The left affine is/so transfer function (4.6) is now given by

$$\begin{aligned} &\left[\begin{array}{c|cc} \mathfrak{A}''(z) & \mathfrak{B}''_{\mathcal{Y}}(z) & \mathfrak{B}''_{\mathcal{U}}(z) \\ \hline \mathfrak{C}''(z) & \mathfrak{D}''_{\mathcal{Y}}(z) & \mathfrak{D}''_{\mathcal{U}}(z) \end{array}\right] \\ &= \left[\begin{array}{c|cc} \mathfrak{A}_1(z) & \mathfrak{B}_1(z)\widetilde{\Theta}_{21} & \mathfrak{B}_1(z)\widetilde{\Theta}_{22} \\ \hline \mathfrak{C}_1(z) & -\widetilde{\Theta}_{11} + \mathfrak{D}_1(z)\widetilde{\Theta}_{21} & -\widetilde{\Theta}_{12} + \mathfrak{D}_1(z)\widetilde{\Theta}_{22} \end{array}\right], \quad z \in \Lambda_{A_1}, \end{aligned} \tag{4.10}$$

the left analogue of (3.14) is

$$\begin{aligned} &\begin{bmatrix} -1_{\mathcal{X}} & \mathfrak{B}_1(z)\widetilde{\Theta}_{21} \\ 0 & -\widetilde{\Theta}_{11} + \mathfrak{D}_1(z)\widetilde{\Theta}_{21} \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & 0 \\ 0 & -\Theta_{11} \end{bmatrix} \\ &\qquad + \begin{bmatrix} \mathfrak{A}_1(z) & \mathfrak{B}_1(z)\widetilde{\Theta}_{22} \\ \mathfrak{C}_1(z) & -\widetilde{\Theta}_{12} + \mathfrak{D}_1(z)\widetilde{\Theta}_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -\Theta_{21} \end{bmatrix} = 1_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y}_1 \end{smallmatrix}\right]}, \quad z \in \Lambda_{A_1}, \end{aligned} \tag{4.11}$$

the left analogue of (3.15) is

$$-\mathfrak{D}''_{\mathcal{Y}}(z)\Theta_{11} - \mathfrak{D}''_{\mathcal{U}}(z)\Theta_{21} = 1_{\mathcal{U}}, \qquad z \in \Lambda_{A_1}, \tag{4.12}$$

where the left affine i/o transfer function is given by

$$\begin{bmatrix}\mathfrak{D}''_{\mathcal{Y}}(z) & \mathfrak{D}''_{\mathcal{U}}(z)\end{bmatrix} = \begin{bmatrix}-\widetilde{\Theta}_{11} + \mathfrak{D}_1(z)\widetilde{\Theta}_{21} & -\widetilde{\Theta}_{12} + \mathfrak{D}_1(z)\widetilde{\Theta}_{22}\end{bmatrix}, \; z \in \Lambda_{A_1}. \tag{4.13}$$

By definition, the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is detectable if it has a stable output nulling representation. By decomposing this representation in accordance with a given i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ we get a stable left affine i/s/o affine i/s/o representation $\Sigma^{l}_{i/s/o}$ given by (1.12). The same comments that we made for the right affine case apply to this case, too. In particular, we get the left analogues of conditions 1)–4) listed at the end of Section 3. Among others we conclude that the left affine i/o transfer function of $\Sigma$, corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$, obtained from a stable left affine i/s/o representation of $\Sigma$ is holomorphic and bounded in $\mathbb{D}$.

The final comment on the LFT-stabilizable case in Section 3 is also valid in the left affine setting, with right coprimeness replaced by left coprimeness in $H^\infty(\mathbb{D})$.

**Remark 4.1.** It is not difficult to see that for each i/s/o representation $\Sigma^{1}_{i/s/o} = \left(\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1\right)$ of $\Sigma = (V; \mathcal{X}, \mathcal{W})$ the two operators $G$ and $\widetilde{G}$ given by (with $\Theta$ and $\widetilde{\Theta}$ defined as in (3.9) and (3.10))

$$\begin{aligned} G &= \left[\begin{array}{cc|cc} -1_{\mathcal{X}} & 0 & A_1 & B_1 \\ 0 & -\Theta_{11} & \Theta_{11}C_1 & \Theta_{11}D_1 + \Theta_{12} \\ \hline 0 & 0 & 1_{\mathcal{X}} & 0 \\ 0 & -\Theta_{21} & \Theta_{21}C_1 & \Theta_{21}D_1 + \Theta_{22} \end{array}\right], \\ \widetilde{G} &= \left[\begin{array}{cc|cc} -1_{\mathcal{X}} & B_1\widetilde{\Theta}_{21} & A_1 & B_1\widetilde{\Theta}_{22} \\ 0 & -\widetilde{\Theta}_{11} + D_1\widetilde{\Theta}_{21} & C_1 & -\widetilde{\Theta}_{12} + D_1\widetilde{\Theta}_{22} \\ \hline 0 & 0 & 1_{\mathcal{X}} & 0 \\ 0 & \widetilde{\Theta}_{21} & 0 & \widetilde{\Theta}_{22} \end{array}\right], \end{aligned} \tag{4.14}$$

are inverses of each other. This implies both (3.12) and (4.9) (plus a number of additional identities). In particular, this means that the two pairs of operators appearing in (3.11) and (4.8) are not just right or left coprime, respectively, but they are actually bi-coprime with respect to the same Bezout factors that appear in (3.12) and (4.9). A similar remark applies to the corresponding Bezout identities (3.14) and (4.11) for the affine transfer functions: Define $\Gamma(z)$ and $\widetilde{\Gamma}(z)$ by

$$\Gamma(z) = \left[\begin{array}{cc|cc} -1_{\mathcal{X}} & 0 & \mathfrak{A}_1(z) & \mathfrak{B}_1(z) \\ 0 & -\Theta_{11} & \Theta_{11}\mathfrak{C}_1(z) & \Theta_{11}\mathfrak{D}_1(z)+\Theta_{12} \\ \hline 0 & 0 & 1_{\mathcal{X}} & 0 \\ 0 & -\Theta_{21} & \Theta_{21}\mathfrak{C}_1(z) & \Theta_{21}\mathfrak{D}_1(z)+\Theta_{22} \end{array}\right]$$

$$\widetilde{\Gamma}(z) = \left[\begin{array}{cc|cc} -1_{\mathcal{X}} & \mathfrak{B}_1(z)\widetilde{\Theta}_{21} & \mathfrak{A}_1(z) & \mathfrak{B}_1(z)\widetilde{\Theta}_{22} \\ 0 & -\widetilde{\Theta}_{11}+\mathfrak{D}_1(z)\widetilde{\Theta}_{21} & \mathfrak{C}_1(z) & -\widetilde{\Theta}_{12}+\mathfrak{D}_1(z)\widetilde{\Theta}_{22} \\ \hline 0 & 0 & 1_{\mathcal{X}} & 0 \\ 0 & \widetilde{\Theta}_{21} & 0 & \widetilde{\Theta}_{22} \end{array}\right]. \tag{4.15}$$

Then $\Gamma(z)$ and $\widetilde{\Gamma}(z)$ are inverses of each other.

We will end this section by reformulating Theorem III.3.7 on the positivity properties of the i/o transfer function of an output nulling represenation of a backward passive s/s system by reinterpreting the transfer function as a left affine i/o transfer function. We again discuss only the transmission and impedance settings.

**Proposition 4.2.** *Let $\Sigma^l_{i/s/o}$ be a left affine i/s/o representation with main operator $A''$ and left affine i/o transfer function $\begin{bmatrix}\mathfrak{D}''_{\mathcal{Y}}(z) & \mathfrak{D}''_{\mathcal{U}}(z)\end{bmatrix}$ of a backward passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$, corresponding to an orthogonal decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of the signal space. Then the kernel*

$$K_{\widetilde{\mathfrak{D}''_{\mathcal{Y}},\mathfrak{D}''_{\mathcal{U}}}}(z,\zeta) = \frac{\mathfrak{D}''_{\mathcal{U}}(z)\mathfrak{D}''_{\mathcal{U}}(\zeta)^* - \mathfrak{D}''_{\mathcal{Y}}(z)\mathfrak{D}''_{\mathcal{Y}}(\zeta)^*}{1-z\overline{\zeta}}, \qquad z,\zeta\in\Omega''_+ \tag{4.16}$$

*is positive definite on the set $\Omega''_+\times\Omega''_+$, where $\Omega''_+ = \Lambda_{A''}\cap\mathbb{D}$. If $\Sigma$ is passive, then we may choose the output nulling variable representation $\Sigma_{s/s/on}$ to be stable, in which case $\mathbb{D}\subset\Lambda_{A''}$, $\Omega''_+=\mathbb{D}$, $\mathfrak{D}''_{\mathcal{Y}}\in H^\infty(\mathbb{D};\mathcal{Y},\mathcal{K})$ and $\mathfrak{D}''_{\mathcal{U}}\in H^\infty(\mathbb{D};\mathcal{U},\mathcal{K})$. If $\Sigma^l_{i/s/o}$ arises from a scattering i/s/o representation, then the functions $\mathfrak{D}''_{\mathcal{Y}}$ and $\mathfrak{D}''_{\mathcal{U}}$ are even left coprime in $H^\infty(\mathbb{D})$.*

*Proof.* This proof is very similar to the proof of Proposition 3.1, with Theorem III.3.6 replaced by Theorem III.3.7 and (3.20) replaced by

$$\mathfrak{D}''(z)\mathfrak{D}''(\zeta)^* = \mathfrak{D}''_{\mathcal{U}}(z)\mathfrak{D}''_{\mathcal{U}}(\zeta)^* - \mathfrak{D}''_{\mathcal{Y}}(z)\mathfrak{D}''_{\mathcal{Y}}(\zeta)^*, \qquad z,\zeta\in\Omega''_+. \quad \square$$

**Proposition 4.3.** *Let $\Sigma^l_{i/s/o}$ be a left affine i/s/o representation with main operator $A''$ and left affine i/o transfer function $\begin{bmatrix}\mathfrak{D}''_{\mathcal{F}}(z) & \mathfrak{D}''_{\mathcal{E}}(z)\end{bmatrix}$ of a backward passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$, corresponding to a Lagrangean decomposition $\mathcal{W} = \mathcal{F}\overset{\Psi}{\dotplus}\mathcal{E}$ of the signal space. Then the kernel*

$$K_{\widetilde{\mathfrak{D}''_{\mathcal{F}},\mathfrak{D}''_{\mathcal{E}}}}(z,\zeta) = \frac{\mathfrak{D}''_{\mathcal{E}}(z)\Psi^*\mathfrak{D}''_{\mathcal{F}}(\zeta)^* + \mathfrak{D}''_{\mathcal{F}}(z)\Psi\mathfrak{D}''_{\mathcal{E}}(\zeta)^*}{1-z\overline{\zeta}}, \qquad z,\zeta\in\Omega''_+ \tag{4.17}$$

*is positive definite on the set* $\Omega''_+ \times \Omega''_+$*, where* $\Omega''_+ = \Lambda_{A''} \cap \mathbb{D}$*. If* $\Sigma$ *is passive, then we may choose the output nulling variable representation* $\Sigma_{s/s/on}$ *to be stable, in which case* $\mathbb{D} \subset \Lambda_{A''}$*,* $\Omega''_+ = \mathbb{D}$*,* $\mathfrak{D}''_{\mathcal{F}} \in H^\infty(\mathbb{D}; \mathcal{F}, \mathcal{K})$ *and* $\mathfrak{D}''_{\mathcal{E}} \in H^\infty(\mathbb{D}; \mathcal{E}, \mathcal{K})$*. If* $\Sigma^l_{i/s/o}$ *arises from a scattering i/s/o representation, then the functions* $\mathfrak{D}''_{\mathcal{F}}$ *and* $\mathfrak{D}''_{\mathcal{E}}$ *are even left coprime in* $H^\infty(\mathbb{D})$*.*

*Proof.* This proof is very similar to the proof of Proposition 3.2, with Theorem III.3.6 replaced by Theorem III.3.7 and (3.22) replaced by

$$\mathfrak{D}''(z)\mathfrak{D}''(\zeta)^* = \mathfrak{D}''_{\mathcal{E}}(z)\Psi^*\mathfrak{D}''_{\mathcal{F}}(\zeta)^* + \mathfrak{D}''_{\mathcal{F}}(z)\Psi\mathfrak{D}''_{\mathcal{E}}(\zeta)^*, \qquad z, \zeta \in \Omega''_+. \quad \square$$

# 5 Generalized Transfer Functions

The equations (1.8) can be rewritten as a system of equations for the formal power series of the corresponding sequences as

$$\begin{aligned} (1_{\mathcal{X}} - zA')\hat{x}(z) - zB'\hat{\ell}(z) &= x_0, \\ C'_{\mathcal{Y}}\hat{x}(z) + D'_{\mathcal{Y}}\hat{\ell}(z) &= \hat{y}(z), \\ C'_{\mathcal{U}}\hat{x}(z) + D'_{\mathcal{U}}\hat{\ell}(z) &= \hat{u}(z). \end{aligned} \tag{5.1}$$

Let us take a closer look at these equations on some domain $\Omega \subset \mathbb{C}$ with values in the respective Hilbert spaces (without trying to interpret $\hat{x}(z)$ etc. as formal power series). Let $\Omega(\Sigma; \mathcal{Y}, \mathcal{U})$ be the set defined in (1.10). Then from (5.1) we can solve $\left[\begin{smallmatrix} \hat{x}(z) \\ \hat{y}(z) \end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix} \hat{x}_0 \\ \hat{u}(z) \end{smallmatrix}\right]$ as follows: Clearly

$$\begin{bmatrix} \hat{x}(z) \\ \hat{\ell}(z) \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} - zA' & -zB' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix}^{-1} \begin{bmatrix} \hat{x}_0 \\ \hat{u}(z) \end{bmatrix} \text{ and } \hat{y}(z) = \begin{bmatrix} C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} \hat{x}(z) \\ \hat{\ell}(z) \end{bmatrix},$$

and hence

$$\begin{bmatrix} \hat{x}(z) \\ \hat{y}(z) \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} - zA' & -zB' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix}^{-1} \begin{bmatrix} \hat{x}_0 \\ \hat{u}(z) \end{bmatrix}, \; z \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y}). \tag{5.2}$$

As the following theorem says, the set $\Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ defined above and the operator function on the right-hand side of (5.2) do not depend on the particular choice of right affine i/s/o representation of $\Sigma$ (only on $\Sigma$ itself and the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$). Moreover, if the decomposition is admissible, then it coincides with the is/so transfer function of $\Sigma$ with respect to the same decomposition, as will be shown in Theorem 5.2. For this reason we shall call the opearator-valued function $\left[\begin{smallmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{smallmatrix}\right]$ defined by (1.11) on

$\Omega(\Sigma;\mathcal{Y},\mathcal{U})$ the *generalized is/so transfer function of* $\Sigma$ *corresponding to the decomposition* $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$. (Note that the domain of this function may be empty.) The bottom right corner $\mathfrak{D}(z)$ is called the *generalized i/o transfer function*, and the top left corner $\mathfrak{A}(z)$ is called the *generalized resolvent* of $\Sigma$ corresponding to this decomposition.

**Theorem 5.1.** *Let* $\Sigma=(V;\mathcal{X},\mathcal{W})$ *be a s/s system, and let* $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$ *be an arbitrary direct sum decomposition of the signal space* $\mathcal{W}$. *Then the domain* $\Omega(\Sigma;\mathcal{U},\mathcal{Y})$ *defined in* (1.10) *and the is/so transfer function* $\left[\begin{smallmatrix}\mathfrak{A}(z)&\mathfrak{B}(z)\\\mathfrak{C}(z)&\mathfrak{D}(z)\end{smallmatrix}\right]$ *defined in* (1.11) *do not depend on the particular right affine i/s/o representation* $\Sigma^r_{i/s/o}$ *used in these definitions.*

*Proof.* Let $\Sigma^{r,1}_{i/s/o}$ in (3.1) be another right affine i/s/o representation of $\Sigma$, and let $M'$ and $K'$ be the operators in (4.2). Then it follows from (3.2) that

$$\begin{bmatrix}1_{\mathcal{X}}-zA_1' & -zB_1'\\ (C_1')_{\mathcal{Y}} & (D_1')_{\mathcal{Y}}\\ 1_{\mathcal{X}} & 0\\ (C_1')_{\mathcal{U}} & (D_1')_{\mathcal{U}}\end{bmatrix}=\begin{bmatrix}1_{\mathcal{X}}-zA' & -zB'\\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}}\\ 1_{\mathcal{X}} & 0\\ C'_{\mathcal{U}} & D'_{\mathcal{U}}\end{bmatrix}\begin{bmatrix}1_{\mathcal{X}} & 0\\ K' & M'\end{bmatrix}. \tag{5.3}$$

In particular, $\left[\begin{smallmatrix}1_{\mathcal{X}}-zA_1' & -zB_1'\\ (C_1')_{\mathcal{U}} & (D_1')_{\mathcal{U}}\end{smallmatrix}\right]=\left[\begin{smallmatrix}1_{\mathcal{X}}-zA' & -zB'\\ C'_{\mathcal{U}} & D'_{\mathcal{U}}\end{smallmatrix}\right]\left[\begin{smallmatrix}1_{\mathcal{X}} & 0\\ K' & M'\end{smallmatrix}\right]$ where $\left[\begin{smallmatrix}1_{\mathcal{X}} & 0\\ K' & M'\end{smallmatrix}\right]$ is invertible, so $\left[\begin{smallmatrix}1_{\mathcal{X}}-zA_1' & -zB_1'\\ (C_1')_{\mathcal{U}} & (D_1')_{\mathcal{U}}\end{smallmatrix}\right]$ is invertible if and only if $\left[\begin{smallmatrix}1_{\mathcal{X}}-zA' & -zB'\\ C'_{\mathcal{U}} & D'_{\mathcal{U}}\end{smallmatrix}\right]$ is invertible, and

$$\begin{aligned}&\begin{bmatrix}1_{\mathcal{X}} & 0\\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}}\end{bmatrix}\begin{bmatrix}1_{\mathcal{X}}-zA' & -zB'\\ C'_{\mathcal{U}} & D'_{\mathcal{U}}\end{bmatrix}^{-1}\\ &=\begin{bmatrix}1_{\mathcal{X}} & 0\\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}}\end{bmatrix}\begin{bmatrix}1_{\mathcal{X}} & 0\\ K' & M'\end{bmatrix}\begin{bmatrix}1_{\mathcal{X}} & 0\\ K' & M'\end{bmatrix}^{-1}\begin{bmatrix}1_{\mathcal{X}}-zA' & -zB'\\ C'_{\mathcal{U}} & D'_{\mathcal{U}}\end{bmatrix}^{-1}\\ &=\begin{bmatrix}1_{\mathcal{X}} & 0\\ (C_1')_{\mathcal{Y}} & (D_1')_{\mathcal{Y}}\end{bmatrix}\begin{bmatrix}1_{\mathcal{X}}-zA_1' & -zB_1'\\ (C_1')_{\mathcal{U}} & (D_1')_{\mathcal{U}}\end{bmatrix}^{-1}. \quad\square\end{aligned}$$

**Theorem 5.2.** *Let* $\Sigma=(V;\mathcal{X},\mathcal{W})$ *be a s/s system. Then the decomposition* $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$ *is admissible if and only if* $0\in\Omega(\Sigma;\mathcal{U},\mathcal{Y})$. *In this case* $\Omega(\Sigma;\mathcal{U},\mathcal{U})=\Lambda_A$, *where* $A$ *is the main operator of the i/s/o representation* $\Sigma_{i/s/o}$ *of* $\Sigma$ *corresponding to the decomposition* $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$, *and the generalized is/so transfer function of* $\Sigma$ *corresponding to the decomposition* $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$ *defined in* (1.11) *coincides with the standard is/so transfer function of* $\Sigma_{i/s/o}$ *defined in* (1.3).

*Proof.* By definition, $0\in\Omega(\Sigma;\mathcal{U},\mathcal{Y})$ if and only if the operator $\left[\begin{smallmatrix}1 & 0\\ C'_{\mathcal{U}} & D'_{\mathcal{Y}}\end{smallmatrix}\right]$ has a bounded inverse, or equivalently, if and only if $D'_{\mathcal{U}}$ has a bounded inverse.

By Theorem I.5.9, this is equivalent to the admissibility of the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$.

Assume that the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is admissible. Then the i/s/o representation $\Sigma_{i/s/o} = \big(\big[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\big]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\big)$ of $\Sigma$ corresponding to this representation may be interpreted as a right affine representation $\Sigma^r_{i/s/o} = \Big(\Big[\begin{smallmatrix} A & B \\ C & D \\ 0 & 1_\mathcal{U} \end{smallmatrix}\Big]; \mathcal{X}, \mathcal{U}, \big[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\big]\Big)$. By using this specific representation in (1.11) one finds that $\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) = \Lambda_A$, and that the is/so transfer function computed from (1.11) coincies with the one computed from (1.3), i.e., for all $z \in \Lambda_A$,

$$\begin{bmatrix} 1_\mathcal{X} & 0 \\ C & D \end{bmatrix} \begin{bmatrix} 1_\mathcal{X} - zA & -zB \\ 0 & 1_\mathcal{U} \end{bmatrix}^{-1} = \begin{bmatrix} (1_\mathcal{X} - zA)^{-1} & z(1_\mathcal{X} - zA)^{-1}B \\ C(1_\mathcal{X} - zA)^{-1} & zC(1_\mathcal{X} - zA)^{-1}B + D \end{bmatrix}. \qquad \square$$

**Theorem 5.3.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system, and let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be an i/o decomposition of $\mathcal{W}$ for which $\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \neq \emptyset$. Let $\big[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\big]$ be the generalized is/so transfer function of the s/s system $\Sigma$ defined in* (1.11)*, and let*

$$\mathfrak{T}(z) = \begin{bmatrix} z\mathfrak{A}(z) & \mathfrak{B}(z) \\ z\mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1_\mathcal{U} \end{bmatrix}, \qquad z \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y}).$$

*Then $\mathfrak{T}$ satisfies the generalized resolvent identity*

$$\begin{aligned} z\zeta[\mathfrak{T}(z) - \mathfrak{T}(\zeta)] &= (z-\zeta)\mathfrak{T}(z) \begin{bmatrix} 1_\mathcal{X} & 0 \\ 0 & 0 \end{bmatrix} \mathfrak{T}(\zeta) \\ &= (z-\zeta)\mathfrak{T}(\zeta) \begin{bmatrix} 1_\mathcal{X} & 0 \\ 0 & 0 \end{bmatrix} \mathfrak{T}(z), \quad z,\ \zeta \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y}). \end{aligned} \tag{5.4}$$

*This identity is equivalent to the following four identites, valid for all $z$, $\zeta \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y})$:*

$$\begin{aligned} z\mathfrak{A}(z) - \zeta\mathfrak{A}(\zeta) &= (z-\zeta)\mathfrak{A}(z)\mathfrak{A}(\zeta) = (z-\zeta)\mathfrak{A}(\zeta)\mathfrak{A}(z), \\ \zeta\mathfrak{B}(z) - \zeta\mathfrak{B}(\zeta)) &= (z-\zeta)\mathfrak{A}(z)\mathfrak{B}(\zeta) = (z-\zeta)\mathfrak{A}(\zeta)\mathfrak{B}(z), \\ z\mathfrak{C}(z) - \zeta\mathfrak{C}(\zeta) &= (z-\zeta)\mathfrak{C}(z)\mathfrak{A}(\zeta) = (z-\zeta)\mathfrak{C}(\zeta)\mathfrak{A}(z), \\ z\zeta\mathfrak{D}(z) - z\zeta\mathfrak{D}(\zeta) &= z(z-\zeta)\mathfrak{C}(z)\mathfrak{B}(\zeta) = \zeta(z-\zeta)\mathfrak{C}(\zeta)\mathfrak{B}(z). \end{aligned} \tag{5.5}$$

The same identies play a central role in Opmeer's definition of a continuous time resolvent linear system in [Opm05] (with $z$ and $\zeta$ replaced by $1/z$ and $1/\zeta$).

*Proof.* If $0 \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ and $z = 0$ or $\zeta = 0$, then it is easy to see that both sides of (5.4) are zero (note that $\mathfrak{T}(0) = \big[\begin{smallmatrix} 0 & 0 \\ 0 & \mathfrak{D}(0) \end{smallmatrix}\big]$). We therefore suppose in the sequel that $z \neq 0$ and $\zeta \neq 0$. In this case we have

$$\begin{aligned}\mathfrak{T}(z) - \mathfrak{T}(\zeta) &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{bmatrix} \left( \begin{bmatrix} 1/z - A' & -B' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix}^{-1} - \begin{bmatrix} 1/\zeta - A' & -B' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix}^{-1} \right) \\ &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} 1/z - A' & -B' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix}^{-1} \begin{bmatrix} 1/\zeta - 1/z & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\zeta - A' & -B' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix}^{-1} \\ &= (1/\zeta - 1/z)\mathfrak{T}(z) \begin{bmatrix} \zeta\mathfrak{A}(\zeta) & \mathfrak{B}(\zeta) \\ 0 & 0 \end{bmatrix} = \frac{z-\zeta}{z\zeta}\mathfrak{T}(z) \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & 0 \end{bmatrix} \mathfrak{T}(\zeta).\end{aligned}$$

In the last formula on the last line we can exchange the places of $\mathfrak{T}(z)$ and $\mathfrak{T}(\zeta)$ by simply interchanging the places of the two inverses on the second line. Thus, (5.4) holds.

We leave the easy verification that (5.4) is equivalent to (5.5) to the reader. □

**Theorem 5.4.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system with the right affine representation $\Sigma^r_{i/s/o}$ given by* (1.7) *and with the is/so transfer function* (3.7) *corresponding to a decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. Then a point $z \in \Lambda_{A'}$ belongs to the domain $\Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ defined in* (1.10) *if and only if $\mathfrak{D}'_{\mathcal{U}}(z)$ has a bounded inverse. On the set*

$$\Omega(\Sigma^r_{i/s/o}) = \Lambda_{A'} \cap \Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \tag{5.6}$$

*the generalized is/so transfer function of $\Sigma$ corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is given by*

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'_{\mathcal{Y}}(z) & \mathfrak{D}'_{\mathcal{Y}}(z) \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \mathfrak{C}'_{\mathcal{U}}(z) & \mathfrak{D}'_{\mathcal{U}}(z) \end{bmatrix}^{-1}, \quad z \in \Omega(\Sigma^r_{i/s/o}). \tag{5.7}$$

*In particular, the generalized i/o transfer function of $\Sigma$ corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is given by*

$$\mathfrak{D}(z) = \mathfrak{D}'_{\mathcal{Y}}(z)\mathfrak{D}'_{\mathcal{U}}(z)^{-1}, \quad z \in \Omega(\Sigma^r_{i/s/o}). \tag{5.8}$$

*Proof.* For all $z \in \Lambda_{A'}$ we may use a LU (lower/upper triangular) factorization to get

$$\begin{bmatrix} 1_{\mathcal{X}} - zA' & -zB' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{U}}\mathfrak{A}'(z) & \mathfrak{D}'_{\mathcal{U}}(z) \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} - zA' & -z\mathfrak{B}'(z) \\ 0 & 1_{\mathcal{L}} \end{bmatrix}.$$

Consequently, for $z \in \Lambda_{A'}$, $\left[\begin{smallmatrix} 1_{\mathcal{X}} - zA' & -zB' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{smallmatrix}\right]$ has a bounded inverse if and only if $\mathfrak{D}'_{\mathcal{U}}(z)$ has a bounded inverse. By inverting the operators in the above formula and multiplying the result to the left by $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{smallmatrix}\right]$ we get (5.7). □

Our definition of a generalized transfer function has been based on the use of a right affine i/s/o representation of the system. It is also possible to use left affine i/s/o representations. We begin the discussion of the left version with three lemmas that lead up to Theorem 5.8 below.

**Lemma 5.5.** *Let* $\mathfrak{S} = (V; \mathcal{X}, \mathcal{W})$ *be a s/s system, and define* $\mathfrak{V}(z)$ *by*

$$\mathfrak{V}(z) = \left\{ \begin{bmatrix} x \\ x - z\dot{x} \\ w \end{bmatrix} \,\middle|\, \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in V \right\} = \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ -z & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V. \qquad z \in \mathbb{C}. \tag{5.9}$$

*If* $\Sigma_{dv/s/s} = \left( \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$ *is a driving variable representation and* $\Sigma_{s/s/on} = \left( \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}; \mathcal{X}, \mathcal{W}, \mathcal{K} \right)$ *is an output nulling representation of* $\Sigma$, *then*

$$\mathfrak{V}(z) = \mathcal{R}\left( \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} - zA' & -zB' \\ C' & D' \end{bmatrix} \right) = \mathcal{N}\left( \begin{bmatrix} 1_{\mathcal{X}} - zA'' & -1_{\mathcal{X}} & -zB'' \\ C'' & 0 & D'' \end{bmatrix} \right). \tag{5.10}$$

*Proof.* This follows from formulas (5.9) and the definitions of a s/s node and the properties of driving variable and output nulling representations of this node. □

**Lemma 5.6.** *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a s/s system with a right affine i/s/o representation* $\Sigma^r_{i/s/o}$ *given by* (1.7) *and a left affine i/s/o representation* $\Sigma^l_{i/s/o}$ *given by* (1.12) *corresponding to the same i/o decomposition* $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$. *Define*

$$M(z) = \begin{bmatrix} M_1(z) \\ M_2(z) \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \\ \hline 1_{\mathcal{X}} - zA' & -zB' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix} \in \mathcal{B}\left( \begin{bmatrix} \mathcal{X} \\ \mathcal{L} \end{bmatrix}; \begin{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \\ \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \end{bmatrix} \right), \tag{5.11}$$

$$\begin{aligned} E(z) &= \begin{bmatrix} E_1(z) & E_2(z) \end{bmatrix} \\ &= \left[ \begin{array}{cc|cc} 1_{\mathcal{X}} - zA'' & -zB''_{\mathcal{Y}} & -1_{\mathcal{X}} & -zB''_{\mathcal{U}} \\ C'' & D''_{\mathcal{Y}} & 0 & D''_{\mathcal{U}} \end{array} \right] \in \mathcal{B}\left( \begin{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \\ \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{K} \end{bmatrix} \right). \end{aligned} \tag{5.12}$$

*Then, for every* $z \in \mathbb{C}$,

1) $M(z)$ *is injective,*

2) $E(z)$ *is surjective,*

3) $\mathcal{R}(M(z)) = \mathcal{N}(E(z)) = \begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 & 0 \\ 0 & 0 & 1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}} \end{bmatrix} \mathfrak{V}(z)$.

*Proof.* Let $C' = \begin{bmatrix} C'_{\mathcal{Y}} \\ C'_{\mathcal{U}} \end{bmatrix}$, $D' = \begin{bmatrix} D'_{\mathcal{Y}} \\ D'_{\mathcal{U}} \end{bmatrix}$, $B'' = \begin{bmatrix} B''_{\mathcal{Y}} & B''_{\mathcal{U}} \end{bmatrix}$, $D'' = \begin{bmatrix} D''_{\mathcal{Y}} & D''_{\mathcal{U}} \end{bmatrix}$. The injectivity of $M(z)$ follows from the injectivity of $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]$, which in turn follows from the injectivity of $D'$. The surjectivity of $E(z)$ follows from the surjectivity of $\left[\begin{smallmatrix} -1_{\mathcal{X}} & -zB'' \\ 0 & D'' \end{smallmatrix}\right]$, which follows from the surjectivity of $D''$. Finally, Claim 3) follows from Lemma 5.5. □

**Lemma 5.7.** *Let $\mathcal{K} = \left[\begin{smallmatrix} \mathcal{K}_1 \\ \mathcal{K}_1 \end{smallmatrix}\right]$ be the product of two Banach spaces, and let $\mathcal{H}$ and $\mathcal{K}$ be two Banach spaces.*

1) *Let $\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right] \in \mathcal{B}(\mathcal{H}; \left[\begin{smallmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{smallmatrix}\right])$ be injective. Then $M_2$ has an inverse in $\mathcal{B}(\mathcal{K}_2; \mathcal{H})$ if and only if the range of $\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right]$ is the graph of a bounded operator $A \in \mathcal{B}(\mathcal{K}_2; \mathcal{K}_1)$, i.e., if and only if*

$$\mathcal{R}\left(\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} A \\ 1_{\mathcal{K}_2} \end{bmatrix}\right) \quad \text{for some } A \in \mathcal{B}(\mathcal{K}_2; \mathcal{K}_1). \tag{5.13}$$

*In this case $A = M_1 M_2^{-1}$, and $A$ is determined uniquely by the range of $\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right]$.*

2) *Let $\begin{bmatrix} E_1 & E_2 \end{bmatrix} \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{smallmatrix}\right]; \mathcal{G})$ be surjective. Then $E_1$ has an inverse in $\mathcal{B}(\mathcal{G}; \mathcal{K}_1)$ if and only if the kernel of $\begin{bmatrix} E_1 & E_2 \end{bmatrix}$ is the graph of a bounded operator $A \in \mathcal{B}(\mathcal{K}_2; \mathcal{K}_1)$, i.e., if and only if*

$$\mathcal{N}\left(\begin{bmatrix} E_1 & E_2 \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} A \\ 1_{\mathcal{K}_2} \end{bmatrix}\right) \quad \text{for some } A \in \mathcal{B}(\mathcal{K}_2; \mathcal{K}_1). \tag{5.14}$$

*In this case $A = -E_1^{-1} E_2$, and $A$ is determined uniquely by the null space of $\begin{bmatrix} E_1 & E_2 \end{bmatrix}$.*

3) *Let $\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right] \in \mathcal{B}(\mathcal{H}; \left[\begin{smallmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{smallmatrix}\right])$ be injective, let $\begin{bmatrix} E_1 & E_2 \end{bmatrix} \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{smallmatrix}\right]; \mathcal{G})$ be surjective, and suppose that*

$$\mathcal{R}\left(\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right]\right) = \mathcal{N}\left(\begin{bmatrix} \mathcal{E}_1 & \mathcal{E}_2 \end{bmatrix}\right). \tag{5.15}$$

*Then $M_2$ has an inverse in $\mathcal{B}(\mathcal{K}_2; \mathcal{H})$ if and only if $E_1$ has an inverse in $\mathcal{B}(\mathcal{G}; \mathcal{K}_1)$, in which case $M_1 M_2^{-1} = -E_1^{-1} E_2$.*

*Proof.* Proof of 1): It is easy to see that the bounded invertability of $M_2$ implies (5.13) with $A = M_1 M_2^{-1}$, so it suffices to prove the converse direction. Assume (5.13), and let $\left[\begin{smallmatrix} k_1 \\ k_2 \end{smallmatrix}\right] \in \mathcal{R}\left(\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right]\right) = \mathcal{R}\left(\left[\begin{smallmatrix} A \\ 1_{\mathcal{K}_2} \end{smallmatrix}\right]\right)$. Then $k_2 = 0$ implies $k_1 = 0$. Writing $\left[\begin{smallmatrix} k_1 \\ k_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right] h$ for some $h \in \mathcal{H}$ we find that $\mathcal{N}(M_2) \subset \mathcal{N}(M_1)$. Thus, $\mathcal{N}\left(\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right]\right) = \mathcal{N}(M_2)$. Since $\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right]$ is supposed to be injective this implies that also $M_2$ is injective. It is also clear that $M_2$ must be surjective since $\mathcal{R}\left(\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right]\right) = \mathcal{R}\left(\left[\begin{smallmatrix} A \\ 1_{\mathcal{K}_2} \end{smallmatrix}\right]\right)$. Thus $M_2$ is both injective and surjective, and by the closed graph theorem, it has a bounded inverse.

Proof of 2): It is again easy to see that boundeed invertability of $E_1$ implies (5.14) with $A = -E_1^{-1}E_2$, so it again suffices to prove the converse. Assume (5.14). Then $E_1$ must be injective, since the graph of $A$ cannot contain a nontrivial subspace of the type $\left\{\left[\begin{smallmatrix} k_1 \\ 0 \end{smallmatrix}\right] \;\middle|\; k_1 \in \mathcal{N}(E_1)\right\}$. The assumption (5.14) implies furthermore that for all $k_2 \in E_2$ we have $\left[\begin{smallmatrix} Ak_2 \\ k_2 \end{smallmatrix}\right] \in \mathcal{N}\left(\begin{bmatrix} E_1 & E_2 \end{bmatrix}\right)$, i.e., $(E_1A + E_2)k_2 = 0$, so that $E_2 = -E_1A$. In particular, $\mathcal{R}(E_2) \subset \mathcal{R}(E_1)$, which means that $\mathcal{R}\left(\begin{bmatrix} E_1 & E_2 \end{bmatrix}\right) = \mathcal{R}(E_1)$. By assumption, $\begin{bmatrix} E_1 & E_2 \end{bmatrix}$ is surjcetive, hence is so is $E_1$. By the closed graph theorem, $E_1$ has a bounded inverse.

The claim 3) follows immediately from 1) and 2). □

**Theorem 5.8.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system with a right affine i/s/o representation $\Sigma^r_{i/s/o}$ given by* (1.7) *and a left affine i/s/o representation $\Sigma^l_{i/s/o}$ given by* (1.12) *corresponding to the same i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. Then the following statements hold.*

1) *The definitions given in* (1.10) *and* (1.14) *of the domain $\Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ of the generalized transfer function are equivalent.*

2) *The definitions given in* (1.11) *and* (1.15) *of the generalized is/so transfer function of $\Sigma$, corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$, are equivalent.*

3) *$z \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ if and only if the subspace $\mathfrak{V}(z)$ defined in* (5.9) *is the generating subspace of a s/s node $\mathfrak{S}(z) = (\mathfrak{V}(z); \mathcal{X}, \mathcal{W})$ and the i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is admissible for the s/s system $\mathfrak{S}(z)$, or equivalently, if and only if*

$$\mathfrak{V}(z) = \mathcal{R}\left(\begin{bmatrix} \mathfrak{A}_z & \mathfrak{B}_z \\ 1_{\mathcal{X}} & 0 \\ \mathfrak{C}_z & \mathfrak{D}_z \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & \mathfrak{A}_z & 0 & \mathfrak{B}_z \\ 0 & \mathfrak{C}_z & -1_{\mathcal{Y}} & \mathfrak{D}_z \end{bmatrix}\right) \tag{5.16}$$

*for some bounded linear operators $\mathfrak{A}_z$, $\mathfrak{B}_z$, $\mathfrak{C}_z$, and $\mathfrak{D}_z$.*

4) *The generalized is/so transfer function of $\Sigma$ evaluated at a point $z \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ is given by $\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \left[\begin{smallmatrix} \mathfrak{A}_z & \mathfrak{B}_z \\ \mathfrak{C}_z & \mathfrak{D}_z \end{smallmatrix}\right]$, so that i/s/o representation of $\mathfrak{S}(z)$ corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is given by*

$$\mathfrak{S}_{i/s/o}(z) = \left(\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right). \tag{5.17}$$

*Thus,*

$$\mathfrak{V}(z) = \mathcal{R}\left(\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ 1_{\mathcal{X}} & 0 \\ \mathfrak{C}(z) & \mathfrak{D}(z) \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & \mathfrak{A}(z) & 0 & \mathfrak{B}(z) \\ 0 & \mathfrak{C}(z) & -1_{\mathcal{Y}} & \mathfrak{D}(z) \end{bmatrix}\right). \tag{5.18}$$

*Proof.* Assertions 1)–2) follow from Lemmas 5.5–5.7. The remaining assertions follow from 1) and 2) and the definitions of a s/s node, of admissibility of an i/o decomposition of the signal space, and of the interpretation of an i/s/o representation as either a driving variable representation or an output nulling representation. □

**Theorem 5.9.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system with the left affine representation $\Sigma^l_{i/s/o}$ given by* (1.12) *and with the is/so transfer function* (4.6) *corresponding to a decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$. Then a point $z \in \Lambda_{A''}$ belongs to the domain $\Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ defined in* (1.14) *if and only if $\mathfrak{D}''_{\mathcal{Y}}(z)$ has a bounded inverse. On the set*

$$\Omega(\Sigma^l_{i/s/o}) = \Lambda_{A''} \cap \Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \tag{5.19}$$

*the generalized is/so transfer function of $\Sigma$ is given by*

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & -\mathfrak{B}''_{\mathcal{Y}}(z) \\ 0 & -\mathfrak{D}''_{\mathcal{Y}}(z) \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{A}''(z) & \mathfrak{B}''_{\mathcal{U}}(z) \\ \mathfrak{C}''(z) & \mathfrak{D}''_{\mathcal{U}}(z) \end{bmatrix}, \quad z \in \Omega(\Sigma^l_{i/s/o}). \tag{5.20}$$

*In particular, the generalized i/o transfer function is given by*

$$\mathfrak{D}(z) = -\mathfrak{D}''_{\mathcal{Y}}(z)^{-1}\mathfrak{D}''_{\mathcal{U}}(z), \quad z \in \Omega(\Sigma^l_{i/s/o}). \tag{5.21}$$

*Proof.* The proof of this theorem is similar to the proof of Theorem 5.4, taking into account assertions 1) and 2) of Theorem 5.8, and we leave it to the reader. □

The generalized is/so transfer function of a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with respect to the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ does not have the same direct interpretation as the standard transfer function has, in the sense that it need not give the the formal power series of the state sequence $x(\cdot)$ and the output sequence $y(\cdot)$ in terms of the initial state $x_0$ and the input sequence $u(\cdot)$. However, the following result is true.

**Proposition 5.10.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system, and define $\mathfrak{V}(z)$ by* (5.9)*. Let $(x(\cdot), w(\cdot))$ be a trajectory of $\Sigma$ on $\mathbb{Z}^+$, let $R$ be the radius of* *convergence of the z-transform of* $\left[\begin{smallmatrix} x(\cdot) \\ w(\cdot) \end{smallmatrix}\right]$, *and suppose that* $R > 0$. *Then*

$$\begin{bmatrix} \hat{x}(z) \\ x_0 \\ \hat{w}(z) \end{bmatrix} \in \mathfrak{V}(z), \qquad |z| < R. \tag{5.22}$$

*In particular, if* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *and if we split* $\hat{w}(z)$ *into* $\hat{w}(z) = \hat{y}(z) + \hat{u}(z)$ *where* $\hat{y}(z) \in \mathcal{Y}$ *and* $\hat{u}(z) \in \mathcal{U}$, *then*

$$\begin{bmatrix} \hat{x}(z) \\ \hat{y}(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} \begin{bmatrix} x(0) \\ \hat{u}(z) \end{bmatrix} \text{ for all } z \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \text{ with } |z| < R \tag{5.23}$$

*(whenever such* $z$ *exist).*

*Proof.* Formula (5.22) follows from (2.1) and (5.9), and (5.23) then follows from Theorem 5.8. □

It is also possible to give following alternative characterization of the is/so transfer function of $\Sigma$, that is valid away from the origin.

**Proposition 5.11.** *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a s/s system, and let* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. *A point* $z \neq 0$ *belongs to the domain* $\Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ *of the generalized is/so transfer function, corresponding to the decomposition* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$, *if and only if the following property holds: For every* $v_0 \in \mathcal{X}$ *and every* $u_0 \in \mathcal{U}$ *there exists a unique* $x_0 \in \mathcal{X}$ *and a unique* $y_0 \in \mathcal{U}$ *with the property that the sequence*

$$(x(n), w(n), v(n)) = z^{-n}(x_0, w_0, v_0), \quad n \in \mathbb{Z}, \tag{5.24}$$

*with* $w_0 = u_0 + y_0$ *satisfies*

$$\begin{bmatrix} x(n+1) - v(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}. \tag{5.25}$$

*In this case* $x_0$ *and* $y_0$ *are given by*

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} \begin{bmatrix} v_0 \\ u_0 \end{bmatrix}. \tag{5.26}$$

*Proof.* We begin by observing that (5.25) and (5.26) hold if and only if $\left[\begin{smallmatrix} x_0 \\ v_0 \\ w_0 \end{smallmatrix}\right] \in \mathfrak{V}(z)$, where $\mathfrak{V}(z)$ is the subspace defined in (5.9). The requirement that to each $x_0 \in \mathcal{X}$ and $u_0 \in \mathcal{U}$ there is a unique $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$ such that $\left[\begin{smallmatrix} x_0 \\ v_0 \\ u_0 + y_0 \end{smallmatrix}\right] \in \mathfrak{V}(z)$ is equivalent to the requirement that $\mathfrak{V}(z)$ has a graph representation of the type (5.16), and by Theorem 5.8, this is equivalent to the condition $z \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y})$. Formula (5.26) also follows directly from Theorem 5.8. □

**Proposition 5.12.** *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *and* $\Sigma^1 = (V_1; \mathcal{X}_1, \mathcal{W})$ *be two similar s/s systems with similarity operator* $R \in \mathcal{B}(\mathcal{X}, \mathcal{X}_1)$*, and let* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$*. Then* $\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) = \Omega(\Sigma^1; \mathcal{U}, \mathcal{Y})$ *and, if* $\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \neq \emptyset$*, then the generalized is/so transfer functions of these two systems, corresponding to the decomposition* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$*, are connected by the relation*

$$\begin{bmatrix} \mathfrak{A}_1(z) & \mathfrak{B}_1(z) \\ \mathfrak{C}_1(z) & \mathfrak{D}_1(z) \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} \begin{bmatrix} R^{-1} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}.$$

*Proof.* This follows immediately from Proposition 5.11 and the fact that $(x_1(\cdot), w(\cdot), v(\cdot)) = (Rx(\cdot), w(\cdot), v(\cdot))$ satisfies (5.25) vith $V$ replaced by $V_1$ if and only if $(x(\cdot), u(\cdot), y(\cdot))$ satisfies (5.25). □

# 6 Realizations of Generalized Transfer Functions

There is another subspace which is closely related to the subspace $\mathfrak{V}(z)$ defined in Lemma 5.5, namely the subspace

$$V(z) = \left\{ \begin{bmatrix} \dot{x} \\ x - z\dot{x} \\ w \end{bmatrix} \;\middle|\; \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in V \right\} = \begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 \\ -z & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V, \; z \in \mathbb{C}, \tag{6.1}$$

which in some respects behave better than $\mathfrak{V}(z)$. In particular, $V(0) = V$. Results similar to those given in Lemmas 5.5–5.6 are valid with $\mathfrak{V}(z)$ replaced by $V(z)$. One way to derive these results is to observe that

$$\mathfrak{V}(z) = \begin{bmatrix} z & 1_{\mathcal{X}} & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V(z), \qquad z \in \mathbb{C}. \tag{6.2}$$

For example, the analogue of (5.10) is

$$V(z) = \mathcal{R}\left( \begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} - zA' & -zB' \\ C' & D' \end{bmatrix} \right) = \mathcal{N}\left( \begin{bmatrix} zA'' - 1_{\mathcal{X}} & A'' & B'' \\ zC'' & C'' & D'' \end{bmatrix} \right). \tag{6.3}$$

**Lemma 6.1.** *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a s/s system, and define* $\mathfrak{V}(z)$ *and* $V(z)$ *as in* (5.9) *and* (6.1)*.*

1) $\mathfrak{V}(z)$ *is the generating subspace of a s/s node* $\mathfrak{S}(z) = (\mathfrak{V}(z); \mathcal{X}, \mathcal{W})$ *if and only if* $V(z)$ *is the generating subspace of a s/s node* $\Sigma(z) = (V(z); \mathcal{X}, \mathcal{W})$*.*

2) $V(0) = V$*, and hence both* $V(0)$ *and* $\mathfrak{V}(0)$ *generate s/s systems* $\Sigma(0) = \Sigma$ *and* $\mathfrak{S}(0)$*, respectively.*

3) *Suppose that the equivalent conditions in 1) hold. Then the following conditions are equivalent:*

   (a) $z \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y})$;

   (b) *The decomposition* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is admissible for* $\mathfrak{S}(z)$;

   (c) *The decomposition* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is admissible for* $\Sigma(z)$.

   *If these equivalent conditions hold, then the corresponding i/s/o representation of* $\mathfrak{V}(z)$ *is given by* (5.17), *and the corresponding i/s/o representation of* $\Sigma(z)$ *is given by*

$$\Sigma_{i/s/o}(z) = \left( \begin{bmatrix} \frac{1}{z}(\mathfrak{A}(z) - 1_{\mathcal{X}}) & \frac{1}{z}\mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right) \tag{6.4}$$

   *if* $z \neq 0$, *and by taking the limit in* (6.4) *as* $z \to 0$ *if* $z = 0$.

We leave the easy proof to the reader.

**Remark 6.2.** The four block function appearing in the i/s/o representation $\Sigma_{i/s/o}(z)$ of $\Sigma(z)$ given above can be interpreted as a is/so transfer function in the following sense: Let $(x(\cdot), y(\cdot), u(\cdot))$ satisfy

$$\begin{bmatrix} x(n) \\ x(n-1) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^+, \quad x(-1) \ = x_{-1}, \tag{6.5}$$

and let $\hat{x}(\cdot)$, $\hat{u}(\cdot)$, and $\hat{y}(\cdot)$ be the formal power series of $x(\cdot)$, $u(\cdot)$, and $y(\cdot)$. Then

$$\begin{aligned} \hat{x}(z) &= \frac{1}{z}(\mathfrak{A}(z) - 1)x_{-1} + \frac{1}{z}\mathfrak{B}(z)\hat{u}(z), \\ \hat{y}(z) &= \mathfrak{C}(z)x_{-1} + \mathfrak{D}(z)\hat{u}(z). \end{aligned}$$

**Theorem 6.3.** *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a s/s system, and let* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{Y}$ *be an i/s/o decomposition of* $\mathcal{W}$ *for which* $\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \neq \emptyset$. *Let* $\zeta \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y})$, *define* $V(\zeta)$ *by* (6.1) *with* $z$ *replaced by* $\zeta$, *and define*

$$\begin{aligned} A_\zeta &= \frac{1}{\zeta}\,(\mathfrak{A}(\zeta) - 1_{\mathcal{X}}), \qquad & B_\zeta &= \frac{1}{\zeta}\,\mathfrak{B}(\zeta), \\ C_\zeta &= \mathfrak{C}(\zeta), & D_\zeta &= \mathfrak{D}(\zeta), \end{aligned} \tag{6.6}$$

*if* $\zeta \neq 0$ *and by taking the limit as* $\zeta \to 0$ *if* $\zeta = 0$. *Then*

1) *the subspace* $V$ *is given by*

$$\begin{aligned} V &= \begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 & 0 \\ \zeta & 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & 1_{\mathcal{Y}} & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}} \end{bmatrix} V(\zeta) = \mathcal{R}\left(\begin{bmatrix} A_\zeta & B_\zeta \\ 1_{\mathcal{X}}+\zeta A_\zeta & \zeta B_\zeta \\ C_\zeta & D_\zeta \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) \\ &= \mathcal{N}\left(\begin{bmatrix} -(1_{\mathcal{X}}+\zeta A_\zeta) & A_\zeta & 0 & B_\zeta \\ -\zeta C_\zeta & C_\zeta & -1_{\mathcal{Y}} & D_\zeta \end{bmatrix}\right), \end{aligned} \tag{6.7}$$

*and for all* $z \in \mathbb{C}$ *the subspace* $V(z)$*, defined in* (6.1)*, has the representation*

$$\begin{aligned} V(z) &= \begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 & 0 \\ \zeta-z & 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & 1_{\mathcal{Y}} & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}} \end{bmatrix} V(\zeta) = \mathcal{R}\left(\begin{bmatrix} A_\zeta & B_\zeta \\ 1_{\mathcal{X}}+(\zeta-z)A_\zeta & (\zeta-z)B_\zeta \\ C_\zeta & D_\zeta \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) \\ &= \mathcal{N}\left(\begin{bmatrix} -(1_{\mathcal{X}}+(\zeta-z)A_\zeta) & A_\zeta & 0 & B_\zeta \\ -(\zeta-z)C_\zeta & C_\zeta & -1_{\mathcal{Y}} & D_\zeta \end{bmatrix}\right). \end{aligned} \tag{6.8}$$

2) $z \in \Omega(\Sigma;\mathcal{U},\mathcal{Y})$ *if and only if* $z-\zeta \in \Lambda_{A_\zeta}$*, and the generalized is/so transfer function* $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *of* $\Sigma$ *can be recovered from the transfer function* $\left[\begin{smallmatrix} \mathfrak{A}_\zeta & \mathfrak{B}_\zeta \\ \mathfrak{C}_\zeta & \mathfrak{D}_\zeta \end{smallmatrix}\right]$ *of the i/s/o system* $\Sigma^\zeta_{i/s/o} := \left(\left[\begin{smallmatrix} A_\zeta & B_\zeta \\ C_\zeta & D_\zeta \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ *(and conversely) from the formula*

$$\begin{aligned} &\begin{bmatrix} \frac{1}{z}\left(\mathfrak{A}(z)-1_{\mathcal{X}}\right) & \frac{1}{z}\mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{z-\zeta}\left(\mathfrak{A}_\zeta(z-\zeta)-1_{\mathcal{X}}\right) & \frac{1}{z-\zeta}\mathfrak{B}_\zeta(z-\zeta) \\ \mathfrak{C}_\zeta(z-\zeta) & \mathfrak{D}_\zeta(z-\zeta) \end{bmatrix}, \quad z \in \Omega(\Sigma;\mathcal{U},\mathcal{Y}), \end{aligned} \tag{6.9}$$

*if* $z \neq 0$ *and* $z \neq \zeta$ *and by taking limits in this formual as* $z \to 0$ *if* $z = 0$ *and as* $z \to \zeta$ *if* $z = \zeta$*. In particular,* $\Sigma^\zeta_{i/s/o}$ *is an i/s/o realization of the shifted function* $z \mapsto \mathfrak{D}(\zeta+z)$*, i.e.,*

$$\mathfrak{D}_\zeta(z) = \mathfrak{D}(z+\zeta), \qquad z \in \Lambda_{A_\zeta}. \tag{6.10}$$

*Proof.* Assertion 1) follows from (6.1) and (6.3).

By Theorem 5.8, $z \in \Omega(\Sigma;\mathcal{U},\mathcal{Y})$ if and only if (5.16) holds for some bounded linear operators $\mathfrak{A}_z$, $\mathfrak{B}_z$, $\mathfrak{C}_z$, and $\mathfrak{D}_z$, in which case the generalized is/so transfer function of $\Sigma$ is given by $\left[\begin{smallmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathfrak{A}_z & \mathfrak{B}_z \\ \mathfrak{C}_z & \mathfrak{D}_z \end{smallmatrix}\right]$. By (6.2) and (6.8),

$$\mathfrak{V}(z) = \mathcal{R}\left(\begin{bmatrix} 1_{\mathcal{X}}+\zeta A_\zeta & \zeta B_\zeta \\ 1_{\mathcal{X}}-(z-\zeta)A_\zeta & -(z-\zeta)B_\zeta \\ C_\zeta & D_\zeta \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right). \tag{6.11}$$

By Lemmas 5.7 and 5.6, $\mathfrak{V}(z)$ has a graph representation of the type (5.16) if and only if the operator $\left[\begin{smallmatrix} 1_{\mathcal{X}}-(z-\zeta)A_\zeta & -(z-\zeta)B_\zeta \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ has a bounded inverse, or equivalently, if and only if $z-\zeta \in \Lambda_{A_\zeta}$, in which case

$$\begin{bmatrix} \mathfrak{A}_z & \mathfrak{B}_z \\ \mathfrak{C}_z & \mathfrak{D}_z \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} + \zeta A_\zeta & \zeta B_\zeta \\ C_\zeta & D_\zeta \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} - (z-\zeta)A_\zeta & -(z-\zeta)B_\zeta \\ 0 & 1_{\mathcal{U}} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} (1+\zeta A_\zeta)(1-(z-\zeta)A_\zeta)^{-1} & \frac{z}{z-\zeta}\mathfrak{B}_\zeta(z-\zeta) \\ \mathfrak{C}_\zeta(z-\zeta) & \mathfrak{D}_\zeta(z-\zeta) \end{bmatrix}.$$

This identity is equivalent to (6.9). □

**Remark 6.4.** If $\Sigma = (V;\mathcal{X},\mathcal{W})$ is a s/s system and $\Omega(\Sigma;\mathcal{U},\mathcal{Y}) \neq \emptyset$ for the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$, then the trajectories $(x(\cdot), w(\cdot))$ satisfy a system of equation of the type

$$\begin{aligned} Ex(n+1) &= Ax(n) + Bu(n), \\ y(n) + Gx(n+1) &= Cx(n) + Du(n), \quad n \in \mathbb{Z}^+, \end{aligned} \tag{6.12}$$

for some bounded linear operators $A$, $B$, $C$, $D$, $E$, and $G$. To see this it suffices to fix some $\zeta \in \Omega(\Sigma;\mathcal{U},\mathcal{Y})$ and define (see (6.6) and the kernel representation of the subspace $V$ given in (6.7))

$$\begin{aligned} E &= 1_{\mathcal{X}} + \zeta A_\zeta, & A &= A_\zeta, & B &= B_\zeta, \\ G &= \zeta C_\zeta, & C &= C_\zeta, & D &= D_\zeta. \end{aligned}$$

Note that with this choice we have the additional relationsships $A = \frac{1}{\zeta}(E - 1_{\mathcal{X}})$ and $C = \frac{1}{\zeta}G$ with the same constant $\zeta$ in both equations. The variable $x(0)$ in (6.12) is free in the sense that for any given $x_0 \in \mathcal{X}$ it is possible to find some trajectory for which $x(0) = x_0$ (but it is not always possible to choose $u(0)$ independently of $x_0$); every s/s system has this property, and it has nothing to do with the assumption that $\Omega(\Sigma;\mathcal{U},\mathcal{Y}) \neq \emptyset$. However, when $\Omega(\Sigma;\mathcal{U},\mathcal{Y}) \neq \emptyset$ it is also true that the input sequence $u(\cdot)$ is free in the sense that to every sequence $u(\cdot)$ there corresponds at least one trajectory of $\Sigma$ (but it is not always possible to choose $x(0)$ independently of $u(0)$); this follows from the range representation of the subspace $V$ given in (6.7). The pair $\left[\begin{smallmatrix} x(0) \\ u(0) \end{smallmatrix}\right]$ is free in the sense that these two variable can be chosen freely independently of each other if and only if the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is admissible, and in this case we may take $E = 1_{\mathcal{X}}$ and $G = 0$.

In the sequel the set of points where the two equivalent conditions in part 1) of Lemma 6.1 hold will be important, and we therefore introduce the following notation:

**Notation 6.5.** Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system. We denote the the set of points $z \in \mathbb{C}$ for which $V(z)$ defined in (6.1) is the generating subspace of a s/s node $\Sigma(z) = (V(z), \mathcal{X}, \mathcal{W})$ by $\Omega(\Sigma)$.

By Lemma 6.1, $0 \in \Omega(\Sigma)$, so that $\Omega(\Sigma) \neq \emptyset$.

**Lemma 6.6.** *The set $\Omega(\Sigma)$ defined above is the union of the sets $\Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ over all i/o decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of the signal space. In particular, $\Omega(\Sigma)$ is an open subset of $\mathbb{C}$.*

*Proof.* By Theorem 5.8 and Lemma 6.1, $\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \subset \Omega(\Sigma)$ for every possible decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. On the other hand, every s/s system has an admissible decomposition, so that $\Omega(\Sigma)$ is contained in the above union. □

**Lemma 6.7.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system with a driving variable representation $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ and an output nulling representation $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}\right)$. Then the following conditions are equivalent.*

1) $z \in \Omega(\Sigma)$,

2) *The following condition is valid for at least one driving variable representation $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ and one output nulling representation $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}\right)$ of $\Sigma$:*

$$\begin{aligned} &\begin{bmatrix} 1_{\mathcal{X}} - zA' & B' \end{bmatrix} \text{ is surjective and} \\ &\begin{bmatrix} 1_{\mathcal{X}} - zA'' \\ C'' \end{bmatrix} \text{ is injective and has closed range.} \end{aligned} \tag{6.13}$$

3) *Condition* (6.13) *is valid for all driving variable representations $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ and all output nulling representations of $\Sigma$.*

*Proof.* It is easy to see that (6.13) can be rewritten in the equvalent form (for all $z \in \mathbb{C}$)

$$\begin{aligned} &\begin{bmatrix} 1_{\mathcal{X}} - zA' & -zB' \end{bmatrix} \text{ is surjective and} \\ &\begin{bmatrix} 1_{\mathcal{X}} - zA'' \\ -zC'' \end{bmatrix} \text{ is injective and has closed range.} \end{aligned} \tag{6.14}$$

By Lemma 6.1, conditions 1)–3) all hold for $z = 0$, so in the sequel we may assume that $z \neq 0$.

If we have two different driving variable representations $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ and $\Sigma^1_{dv/s/s} = \left(\left[\begin{smallmatrix} A'_1 & B'_1 \\ C'_1 & D'_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}_1, \mathcal{W}\right)$ of $\Sigma$, then it follows from Theorem 2.1 that there exists a bounded linear operator $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ K' & M' \end{smallmatrix}\right]$ such that $\begin{bmatrix} 1_{\mathcal{X}} - zA'_1 & -zB'_1 \end{bmatrix} =$

$\begin{bmatrix}1_{\mathcal{X}} - zA' & -zB'\end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ K' & M' \end{bmatrix}$. This implies that if the first half of (6.14) holds for one driving variable representation of $\Sigma$, then it holds for all driving variable representations of $\Sigma$. A similar argument with Theorem 2.1 replaced by Theorem 2.2 shows that if the second half of (6.14) holds for one output nuling representation of $\Sigma$, then it holds for all output nulling representations of $\Sigma$. Thus, 2) and 3) are equivalent, and we may in the sequel assume without loss of generality that the two representations $\Sigma_{dv/s/s}$ and $\Sigma_{s/s/on}$ are induced by one and the same i/s/o representations $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ for some i/o decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ of $\mathcal{W}$. Then (6.3) becomes

$$V(z) = \mathcal{R}\left(\begin{bmatrix} A & B \\ 1_{\mathcal{X}} - zA & -zB \\ C & D \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} zA - 1_{\mathcal{X}} & A & 0 & B \\ zC & C & -1_{\mathcal{Y}} & D \end{bmatrix}\right). \tag{6.15}$$

Below when we refer to conditions (i)–(iv) we mean the conditions listed at the beginning of Section 2. The subspace $V(z)$ is always closed, are required by condition (i), since $V(z)$ is the kernel of a bounded linear operator (see (6.15)). Condition (ii) holds if and only if $\begin{bmatrix}1_{\mathcal{X}} - zA & -zB\end{bmatrix}$ is surjective, and condition (iii) holds if and only if $\left[\begin{smallmatrix} 1_{\mathcal{X}} - zA \\ -zC \end{smallmatrix}\right]$ is injective. Finally, condition (iv) holds if and only if the operator $\left[\begin{smallmatrix} 1_{\mathcal{X}} - zA & -zB \\ C & D \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ has closed range, and this is true if and only if the operator $\left[\begin{smallmatrix} 1_{\mathcal{X}} - zA \\ -zC \end{smallmatrix}\right]$ has closed range. Thus, 1)–3) are equivalent. □

Theorem 6.3 has the following converse:

**Theorem 6.8.** *Let $\mathcal{W}$ be a Kreĭn space, let $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$, let $\zeta \in \mathbb{C}$, and $\left[\begin{smallmatrix} A_\zeta & B_\zeta \\ C_\zeta & D_\zeta \end{smallmatrix}\right] \in \mathcal{B}\left(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]\right)$ be an (arbitrary) four block operator. Define $V$ by any one of the two equivalent formulas given in* (6.7)*. Then $\Sigma := (V; \mathcal{X}, \mathcal{W})$ is a s/s system if and only if the following condition holds:*

$$\begin{aligned} &\begin{bmatrix}1_{\mathcal{X}} + \zeta A_\zeta & B_\zeta\end{bmatrix} \text{ is surjective and} \\ &\begin{bmatrix} 1_{\mathcal{X}} + \zeta A_\zeta \\ C_\zeta \end{bmatrix} \text{ is injective and has closed range.} \end{aligned} \tag{6.16}$$

*When this conditions hods, then $\zeta \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ and formulas* (6.6) *hold, and consequently, all the conclusions of Theorem 6.3 are valid.*

*Proof.* Let $\Sigma_\zeta$ be the s/s system whose i/s/o representation with respect to the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is $\Sigma^\zeta_{i/s/o} = \left(\left[\begin{smallmatrix} A_\zeta & B_\zeta \\ C_\zeta & D_\zeta \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$, and let $V_\zeta$ be the generating subspace of this system. Then the space $V$ defined by (6.7) can be interpreted as $V_\zeta(-\zeta)$, where $V_\zeta(z)$ is defined as in (6.1) with $V$

replaced by $V_\zeta$. The conclusion of Theorem 6.8 now follows from Lemma 6.7 with $\Sigma$ replaced by $\Sigma_\zeta$ and $z$ replaced by $-\zeta$. □

**Theorem 6.9.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma^1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be two s/s system. For all $z \in \Omega(\Sigma, \Sigma^1) := \Omega(\Sigma) \cap \Omega(\Sigma^1)$ we define $\Sigma(z)$ and $\Sigma^1(z)$ as in Lemma 6.1 (where we in the latter case replace $\Sigma$ by $\Sigma^1$). If $\Sigma(z_0)$ and $\Sigma^1(z_0)$ are externally equivalent for some $z_0 \in \Omega(\Sigma, \Sigma^1)$, then $\Sigma(z)$ and $\Sigma^1(z)$ are externally equivalent for each $z$ which belongs to the same connected component of $\Omega(\Sigma, \Sigma^1)$ as $z_0$. In particular, if $\Sigma(z_0)$ and $\Sigma^1(z_0)$ are externally equivalent for some $z_0$ in the connected component of $\Omega(\Sigma, \Sigma^1)$ which contains the origin, then $\Sigma$ and $\Sigma^1$ are externally equivalent.*

*Proof.* Define $\Omega$ by

$$\Omega = \big\{\zeta \in \Omega(\Sigma) \cap \Omega(\Sigma^1) \bigm| \Sigma(\zeta) \text{ is externally equivalent to } \Sigma_1(\zeta)\big\}.$$

We claim that $\Omega$ is both (relatively) open and relatively closed in $\Omega(\Sigma) \cap \Omega(\Sigma^1)$.[6] Assume this for the moment. By the assumption, $z_0 \in \Omega$, and by definition, if we denote the connected component of $\Omega(\Sigma) \cap \Omega(\Sigma^1)$ which contains $z_0$ by $\Omega_{z_0}(\Sigma; \Sigma^1)$, then $\Omega_{z_0}(\Sigma; \Sigma^1)$ is the *smallest* (relatively) open and relatively closed subset of $\Omega(\Sigma) \cap \Omega(\Sigma^1)$ which contains zero. Thus, $\Omega_{z_0}(\Sigma; \Sigma^1) \subset \Omega$, which means that $\Sigma(z)$ is externally equivalent to $\Sigma_1(z)$ for every $z \in \Omega_{z_0}(\Sigma; \Sigma^1)$, proving the main claim of the theorem.

The proof of the openness of $\Omega$ is easy. Indeed, suppose that $\zeta \in \Omega$, i.e., suppose that $\zeta \in \Omega(\Sigma)$ and that $\Sigma(\zeta)$ and $\Sigma^1(\zeta)$ are externally equivalent. Then, by Theorem I.7.7, these systems have a common admissible i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of the signal space, and the corresponding i/o transfer functions $\mathfrak{D}_\zeta$ and $\mathfrak{D}^1_\zeta$ of $\Sigma(\zeta)$ and $\Sigma^1(\zeta)$, respectively, are defined on and coincide in some disk $\mathcal{V} = \{z \in \mathbb{C} \mid |z| < \delta\}$. By Lemma 6.1 and Theorem 6.3, the same decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is admissible for both $\Sigma(\xi)$ and $\Sigma^1(\xi)$ for all $\xi \in \mathcal{V}$, and for each $\xi \in \mathcal{V}$ and $z + \xi - \zeta \in \mathcal{V}$, the corresponding i/o transfer functions of $\Sigma(\xi)$ and $\Sigma^1(\xi)$, evaluated at $z$, are given by $\mathfrak{D}_\xi(z) = \mathfrak{D}_\zeta(z + \xi - \zeta)$ and $\mathfrak{D}^1_\xi(z) = \mathfrak{D}^1_\zeta(z + \xi - \zeta)$, respectively. In particular, for all $\xi$ with $|\xi - \zeta| < \delta/2$ we have $\mathfrak{D}_\xi(z) = \mathfrak{D}^1_\xi(z)$ for all $z$ with $|z| < \delta/2$, and so by Theorem I.7.7, $\Sigma(\xi)$ and $\Sigma^1(\xi)$ are externally equivalent. This proves that $\Omega$ is open.

We next prove that $\Omega$ is a relatively closed subset of $\Omega(\Sigma) \cap \Omega(\Sigma^1)$. Let $\zeta_n \in \Omega$, $n \in \mathbb{Z}^+$, and let $\zeta_n \to \zeta$ as $n \to \infty$ with $\zeta \in \Omega(\Sigma) \cap \Omega(\Sigma^1)$. We claim that this implies that $\zeta \in \Omega$.

[6] In this case "relatively open" is the same as open since $\Omega(\Sigma) \cap \Omega(\Sigma^1)$ is open. "Relatively closed" means that $\overline{\Omega} \cap (\Omega(\Sigma) \cap \Omega(\Sigma^1)) = \Omega$.

Since $\zeta \in \Omega(\Sigma)$, there exists a decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ which is admissible for $\Sigma(\zeta)$, and hence for $\Sigma(\zeta_n)$ for all sufficiently large $n$. Let $\mathfrak{D}_\zeta$ and $\mathfrak{D}_{\zeta_n}$ be the i/o transfer functions of $\Sigma(\zeta)$ and $\Sigma(\zeta_n)$, respectively, with respect to this decomposition. Then by Theorem 6.3, $\mathfrak{D}_{\zeta_n}(0) = \mathfrak{D}_\zeta(\zeta_n - \zeta)$. We can repeat the same argument with $\Sigma$ replaced by $\Sigma^1$ to get another decomposition $\mathcal{W} = \mathcal{Y}' \dotplus \mathcal{U}'$ which is admissible for $\Sigma^1(\zeta)$, hence for $\Sigma^1(\zeta_n)$ for all sufficiently large $n$, and such that the i/o transfer functions $(\mathfrak{D}^1)'_\zeta$ and $(\mathfrak{D}^1)'_{\zeta_n}$ of $\Sigma^1(\zeta)$ and $\Sigma^1(\zeta_n)$, respectively, with respect to this second decomposition satisfy $(\mathfrak{D}^1)'_{\zeta_n}(0) = (\mathfrak{D}^1)'_\zeta(\zeta_n - \zeta)$.

Recall that $\zeta_n \in \Omega$ for all $n$, i.e., that $\Sigma(\zeta_n)$ is externally equivalent to $\Sigma^1(\zeta_n)$. By Theorem I.7.7, $\Sigma(\zeta_n)$ and $\Sigma^1(\zeta_n)$ have the same admissible decompositions and the same feedthrough operators (with respect to all possible decompositions of $\mathcal{W}$). In particular, the decomposition $\mathcal{W} = \mathcal{Y}' \dotplus \mathcal{U}'$ is admissible also for $\Sigma(\zeta_n)$, and $\mathfrak{D}'_{\zeta_n}(0) = (\mathfrak{D}^1)'_{\zeta_n}(0)$ where $\mathfrak{D}'_{\zeta_n}(0)$ is the feedthrough operator of $\Sigma(\zeta_n)$ with respect to this decomposition.

Define $\Theta$ and $\widetilde{\Theta}$ by (3.9) and (3.10), replacing $\mathcal{U}_1$ by $\mathcal{U}'$ and $\mathcal{Y}_1$ by $\mathcal{Y}'$. Then it follows from (3.9) and (3.10) together with Theorem I.6.5 that

$$\mathfrak{D}'_{\zeta_n}(0) = (\widetilde{\Theta}_{11}\mathfrak{D}_{\zeta_n}(0) + \widetilde{\Theta}_{12})(\widetilde{\Theta}_{21}\mathfrak{D}_{\zeta_n}(0) + \widetilde{\Theta}_{22})^{-1}.$$

This combined with the fact that $\Theta$ and $\widetilde{\Theta}$ are inverses of each other implies that

$$\Theta_{21}\mathfrak{D}'_{\zeta_n}(0) + \Theta_{22} = (\widetilde{\Theta}_{21}\mathfrak{D}_{\zeta_n}(0) + \widetilde{\Theta}_{22})^{-1}.$$

Consequently, taking into account that $\mathfrak{D}'_{\zeta_n}(0) = (\mathfrak{D}^1)'_{\zeta_n}(0)$, we get

$$\begin{aligned}
(\Theta_{21}(\mathfrak{D}^1)'_{\zeta_n}(0) + \Theta_{22})(\widetilde{\Theta}_{21}\mathfrak{D}_{\zeta_n}(0) + \widetilde{\Theta}_{22}) &= 1_{\mathcal{U}},\\
(\widetilde{\Theta}_{21}\mathfrak{D}_{\zeta_n}(0) + \widetilde{\Theta}_{22})(\Theta_{21}(\mathfrak{D}^1)'_{\zeta_n}(0) + \Theta_{22}) &= 1_{\mathcal{U}'}.
\end{aligned}$$

As $n \to \infty$ we have $\mathfrak{D}_{\zeta_n}(0) = \mathfrak{D}_\zeta(\zeta_n - \zeta) \to \mathfrak{D}_\zeta(0)$ and $(\mathfrak{D}^1)'_{\zeta_n}(0) = (\mathfrak{D}^1)'_\zeta(\zeta_n - \zeta) \to (\mathfrak{D}^1)'_\zeta(0)$, so by passing to the limit in the two identities above we find that $\widetilde{\Theta}_{21}\mathfrak{D}_{\zeta_n}(0) + \widetilde{\Theta}_{22}$ is invertible (with inverse $\Theta_{21}(\mathfrak{D}^1)'_{\zeta_n}(0) + \Theta_{22}$). Therefore, by Theorem I.6.5, the decomposition $\mathcal{W} = \mathcal{Y}' \dotplus \mathcal{U}'$ is admissible for $\Sigma(\zeta)$ (in addition to being admissible for $\Sigma^1(\zeta)$). If we denote the i/o transfer function of $\Sigma(\zeta)$ with respect to the decomposition $\mathcal{W} = \mathcal{Y}' \dotplus \mathcal{U}'$ by $\mathfrak{D}'_\zeta$ , then both $\mathfrak{D}'_\zeta$ and $(\mathfrak{D}^1)'_\zeta$ are analytic at zero, and

$$\mathfrak{D}'_\zeta(\zeta_n - \zeta) - (\mathfrak{D}^1)'_\zeta(\zeta_n - \zeta) = \mathfrak{D}'_{\zeta_n}(0) - (\mathfrak{D}^1)'_{\zeta_n}(0) = 0,$$

where $\zeta_n - \zeta \to 0$ as $n \to \infty$. This implies that $\mathfrak{D}'_\zeta(z) = (\mathfrak{D}^1)'_\zeta(z)$ for all $z$ in some neighborhood of zero. By Theorem I.7.7, $\Sigma(\zeta)$ and $\Sigma^1(\zeta)$ are externally equivalent, i.e., $\zeta \in \Omega$.

We have now proved that $\Omega$ is both open and relatively closed in $\Omega(\Sigma) \cap \Omega(\Sigma^1)$. As we observed at the beginning of the proof, this implies that $\Sigma(z)$ and $\Sigma^1(z)$ are externally equivalent for each $z$ which belongs to the same connected component of $\Omega(\Sigma, \Sigma^1)$ as $z_0$.

The final claim about the external equivalence of $\Sigma$ and $\Sigma^1$ follows from the fact that $\Sigma(0) = \Sigma$ and $\Sigma^1(0) = \Sigma^1$. □

**Corollary 6.10.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma^1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be two s/s system, let $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$, and suppose that the generalized i/o transfer fuctions $\mathfrak{D}$ and $\mathfrak{D}_1$ coincide in the neighborhood of a point $z_0 \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \cap \Omega(\Sigma_1; \mathcal{U}, \mathcal{Y})$. If $z_0$ belongs to the same connected component of $\Omega(\Sigma) \cap \Omega(\Sigma^1)$ as the origin, then $\Sigma$ and $\Sigma_1$ are externally equivalent, i.e., they induce the same behavior. In particular, if both $\Sigma$ and $\Sigma_1$ is minimal, then $\Sigma$ and $\Sigma_1$ are pseudo-similar.*

*Proof.* By Theorem I.7.7, with the terminology of Theorem 6.9, $\Sigma(z)$ and $\Sigma^1(z)$ are externally equivalent, and by Theorem 6.9, $\Sigma$ and $\Sigma^1$ are externally equivalent. The pseudo-similarity of the two system in the minimal case follows from Proposition I.7.11. □

Our following results refer to the set $\Omega_0(\Sigma)$, which is defined as follows.

**Notation 6.11.** Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system. We denote the connected component of $\Omega(\Sigma)$ which contains the origin by $\Omega_0(\Sigma)$.

**Lemma 6.12.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be s/s system, and define $\Sigma(z)$, $z \in \Omega(\Sigma)$ as in Lemma 6.1. Then the reachable subspace of $\Sigma(z)$ and the unobservable subspace of $\Sigma(z)$ are constant in each connected component of $\Omega(\Sigma)$. In particular, for each $z_0 \in \Omega_0(\Sigma)$, the reachable subspace of $\Sigma(z_0)$ coincides with the reachable subspace of $\Sigma$, and the unobservable subspace of $\Sigma(z_0)$ coincides with the unobservable subspace of $\Sigma$.*

*Proof.* We only prove the claim about the reachable subspace, and leave the (slightly easier) proof of the claim about the constancy of the unobservable subspace to the reader.

For each $\zeta \in \Omega(\Sigma)$, let us denote the reachable subspace of $\Sigma(\zeta)$ by $\mathfrak{R}_\zeta$. We claim that the subspace $\mathfrak{R}_\zeta$ is locally constant in the sense that for each $\zeta \in \Omega(\Sigma)$ there is neighborhood $\mathcal{V}$ of $\zeta$ such that $\mathfrak{R}_\xi = \mathfrak{R}_\zeta$ for all $\xi \in \mathcal{V}$.

Let $\zeta \in \Omega(\Sigma)$, and let $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ be an admissible decomposition for $\Sigma(\zeta)$. Then by Lemma 6.1 and Theorem 6.3, the same decomposition is admissible for $\Sigma(\xi)$ for all $\xi$ in some disk $\mathcal{V} = \{\xi \in \mathbb{C} \mid |\xi| < \delta\}$. Let $\Sigma^\xi_{i/s/o} = \left(\begin{bmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be the corresponding i/s/o representation of $\Sigma(\xi)$, and define $\widetilde{\mathfrak{B}}_\xi(z) = (1_\mathcal{X} - zA_\xi)^{-1} B_\xi$, $\xi \in \mathcal{V}$, $z \in \Lambda_{A_\xi}$. By Corollary I.5.5,

$\mathfrak{R}_\zeta = \vee_{k\in\mathbb{Z}^+}\mathcal{R}\left(A_\zeta^k B_\zeta\right)$ (here $\vee$ stands for the closed linar span). Equivalently, $\mathfrak{R}_\zeta = \vee_{k\in\mathbb{Z}^+}\mathcal{R}\left(\widetilde{\mathfrak{B}}_\zeta^{(k)}(0)\right)$. Thus, $x_* \in \mathfrak{R}_\zeta^\perp$ if and only if $(\widetilde{\mathfrak{B}}_\zeta^{(k)}(0)x, x_*)_\mathcal{X} = 0$ for all $x \in \mathcal{X}$ and all $k \in \mathbb{Z}^+$. By the analyticity of the function $z \mapsto (\widetilde{\mathfrak{B}}_\zeta^{(k)}(z)x, x_*)_\mathcal{X}$, this is equivalent to the requirement that $(\widetilde{\mathfrak{B}}_\zeta^{(k)}(z)x, x_*)_\mathcal{X} = 0$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{V}$. By Theorem 6.3, for each $\xi \in \mathcal{V}$ and $z+\xi-\zeta \in \mathcal{V}$, we have $\widetilde{\mathfrak{B}}_\xi(z) = \widetilde{\mathfrak{B}}_\zeta(z+\xi-\zeta)$. In particular, this implies that for all $\xi$ with $|\xi-\zeta| < \delta/2$ we have $x_* \in \mathfrak{R}_\zeta^\perp$ if and only if $(\widetilde{\mathfrak{B}}_\zeta^{(k)}(z)x, x_*)_\mathcal{X} = 0$ for all $x \in \mathcal{X}$ and all $z$ with $|z| < \delta/2$, or eqivalently, if and only if $(\mathfrak{B}_\xi{}^{(k)}(0)x, x_*)_\mathcal{X} = 0$ for all $x \in \mathcal{X}$ and all $k \in \mathbb{Z}^+$. The last condition is equivalent to $x_* \in \mathfrak{R}_\xi^\perp$. Thus, $\mathfrak{R}_\zeta^\perp = \mathfrak{R}_\xi^\perp$, and so $\mathfrak{R}_\zeta = \mathfrak{R}_\xi$. This proves our claim that the space $\mathfrak{R}_\zeta$ is locally constant.

Fix some $z_0 \in \Omega(\Sigma)$, and define

$$\Omega_{z_0} = \left\{\zeta \in \Omega(\Sigma) \;\middle|\; \mathfrak{R}_\zeta = \mathfrak{R}_{z_0}\right\}.$$

The fact that $\mathfrak{R}_\zeta$ is locally constant implies that $\Omega_{z_0}$ is both open and relatively closed in $\Omega(\Sigma)$. As in the proof of Theorem 6.9 we conclude that $\Omega_{z_0}$ contains the full connected component of $\Sigma(\Omega)$ to which $z_0$ belongs. The final claim of Lemma 6.12 follows from the fact that $\Sigma(0) = \Sigma$. □

**Proposition 6.13.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be s/s system, let $z_0 \in \Omega_0(\Sigma)$, and define $\Sigma(z_0)$ as in Lemma 6.1.*

1) *$\Sigma$ is controllable if and only if $\Sigma(z_0)$ is controllable,*

2) *$\Sigma$ is observable if and only if $\Sigma(z_0)$ is observable,*

3) *$\Sigma$ is minimal if and only if $\Sigma(z_0)$ is minimal.*

*Proof.* This follows from Lemma 6.12. □

**Proposition 6.14.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be s/s system, and let $\mathfrak{B}$ be the generalized i/s (input/state) transfer function of $\Sigma$, corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$. Then, for all $z_0 \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \cap \Omega_0(\Sigma)$ the reachable subspace of $\Sigma$ is the closed linear span of $\{\mathcal{R}\left(\mathfrak{B}^{(n)}(z_0)\right) \mid n \in \mathbb{Z}^+\}$. It is also equal to the closed linear span of $\{\mathcal{R}\left(\mathfrak{B}(z)\right) \mid z \in \Omega\}$, where $\Omega$ is an arbitrary open subset of $\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \cap \Omega_0(\Sigma)$.*

*Proof.* By Lemma 6.1, the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is admissible for the system $\Sigma(z_0)$. Denote the i/s transfer function of $\Sigma_{z_0}$ by $\mathfrak{B}_{z_0}$, define $\widetilde{\mathfrak{B}}_{z_0}(z) = z^{-1}\mathfrak{B}_{z_0}(z)$ for $z \neq 0$, and $\widetilde{\mathfrak{B}}_{z_0}(0) = \lim_{z\to 0} z^{-1}\mathfrak{B}_{z_0}(z)$. As we observed in the proof of Lemma 6.12, the reachable subspace $\mathfrak{R}_{z_0}$ of

$\Sigma(z_0)$ is given by $\mathfrak{R}_{z_0} = \vee_{k\in\mathbb{Z}^+}\mathcal{R}\left(\widetilde{\mathfrak{B}}_{z_0}^{(k)}(0)\right)$, which according to Theorem 6.3 is equal to $\vee_{k\in\mathbb{Z}^+}\mathcal{R}\left(\widetilde{\mathfrak{B}}^{(k)}(z_0)\right)$, where $\widetilde{\mathfrak{B}}(z) = z^{-1}\mathfrak{B}(z)$ for $z \neq 0$, and $\widetilde{\mathfrak{B}}(0) = \lim_{z\to 0} z^{-1}\mathfrak{B}(z)$. It is not difficult to show that this is equal to $\vee_{k\in\mathbb{Z}^+}\mathcal{R}\left(\mathfrak{B}^{(k)}(z_0)\right)$. Thus, by Lemma 6.12, the reachable subspace of $\Sigma$ is equal to $\vee_{k\in\mathbb{Z}^+}\mathcal{R}\left(\mathfrak{B}^{(k)}(z_0)\right)$, as claimed.

To prove the final claim it suffices to show that $\mathfrak{R}_{z_0} = \vee_{z\in\mathcal{V}}\mathcal{R}\left(\mathfrak{B}(z)\right)$ whenever $\mathcal{V}$ is a disk centered at $z_0$ and contained in $\Omega(\Sigma;\mathcal{U},\mathcal{Y})$. That $\vee_{z\in\mathcal{V}}\mathcal{R}\left(\mathfrak{B}(z)\right) \subset \mathfrak{R}_{z_0}$ follows immediately from Lemma 6.12, and to prove the opposite inclusion one uses the same technique as we used in the proof of Lemma 6.12 (we have $(\widetilde{\mathfrak{B}}_{z_0}^{(k)}(z)x, x_*)_{\mathcal{X}} = 0$ for all $x \in \mathcal{X}$ and all $z \in \mathcal{V}$ if and only if $x_*$ is orthogonal to $\mathcal{R}\left(\widetilde{\mathfrak{B}}_{z_0}^{(k)}(z)\right)$ for all $z \in \mathcal{V}$). We leave the completion of the proof to the reader. □

**Proposition 6.15.** *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be s/s system, and let $\mathfrak{C}$ be the generalized s/o (state/output) transfer function of $\Sigma$, corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. Then, for all $z \in \Omega(\Sigma;\mathcal{U},\mathcal{Y}) \cap \Omega_0(\Sigma)$, the unobservable subspace of $\Sigma$ is given by $\mathfrak{U} = \cap_{n\in\mathbb{Z}^+}\mathcal{N}\left(\mathfrak{C}^{(n)}(z_0)\right)$, and also by $\mathfrak{U} = \cap_{z\in\Omega}\mathcal{N}\left(\mathfrak{C}(z)\right)$, where $\Omega$ is an arbitrary open subset of $\Omega(\Sigma;\mathcal{U},\mathcal{Y})\cap\Omega_0(\Sigma)$.*

*Proof.* This proof is similar to the proof of Proposition 6.14 (but slightly simpler), and we leave it to the reader. □

**Proposition 6.16.** *If $\Sigma = (V;\mathcal{X},\mathcal{W})$ is a s/s system with a finite-dimensional state space $\mathcal{X}$, then for each decomposition $\mathcal{W} = \mathcal{Y} \oplus \mathcal{U}$ the set $\Omega(\Sigma;\mathcal{U},\mathcal{Y})$ is either empty or its complement contains at most $\dim\mathcal{X}$ points. In the latter case the generalized is/so transfer function is rational. If $\Sigma$ is minimal and $\Omega(\Sigma;\mathcal{U},\mathcal{Y}) \neq \emptyset$, then the McMillan degree of the generalized i/o transfer function is equal to $\dim\mathcal{X}$.*

*Proof.* We shall see later an example where $\Omega(\Sigma;\mathcal{U},\mathcal{Y}) = \emptyset$, so let us suppose that $\Omega(\Sigma;\mathcal{U},\mathcal{Y}) \neq \emptyset$. Pick some $\zeta \in \Omega(\Sigma;\mathcal{U},\mathcal{Y})$, and let $\Sigma_{i/s/o}^{\zeta}$ be the i/s/o system defined in Part 2) of Theorem 6.3. Since $\dim\mathcal{X}$ is finite, it follows immeditately that the (standard) is/so transfer function of $\Sigma_{i/s/o}^{\zeta}$ is rational, and hence, by Theorem 6.3, the generalized is/so transfer function of $\Sigma$ is rational. Moreover, the McMillan degree of the generalized i/o transfer function of $\Sigma$ is equal to the McMillan degree of the i/o transfer function of $\Sigma_{i/s/o}^{\zeta}$, which is equal to $\dim\mathcal{X}$ if $\Sigma_{i/s/o}^{\zeta}$ is minimal. Since $\dim\mathcal{X} < \infty$, the set $\Omega_0(\Sigma)$ is the whole complex plane with the possible exception of a finite number of points, and it follows from Propositions 6.14 and 6.15 that the system $\Sigma_{i/s/o}^{\zeta}$ is minimal (i.e., controllable and observable) if and only if $\Sigma$ is minimal. □

**Proposition 6.17.** *Every rational $\mathcal{B}(\mathcal{U},\mathcal{Y})$-valued function $\mathfrak{D}$ with finite-dimensional $\mathcal{U}$ and $\mathcal{Y}$ has a minimal s/s realization $\Sigma(V;\mathcal{X},\mathcal{W})$ where $\dim\mathcal{X}$ is equal to the McMillan degree of $\mathfrak{D}$ and $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$. This realization is unique up to a similarity transform in the state space.*

*Proof.* Choose some point $\zeta$ which is not a pole of $\mathfrak{D}$. As is well-known, the function $z\mapsto\mathfrak{D}(\zeta+z)$ has a minimal i/s/o realization $\Sigma_{i/s/o}^{\zeta}=\left(\left[\begin{smallmatrix}A_\zeta & B_\zeta\\ C_\zeta & D_\zeta\end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$, where $\dim\mathcal{X}$ is equal to the McMillan degree of $\mathfrak{D}$. Since the realization is minimal it is true for all $z\in\mathbb{C}$ that $\begin{bmatrix}1_{\mathcal{X}}-zA_\zeta & B_\zeta\end{bmatrix}$ is surjective and that $\left[\begin{smallmatrix}1_{\mathcal{X}}-zA_\zeta\\ C_\zeta\end{smallmatrix}\right]$ has a bounded left-inverse (see, e.g., [Sta05, Lemma 9.6.6], and recall that $\mathcal{X}$ is now finite-dimensional). Let $\Sigma$ be the s/s system constructed in Theorem 6.8. The uniqueness claim follows from the fact that any two minimal externally equivalent s/s systems are pseudo-similar (see Theorem I.7.11), which in the case of a finite-dimensional state space is equivalent to similarity. □

**Example 6.18.** We finish this section with a simple finite-dimensional example. Let $\Sigma_{i/s/o}=(\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right];\mathbb{C}^2,\mathbb{C}^2,\mathbb{C}^2)$ be the i/s/o system whose coefficient matrix is

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\left[\begin{array}{cc|cc}0&0&1&0\\0&0&0&1\\ \hline 1&0&0&0\\0&1&0&0\end{array}\right],$$

and let $\Sigma=(V;\mathbb{C}^2,\mathbb{C}^4)$ be the s/s system with signal space $\mathbb{C}^4=\left[\begin{smallmatrix}\mathbb{C}^2\\ C^2\end{smallmatrix}\right]$ for which $\Sigma_{i/s/o}$ is an i/s/o representation. It is easy to see that $\Sigma_{i/s/o}$ is minimal, hence so is $\Sigma$. The is/so transfer function of this system, defined everywhere i $\mathbb{C}$ (except the point at infinity), is given by

$$\begin{bmatrix}\mathfrak{A}(z) & \mathfrak{B}(z)\\ \mathfrak{C}(z) & \mathfrak{D}(z)\end{bmatrix}=\left[\begin{array}{cc|cc}1&0&z&0\\0&1&0&z\\ \hline 1&0&z&0\\0&1&0&z\end{array}\right],$$

and the generating subsapce $V$ of $\Sigma$ has the representations

$$V=\mathcal{R}\left(\left[\begin{array}{cc|cc}0&0&1&0\\0&0&0&1\\ \hline 1&0&0&0\\0&1&0&0\\ \hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\right)=\mathcal{N}\left(\left[\begin{array}{cc|cc|cccc}-1&0&0&0&0&0&1&0\\0&-1&0&0&0&0&0&1\\ \hline 0&0&1&0&-1&0&0&0\\0&0&0&1&0&-1&0&0\end{array}\right]\right).$$

Let us write $w\in\mathbb{C}^4$ as a column vector with elements $w_i$, $i=1,2,3,4$. In our original representation we take $\left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right]$ to be the output and $\left[\begin{smallmatrix}w_3\\ w_4\end{smallmatrix}\right]$ to be the input. If we instead take, e.g., $\left[\begin{smallmatrix} w_1 \\ w_3 \end{smallmatrix}\right]$ to be the output and $\left[\begin{smallmatrix} w_2 \\ w_4 \end{smallmatrix}\right]$ to be the input, then the domain of the corresponding is/so transfer function consists of those points where the matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & -z & 0 \\ 0 & 1 & 0 & -z \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

is invertible. However, the determinant of this matrix is identically zero, so that the generalized is/so transfer function corresponding to this i/o decomposition of $\mathcal{W}$ is nowhere defined. On the other hand, if we, e.g., take $\left[\begin{smallmatrix} w_1 \\ w_4 \end{smallmatrix}\right]$ to be the output and $\left[\begin{smallmatrix} w_2 \\ w_3 \end{smallmatrix}\right]$ to be the input, then the domain of the generalized is/so transfer function is determined by the determinant of the matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & -z & 0 \\ 0 & 1 & 0 & -z \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

which is equal to $-z$. Thus, in this case the generalized is/so transfer function is defined for all $z \neq 0$ (and $z \neq \infty$), and it is given by

$$\left[\begin{array}{cc|cc} 1 & 0 & z & 0 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & z & 0 \\ 0 & -\frac{1}{z} & 0 & \frac{1}{z} \end{array}\right].$$

Finally, if we choose the input vector to be $w$ multiplied by $\left[\begin{smallmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{smallmatrix}\right]$ where $\alpha$ and $\beta$ are two linearly independent vectors, then the domain of the generalized transfer function is determined by the the determinant of the matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & -z & 0 \\ 0 & 1 & 0 & -z \\ \hline \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{array}\right]$$

which is equal to

$$(\alpha_1\beta_2 - \beta_2\alpha_1)z^2 + (\alpha_3\beta_2 - \alpha_2\beta_3 + \alpha_1\beta_4 - \alpha_4\beta_1)z + (\alpha_3\beta_4 - \alpha_4\beta_3).$$

By adjusting the parameters of $\alpha$ and $\beta$ we can make this determinant vanish at any two predescribed points in the complex plane, and thereby get an example where $\Omega(\Sigma;\mathcal{U},\mathcal{Y})$ consist of the whole complex plane minus the set consisting of the two prescribed points. Note that a decomposition $\mathbb{C}^4 = \mathcal{Y}\dotplus\mathcal{U}$ with this input space $\mathcal{U}$ is admissible (i.e., $0\in\Omega(\Sigma;\mathcal{U},\mathcal{Y})$) if and only if $\alpha_3\beta_4\neq\alpha_4\beta_3$.

# 7 Affine Representations and Generalized Si/So Transfer Functions of Adjoint Systems

In this section we shall first study relations between right and left affine i/s/o representations of a s/s system $\Sigma=(V;\mathcal{X},\mathcal{W})$, corresponding to the decomposition $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$, and the left and right affine i/s/o representations of the adjoint s/s system $\Sigma_*=(V_*;\mathcal{X},\mathcal{W}_*)$, corresponding to a decomposition $\mathcal{W}_*=\mathcal{U}_*\dotplus\mathcal{Y}_*$, where $\mathcal{U}_*$ and $\mathcal{Y}_*$ are constructed from $\mathcal{U}$ and $\mathcal{Y}$ as described in the equivalent formulas (2.21) and (2.22).

**Theorem 7.1.** *Let $\Sigma=(V;\mathcal{X},\mathcal{W})$ be a s/s system, let $\Sigma^r_{i/s/o}$ given by* (1.7) *be a a right affine i/s/o representation, and let $\Sigma^l_{i/s/o}$ given by* (1.12) *be a left affine i/s/o representation of $\Sigma$, corresponding to the i/o decomposition $\mathcal{W}=\mathcal{Y}\dotplus\mathcal{U}$. Define $\mathcal{U}_*$ and $\mathcal{Y}_*$ by* (2.21)*. Then $\mathcal{W}_*=\mathcal{U}_*\dotplus\mathcal{Y}_*$,*

$$(\Sigma_*)^l_{i/s/o}=\left(\left[\begin{array}{c|cc}(A')^* & -(C'_\mathcal{U})^\dagger & -(C'_\mathcal{Y})^\dagger\\ -(B')^* & (D'_\mathcal{U})^\dagger & (D'_\mathcal{Y})^\dagger\end{array}\right];\mathcal{X},\left[\begin{smallmatrix}\mathcal{U}_*\\ \mathcal{Y}_*\end{smallmatrix}\right],\mathcal{L}\right) \tag{7.1}$$

*is a left affine i/s/o representation and*

$$(\Sigma_*)^r_{i/s/o}=\left(\left[\begin{array}{cc}(A'')^* & (C'')^*\\ \hline (B''_\mathcal{U})^\dagger & (D''_\mathcal{U})^\dagger\\ (B''_\mathcal{Y})^\dagger & (D''_\mathcal{Y})^\dagger\end{array}\right];\mathcal{X},\mathcal{K},\left[\begin{smallmatrix}\mathcal{U}_*\\ \mathcal{Y}_*\end{smallmatrix}\right]\right) \tag{7.2}$$

*is a right affine i/s/o representation of the adjoint system $\Sigma_*=(V_*;\mathcal{X},\mathcal{W}_*)$, corresponding to the i/o decomposition $\mathcal{W}_*=\mathcal{U}_*\dotplus\mathcal{Y}_*$. In the computations of the adjoints we identify the duals of $\mathcal{L}$, $\mathcal{K}$, and $\mathcal{X}$ with themselves, and the duals of $\mathcal{U}$ and $\mathcal{Y}$ are identified with $\mathcal{U}_*$ and $\mathcal{Y}_*$ via the duality pairings* (2.23) *and* (2.24)*.*

*Proof.* That (7.1) is a left affine i/s/o representation of $\Sigma_*$ follows from Proposition II.4.10, after we realize that the adjoints $(C')^\dagger$ and $(D')^\dagger$ in that proposition split into $(C')^\dagger=\left[(C'_\mathcal{U})^\dagger\ \ (C'_\mathcal{Y})^\dagger\right]$ and $(D')^\dagger=\left[(D'_\mathcal{U})^\dagger\ \ (D'_\mathcal{Y})^\dagger\right]$ because of (2.23) and (2.24) and the facts that $\mathcal{Y}_*$ is orthogonal to $\mathcal{U}$ and $\mathcal{U}_*$ is orthogonal to $\mathcal{Y}$. In the same way it follows from the same proposition that (7.2) is a right affine i/s/o representation of $\Sigma_*$, once we realize that the adjoints $(B'')^\dagger$ and $(D'')^\dagger$ in that proposition split into $(B'')^\dagger = \left[\begin{smallmatrix} (B''_{\mathcal{U}})^\dagger \\ (B''_{\mathcal{Y}})^\dagger \end{smallmatrix}\right]$ and $(D'')^\dagger = \left[\begin{smallmatrix} (D''_{\mathcal{U}})^\dagger \\ (D''_{\mathcal{Y}})^\dagger \end{smallmatrix}\right]$. □

**Theorem 7.2.** *Under the assumption of Theorem 7.1 the left and right affine is/so transfer functions corresponding to the left and right i/s/o representations $(\Sigma_*)^l_{i/s/o}$ and $(\Sigma_*)^r_{i/s/o}$ of $\Sigma_*$ are given in terms of the right and left is/so transfer functions* (3.7) *and* (4.6) *by*

$$\left[\begin{array}{c|cc} \mathfrak{A}'(\overline{z})^* & -\mathfrak{C}'_{\mathcal{U}}(\overline{z})^\dagger & -\mathfrak{C}'_{\mathcal{Y}}(\overline{z})^\dagger \\ -\mathfrak{B}'(\overline{z})^* & \mathfrak{D}'_{\mathcal{U}}(\overline{z})^\dagger & \mathfrak{D}'_{\mathcal{Y}}(\overline{z})^\dagger \end{array}\right], \qquad \overline{z} \in \Lambda_{A'}, \tag{7.3}$$

*and*

$$\left[\begin{array}{cc} \mathfrak{A}''(\overline{z})^* & \mathfrak{C}''(\overline{z})^* \\ \hline \mathfrak{B}''_{\mathcal{U}}(\overline{z})^\dagger & \mathfrak{D}''_{\mathcal{U}}(\overline{z})^\dagger \\ \mathfrak{B}''_{\mathcal{Y}}(\overline{z})^\dagger & \mathfrak{D}''_{\mathcal{Y}}(\overline{z})^\dagger \end{array}\right], \qquad \overline{z} \in \Lambda_{A''}, \tag{7.4}$$

*respectively.*

*Proof.* This follows immediately from Theorem 7.1. □

**Theorem 7.3.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system, and let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a decomposition of $\mathcal{W}$ for which the correspondig set $\Omega(\Sigma; \mathcal{U}, \mathcal{Y})$ is nonempty. Define $\mathcal{U}_*$ and $\mathcal{Y}_*$ by* (2.21)*. Then $\mathcal{W}_* = \mathcal{U}_* \dotplus \mathcal{Y}_*$,*

$$\Omega(\Sigma_*; \mathcal{Y}_*, \mathcal{U}_*) = \left\{ z \in \mathbb{C} \;\middle|\; \overline{z} \in \Omega(\Sigma; \mathcal{U}, \mathcal{Y}) \right\}$$

*(in particular, $\Omega(\Sigma_*; \mathcal{Y}_* \mathcal{U}) \neq \emptyset$), and the generalized is/so transfer functions of $\Sigma$ and $\Sigma_*$, corresponding to the decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ and $\mathcal{W}_* = \mathcal{U}_* \dotplus \mathcal{Y}_*$, respectively, are connected by the relation*

$$\begin{bmatrix} \mathfrak{A}_*(z) & \mathfrak{B}_*(z) \\ z\mathfrak{C}_*(z) & \mathfrak{D}_*(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(\overline{\zeta} z)^* & -z\mathfrak{C}(\overline{z})^\dagger \\ \mathfrak{B}(\overline{z})^* & -\mathfrak{D}(\overline{z})^\dagger \end{bmatrix}, \qquad z \in \Omega(\Sigma_*; \mathcal{Y}_*, \mathcal{U}_*), \tag{7.5}$$

*where the blocks on the left-hand side are taken from the generalized is/so transfer function of $\Sigma_*$ and the blocks on the right-hand side are taken from the generalized is/so transfer function of $\Sigma$*

*Proof.* Define $\Sigma^r_{i/s/o}$, $\Sigma^l_{i/s/o}$, $(\Sigma_*)^l_{i/s/o}$, and $(\Sigma_*)^l_{i/s/o}$ as in Theorem 7.1. By (1.10), $\overline{z} \in \Omega(\Sigma; \mathcal{Y}, \mathcal{U})$ if and only if $\left[\begin{smallmatrix} 1_{\mathcal{X}} - \overline{z}A' & -\overline{z}B' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{smallmatrix}\right]$ has a bounded inverse. By (1.14), applied to the left affine i/s/o representation $(\Sigma_*)^l_{i/s/o}$ of $\Sigma_*$, $z \in \Omega(\Sigma_*; \mathcal{U}_*, \mathcal{Y}_*)$ if and only if $\left[\begin{smallmatrix} 1_{\mathcal{X}} - z(A')^* & z(C'_{\mathcal{U}})^\dagger \\ (B')^* & -(D'_{\mathcal{U}})^\dagger \end{smallmatrix}\right]$ has a bounded inverse. This is true if and only if the adjoint $\left[\begin{smallmatrix} 1_{\mathcal{X}} - \overline{z}A' & B' \\ \overline{z}C'_{\mathcal{U}} & -D'_{\mathcal{U}} \end{smallmatrix}\right]$ of this operator is invertible.

In particular, $0 \in \Omega(\Sigma;\mathcal{U},\mathcal{Y})$ if and only if $0 \in \Omega(\Sigma_*;\mathcal{Y}_*,\mathcal{U}_*)$, because both these conditions are equivalent to the bounded invertibility of $D'_{\mathcal{U}}$. For $z \neq 0$ we observe that

$$\begin{bmatrix} \overline{z} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} - \overline{z}A' & B' \\ \overline{z}C'_{\mathcal{U}} & -D'_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} 1/\overline{z} & 0 \\ 0 & -1_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} - \overline{z}A' & -\overline{z}B' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix},$$

and hence also for $z \neq 0$ we have $z \in \Omega(\Sigma_*;\mathcal{U}_*,\mathcal{Y}_*)$ if and only if $\overline{z} \in \Omega(\Sigma;\mathcal{Y},\mathcal{U})$.

A similar argument can be used to prove (7.5). We have from (1.11) that

$$\begin{bmatrix} \overline{z}\mathfrak{A}(\overline{z}) & \mathfrak{B}(\overline{z}) \\ \overline{z}\mathfrak{C}(\overline{z}) & \mathfrak{D}(\overline{z}) \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C'_{\mathcal{Y}} & D'_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} - \overline{z}A' & -\overline{z}B' \\ C'_{\mathcal{U}} & D'_{\mathcal{U}} \end{bmatrix}^{-1} \begin{bmatrix} \overline{z} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}, \quad \overline{z} \in \Omega(\Sigma;\mathcal{U},\mathcal{Y}).$$

Passing to adjoints we have for $z \neq 0$,

$$\begin{aligned} \begin{bmatrix} z\mathfrak{A}(\overline{z})^* & z\mathfrak{C}(\overline{z})^\dagger \\ \mathfrak{B}(\overline{z})^\dagger & \mathfrak{D}(\overline{z})^\dagger \end{bmatrix} &= \begin{bmatrix} z & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} - z(A')^* & (C'_{\mathcal{U}})^\dagger \\ -z(B')^* & (D'_{\mathcal{U}})^\dagger \end{bmatrix}^{-1} \begin{bmatrix} 1_{\mathcal{X}} & (C'_{\mathcal{Y}})^\dagger \\ 0 & (D'_{\mathcal{Y}})^\dagger \end{bmatrix} \\ &= \begin{bmatrix} 1/z - (A')^* & (C'_{\mathcal{U}})^\dagger \\ (B')^* & -(D'_{\mathcal{U}})^\dagger \end{bmatrix}^{-1} \begin{bmatrix} 1_{\mathcal{X}} & (C'_{\mathcal{Y}})^\dagger \\ 0 & -(D'_{\mathcal{Y}})^\dagger \end{bmatrix}. \end{aligned}$$

On the other hand, (1.15), applied to the left affine i/s/o representation $(\Sigma_*)^l_{i/s/o}$ of $\Sigma_*$, gives

$$\begin{aligned} \begin{bmatrix} z\mathfrak{A}_*(z) & \mathfrak{B}_*(z) \\ z\mathfrak{C}_*(z) & \mathfrak{D}_*(z) \end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{X}} - z(A')^* & z(C'_{\mathcal{U}})^\dagger \\ (B')^* & -(D'_{\mathcal{U}})^\dagger \end{bmatrix}^{-1} \begin{bmatrix} 1_{\mathcal{X}} & -z(C'_{\mathcal{Y}})^\dagger \\ 0 & (D'_{\mathcal{Y}})^\dagger \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \\ &= \begin{bmatrix} 1/z - (A')^* & (C'_{\mathcal{U}})^\dagger \\ (B')^* & -(D'_{\mathcal{U}})^\dagger \end{bmatrix}^{-1} \begin{bmatrix} 1_{\mathcal{X}} & -(C'_{\mathcal{Y}})^\dagger \\ 0 & (D'_{\mathcal{Y}})^\dagger \end{bmatrix}. \end{aligned}$$

Comparing this formula to the one above we find that (7.5) holds for $z \neq 0$. We leave the easy proof of the fact that (7.5) holds for $z = 0$ to the reader. □

**Proposition 7.4.** *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s system, and let $\mathcal{W} = -\mathcal{Y}[\dotplus]\mathcal{U}$ be an orthogonal i/o decomposition of $\mathcal{W}$. Let $\Sigma_* = (V_*;\mathcal{X},\mathcal{W})$ be the adjoint s/s system, and decompose $\mathcal{W}_*$ into $\mathcal{W}_* = -\mathcal{U}\,[\dotplus]\,\mathcal{Y}$. Then the domain $\Omega(\Sigma;\mathcal{U},\mathcal{Y})$ of the generalized is/so transfer function of $\Sigma$ and the domain $\Omega(\Sigma_*;\mathcal{Y},\mathcal{U})$ of the generalized is/so transfer function of $\Sigma_*$ satisfy*

$$\Omega(\Sigma_*;\mathcal{Y},\mathcal{U}) = \{z \in \mathbb{C} \mid \overline{z} \in \Omega(\Sigma;\mathcal{U},\mathcal{Y})\}, \tag{7.6}$$

*and the corresponding is/so transfer functions satisfy*

$$\begin{bmatrix} \mathfrak{A}_*(z) & \mathfrak{B}_*(z) \\ z\mathfrak{C}_*(z) & \mathfrak{D}_*(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(\overline{z})^* & z\mathfrak{C}(\overline{z})^* \\ \mathfrak{B}(\overline{z})^* & \mathfrak{D}(\overline{z})^* \end{bmatrix}, \quad z \in \Omega(\Sigma_*;\mathcal{Y},\mathcal{U}). \tag{7.7}$$

*Proof.* If $0 \in \Omega(\Sigma;\mathcal{U},\mathcal{Y})$ then, by Theorem 5.2, the decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ is admissible, $\Omega(\Sigma;\mathcal{U},\mathcal{Y}) = \Lambda_A$ (where $A$ is the corresponding main operator), and the claim follows from Proposition II.4.11 and the definition of the standard is/so transfer function.

In the general case this proposition can be deduced from Theorem 7.3 as follows by taking $\mathcal{U}_* = \mathcal{Y}^{\langle\perp\rangle} = \mathcal{U}$ and $\mathcal{Y}_* = \mathcal{U}^{\langle\perp\rangle} = \mathcal{Y}$. If we use a superscript $*$ to represent adjoints with respect to the inner products in $\mathcal{U}$ and $\mathcal{Y}$ (as opposed to the adjoints with respect to the duality pairings used in Theorem 7.3), then we get $C^\dagger = -C$ and $D^\dagger = -C^*$, because for all $y \in \mathcal{Y}$ (which we identify with the vector $\mathcal{I}^{-1}y \in \mathcal{W}_*$), $x \in \mathcal{X}$, and $u \in \mathcal{U}$ we have

$$\begin{aligned}(x, C^\dagger y)_{\mathcal{X}} &= [Cx, y]_{\mathcal{W}} = -[Cx, y]_{\mathcal{Y}} = -(x, C^* y)_{\mathcal{X}},\\ [u, D^\dagger y]_{\mathcal{U}} &= [Du, y]_{\mathcal{W}} = -[Du, y]_{\mathcal{Y}} = -(u, D^* y)_{\mathcal{X}}. \quad \square\end{aligned}$$

Our following proposition is a non-orthogonal version of Proposition II.4.11 and Theorem III.5.3.

**Proposition 7.5.** *Let $\Sigma = (V, \mathcal{X}, \mathcal{W})$ be a s/s node, and let $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ be an admissible decomposition of $\mathcal{W}$, with the corresponding i/s/o representation $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ of $\Sigma$. Define $\mathcal{Y}_*$ and $\mathcal{U}_*$ by (2.22). Then $\mathcal{W}_* = \mathcal{U}_* \dot{+} \mathcal{Y}_*$ is an admissible decomposition of $\mathcal{W}_*$ for the adjoint s/s node $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$ of $\Sigma$, and $(\Sigma_*)_{i/s/o} = \left(\left[\begin{smallmatrix} A^* & -C^\dagger \\ B^\dagger & -D^\dagger \end{smallmatrix}\right]; \mathcal{X}, \mathcal{Y}_*, \mathcal{U}_*\right)$ is an i/s/o representation of $\Sigma_*$.*

*Proof.* We already proved above in Lemma 2.3 that $\mathcal{W}_* = \mathcal{Y}_* \dot{+} \mathcal{U}_*$.

Substitute $z = Ax + Bu$ and $y = Cx + Du$ in (2.25) and use (2.25) and the orthogonality between $V$ and $V_*$ to show that $\left[\begin{smallmatrix} z_* \\ x_* \\ w_* \end{smallmatrix}\right] \in V_*$ with $w_* = y_* + u_*$, $y_* \in \mathcal{Y}$, $u_* \in \mathcal{U}_*$, if and only if

$$-(Ax + Bu, x_*)_{\mathcal{X}} + (x, z_*)_{\mathcal{X}} + \langle Cx + Du, y_* \rangle_{\langle \mathcal{Y}, \mathcal{Y}_* \rangle} + \langle u, u_* \rangle_{\langle \mathcal{U}, \mathcal{U}_* \rangle} = 0$$

for all $x \in \mathcal{X}$ and all $u \in \mathcal{U}$. Passing to adjoints we get the equivalent condition

$$\langle x, z_* - A^* x_* + C^\dagger y_* \rangle_{\mathcal{X}} + \langle u, -B^\dagger x_* + D^\dagger y_* + u_* \rangle_{\langle \mathcal{U}, \mathcal{Y}_* \rangle} = 0.$$

Letting $x$ vary over $\mathcal{X}$ and $u$ vary over $\mathcal{U}$ we get

$$\begin{aligned} z_* &= A^* x_* - C^\dagger y_*,\\ u_* &= B^\dagger x_* - D^\dagger y_*. \end{aligned}$$

This is an i/s/o representation for $\Sigma_*$ of the required type. $\square$

# 8 Transmission Matrices of Passive S/S Systems

Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s system and let $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ be an orthogonal i/o decomposition of $\mathcal{W}$ such that

$$\Omega_+ := \Omega(\Sigma;\mathcal{U},\mathcal{Y}) \cap \mathbb{D} \tag{8.1}$$

is nonempty. We shall denote the generalized i/o transfer function of $\Sigma$ by $\mathfrak{D}^{\mathrm{tra}}$, and we call it the *generalized transmission matrix* of $\Sigma$ corresponding to this decomposition. With the help of this transmission matrix we define the following kernels (for $z \neq \zeta_*$ and $z_* \neq \zeta$, and by continuity if $z = \zeta_*$ or $z_* = \zeta$)

$$K_{\mathfrak{D}^{\mathrm{tra}}}(z,\zeta) = \frac{1_{\mathcal{U}} - \mathfrak{D}^{\mathrm{tra}}(z)^*\mathfrak{D}^{\mathrm{tra}}(\zeta)}{1-\zeta\overline{z}}, \qquad z,\zeta \in \Omega_+, \tag{8.2}$$

$$\widetilde{K_{\mathfrak{D}^{\mathrm{tra}}}}(z_*,\zeta_*) = \frac{1_{\mathcal{Y}} - \mathfrak{D}^{\mathrm{tra}}(z_*)\mathfrak{D}^{\mathrm{tra}}(\zeta_*)^*}{1-\overline{\zeta}_* z_*}, \qquad z_*,\zeta_* \in \Omega_+, \tag{8.3}$$

$$\Delta_{\mathfrak{D}^{\mathrm{tra}}}(z,z_*;\zeta,\zeta_*) = \begin{bmatrix} \widetilde{K_{\mathfrak{D}^{\mathrm{tra}}}}(z_*,\zeta_*) & \frac{\mathfrak{D}^{\mathrm{tra}}(\zeta)-\mathfrak{D}^{\mathrm{tra}}(z_*)}{\zeta - z_*} \\ \frac{\mathfrak{D}^{\mathrm{tra}}(\zeta_*)^*-\mathfrak{D}^{\mathrm{tra}}(z)^*}{\overline{\zeta}_*-\overline{z}} & K_{\mathfrak{D}^{\mathrm{tra}}}(z,\zeta) \end{bmatrix}, \; z,z_*,\zeta,\zeta_* \in \Omega_+. \tag{8.4}$$

Note that these kernels are analytic with respect to $\zeta$ and $z_*$, and conjugate analytic with respect of $z$ and $\zeta_*$.

**Theorem 8.1.** *Let $\mathfrak{D}^{\mathrm{tra}}$ be the generalized transmission matrix of a s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ corresponding to an orthogonal decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ for which $\Omega_+$ defined in* (8.1) *is nonempty. Define $K_{\mathfrak{D}^{\mathrm{tra}}}$, $\widetilde{K_{\mathfrak{D}^{\mathrm{tra}}}}$, and $\Delta_{\mathfrak{D}^{\mathrm{tra}}}$ by* (8.2)–(8.4).

1) *If $\Sigma$ is forward passive, then $K_{\mathfrak{D}^{\mathrm{tra}}}$ is a positive definite kernel on $\Omega_+ \times \Omega_+$ with respect to the inner product in the Kreĭn space $\mathcal{U}$, i.e.,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}[K_{\mathfrak{D}^{\mathrm{tra}}}(z_j,z_i)u_i,u_j]_{\mathcal{U}} \geq 0$$

*for every $n \geq 1$, $z_i \in \Omega_+$ and $u_i \in \mathcal{U}$, $1 \leq i \leq n$.*

2) *If $\Sigma$ is backward passive, then $\widetilde{K_{\mathfrak{D}^{\mathrm{tra}}}}$ is a positive definite kernel on $\Omega_+ \times \Omega_+$ with respect to the inner product in the Kreĭn space $\mathcal{Y}$.*

3) *If* $\Sigma$ *is passive, then* $\Delta_{\mathfrak{D}^{\rm tra}}$ *is a positive definite kernel on* $(\Omega_+ \times \Omega_+) \times (\Omega_+ \times \Omega_+)$ *with respect to the inner product in the Kreĭn space* $\mathcal{W} = \mathcal{U}\,[\dot{+}]\,\mathcal{Y}$ *in the sense that*

$$\sum_{k=1}^{m}\sum_{\ell=1}^{m} \frac{[(1_{\mathcal{Y}} - \mathfrak{D}^{\rm tra}(z_{*\ell})\mathfrak{D}^{\rm tra}(z_{*k})^*)y_{*k}, y_{*\ell}]_{\mathcal{Y}}}{1-\overline{z}_{*k}z_{*\ell}} + \sum_{i=1}^{n}\sum_{\ell=1}^{m} \frac{[(\mathfrak{D}^{\rm tra}(z_i) - \mathfrak{D}^{\rm tra}(z_{*\ell}))u_i, y_{*\ell}]_{\mathcal{Y}}}{z_i - z_{*\ell}} + \sum_{k=1}^{m}\sum_{j=1}^{n} \frac{[(\mathfrak{D}^{\rm tra}(z_{*k})^* - \mathfrak{D}^{\rm tra}(z_j)^*)y_{*k}, u_j]_{\mathcal{U}}}{\overline{z}_{*k} - \overline{z}_j} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{[(1_{\mathcal{U}} - \mathfrak{D}^{\rm tra}(z_j)^*\mathfrak{D}^{\rm tra}(z_i))u_i, u_j]_{\mathcal{U}}}{1-z_i\overline{z}_j} \geq 0$$

*for all sequences* $z_i$, $z_{*k} \in \Omega_+$, *and* $u_i \in \mathcal{U}$, *and* $y_{*k} \in \mathcal{Y}$.

*Proof.* Proof of 1): Let $\Sigma^r_{i/s/o}$ be the right affine i/s/o representation of $\Sigma$ corresponding to the i/o decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ given by (1.7), and let $z \in \Omega_+$ and $u \in \mathcal{U}$. Then there exist unique $x_{z,u} \in \mathcal{X}$ and $\ell_{z,u} \in \mathcal{L}$ such that

$$\begin{aligned} (1_{\mathcal{X}} - zA')x_{z,u} - zB'\ell_{z,u} &= 0, \\ C'_{\mathcal{U}}x_{z,u} + D'\ell_{z,u} &= u, \end{aligned} \tag{8.5}$$

and, by the definition of the generalized i/o transfer function,

$$\mathfrak{D}^{\rm tra}(z)u = C'\mathcal{Y}x_{z,u} + D'\mathcal{Y}\ell_{z,u}.$$

Thus,

$$x_{z,u} = z[A'x_{z,u} + B'\ell_{z,u}].$$

Define $\dot{x}_{z,u} = A'x_{z,u} + B'\ell_{z,u}$. Then $x_{z,u} = z\dot{x}_{z,u}$, and hence

$$k_{z,u} := \begin{bmatrix} \dot{x}_{z,u} \\ z\dot{x}_{z,u} \\ \mathfrak{D}^{\rm tra}(z)u \\ u \end{bmatrix} \in V.$$

Consequently,

$$k(\zeta) := \sum_{i=1}^{n} \frac{1}{1-z_i\zeta} k_{z_i,u_i} \in V \text{ for any } z_i \in \Omega_+,\ u_i \in \mathcal{U},\ |\zeta| = 1.$$

The subspace $V$ is nonnegative in $\mathfrak{K}$, since the s/s system is assumed to be forward passive. This means that $[k(\zeta), k(\zeta)]_{\mathfrak{K}} \geq 0$ for every $\zeta$ with $|\zeta| = 1$. This inner product is given by

$$0 \leq [k(\zeta), k(\zeta)]_{\mathfrak{K}} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{-1 + z_i\overline{z}_j}{(1 - z_i\zeta)(1 - \overline{z}_j\overline{\zeta})}(\dot{x}_{z_i,u_i}, \dot{x}_{z_j,u_j})_{\mathcal{X}}$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{[(1_{\mathcal{U}} - \mathfrak{D}^{\rm tra}(z_j)^*\mathfrak{D}^{\rm tra}(z_i))u_i, u_j]_{\mathcal{U}}}{(1 - z_i\zeta)(1 - \overline{z}_j\overline{\zeta})}.$$

By integrating this inequality around the circle $|\zeta| = 1$ and taking into account that

$$\frac{1}{2\pi}\int_{|\zeta|=1} \frac{|d\zeta|}{(1 - z_i\zeta)(1 - \overline{z}_j\overline{\zeta})} = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{(1 - z_i\zeta)(\zeta - \overline{z}_j)} = \frac{1}{1 - z_i\overline{z}_j},$$

we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{[(1_{\mathcal{U}} - \mathfrak{D}^{\rm tra}(z_j)^*\mathfrak{D}^{\rm tra}(z_i))u_i, u_j]_{\mathcal{U}}}{1 - z_i\overline{z}_j} \geq \sum_{i=1}^{n}\sum_{j=1}^{n}(\dot{x}_{z_i,u_i}, \dot{x}_{z_j,u_j})_{\mathcal{X}}$$
$$= \Big\|\sum_{i=1}^{n} \dot{x}_{z_i,u_i}\Big\|_{\mathcal{X}}^2 \geq 0.$$

This proves the positivity of the kernel $K_{\mathfrak{D}^{\rm tra}}$ on $\Omega_+ \times \Omega_+$.

Proof of 2): This follows from Part 1) if we replace $\Sigma$ by the adjoint s/s system $\Sigma_*$ and use the connection between the generalized i/o transfer functions of $\Sigma$ and $\Sigma_*$ expressed in Proposition 7.4.

Proof of 3): Let $m$ and $n$ be positiv integers, let $z_i \in \Omega_+$ and $u_i \in \mathcal{U}$ for $1 \leq i \leq n$, and let $z_{*k} \in \Omega_+$ and $y_{*k} \in \mathcal{Y}$ for $1 \leq k \leq m$. In the proof below we assume that $z_j \neq z_{*k}$ for all $j$ and $k$ (the case where some $z_j = z_{*k}$ for some $j$ and $k$ can be obtained from this case by first taking $z_j \neq z_{*k}$ and then letting $z_j \to z_{*k}$). As in the proof of 1) we find that that there to each $i$ there exists a vector $\dot{x}_{z_i,u_i} \in \mathcal{X}$ such that

$$k_{z_i,u_i} := \begin{bmatrix} \dot{x}_{z_i,u_i} \\ z_i\dot{x}_{z_i,u_i} \\ \mathfrak{D}^{\rm tra}(z_i)u_i \\ u_i \end{bmatrix} \in V$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}[K_{\mathfrak{D}^{\rm tra}}(z_i, z_j)u_i, u_j]_{\mathcal{U}} \geq \Big\|\sum_{i=1}^{n} \dot{x}_{z_i,u_i}\Big\|_{\mathcal{X}}^2 \geq 0. \tag{8.6}$$

By applying the same argument to the adjoint system we find that for each $k$ there exists a vector $\dot{x}_{z_{*k},y_{*k}} \in \mathcal{X}$ such that

$$k_{z_{*k},y_{*k}} := \begin{bmatrix} \dot{x}_{z_{*k},y_{*k}} \\ \overline{z}_{*k}\dot{x}_{z_{*k},y_{*k}} \\ y_{*k} \\ \mathfrak{D}^{\mathrm{tra}}(z_{*k})^* y_{*k} \end{bmatrix} \in V_*$$

and

$$\sum_{k=1}^{m}\sum_{\ell=1}^{m}[K_{\widetilde{\mathfrak{D}^{\mathrm{tra}}}}(z_{*k}, z_{*\ell})y_{*k}, y_{*\ell}]_{\mathcal{Y}} \geq \Big\|\sum_{i=1}^{m}\dot{x}_{z_{*k},y_{*k}}\Big\|_{\mathcal{X}}^2 \geq 0. \tag{8.7}$$

The generating subspace $V_*$ of the adjoint system $\Sigma_*$ is the annihilator of $V$ in the sense that if $\left[\begin{smallmatrix} \dot{x} \\ x \\ w \end{smallmatrix}\right] \in V$ and $\left[\begin{smallmatrix} \dot{x}_* \\ x_* \\ w_* \end{smallmatrix}\right] \in V_*$, then

$$-(\dot{x}, x_*)_{\mathcal{X}} + (x, \dot{x}_*)_{\mathcal{X}} + \langle w, w_*\rangle_{\langle \mathcal{W},\mathcal{W}_*\rangle} = 0.$$

Therefore, we have for all $i \in [1, n]$ and all $\ell \in [1, m]$,

$$(z_i - z_{*\ell})(\dot{x}_{z_i,u_i}, \dot{x}_{z_{*\ell},y_{*\ell}})_{\mathcal{X}} = [(\mathfrak{D}^{\mathrm{tra}}(z_i) - \mathfrak{D}^{\mathrm{tra}}(z_{*\ell}))u_i, y_{*\ell}]_{\mathcal{Y}}.$$

Dividing by $z_i - z_{*\ell}$ and adding over $i$ and $\ell$ we get

$$\sum_{i=1}^{n}\sum_{\ell=1}^{m}\frac{[(\mathfrak{D}^{\mathrm{tra}}(z_i) - \mathfrak{D}^{\mathrm{tra}}(z_{*\ell}))u_i, y_{*\ell}]_{\mathcal{Y}}}{z_i - z_{*\ell}} = \sum_{i=1}^{n}\sum_{\ell=1}^{m}(\dot{x}_{z_i,u_i}, \dot{x}_{z_{*\ell},y_{*\ell}})_{\mathcal{X}}. \tag{8.8}$$

By combining (8.6), (8.7), (8.8),and the corresponding dual equation, we find that

$$\begin{aligned}
&\sum_{k=1}^{m}\sum_{\ell=1}^{m}\frac{[(1_{\mathcal{Y}} - \mathfrak{D}^{\mathrm{tra}}(z_{*\ell})\mathfrak{D}^{\mathrm{tra}}(z_{*k})^*)y_{*k}, y_{*\ell}]_{\mathcal{Y}}}{1 - \overline{z}_{*k}z_{*\ell}} \\
&\quad + \sum_{i=1}^{n}\sum_{\ell=1}^{m}\frac{[(\mathfrak{D}^{\mathrm{tra}}(z_i) - \mathfrak{D}^{\mathrm{tra}}(z_{*\ell}))u_i, y_{*\ell}]_{\mathcal{Y}}}{z_i - z_{*\ell}} \\
&\quad + \sum_{k=1}^{m}\sum_{j=1}^{n}\frac{[(\mathfrak{D}^{\mathrm{tra}}(z_{*k})^* - \mathfrak{D}^{\mathrm{tra}}(\overline{z}_j)^*)y_{*k}, u_j]_{\mathcal{U}}}{\overline{z}_{*k} - \overline{z}_j} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{[(1_{\mathcal{U}} - \mathfrak{D}^{\mathrm{tra}}(z_j)^*\mathfrak{D}^{\mathrm{tra}}(z_i))u_i, u_j]_{\mathcal{U}}}{1 - z_i\overline{z}_j} \\
&\qquad \geq \Big\|\sum_{i=1}^{n}\dot{x}_{z_i,u_i} + \sum_{i=1}^{m}\dot{x}_{z_{*k},y_{*k}}\Big\|_{\mathcal{X}}^2 \geq 0. \quad \square
\end{aligned}$$

**Remark 8.2.** Part 3) is a stronger version of the inequality that is usually found in the literature. Usually one takes $m = n$, and connects the primal variables $z$ and $\zeta$ to the dual variables $z_*$ and $\zeta_*$ to each other by taking $z = \overline{z}_*$ and $\zeta = \overline{\zeta}_*$ (see, e.g., [ADRdS97]). This is, of course, possible only if $\Omega_+ \cap \Omega_+^* \neq \emptyset$, because this forces us to take $z, \zeta \in \Omega_+ \cap \Omega_+^*$. We are essentially following the tradition of Potapov, who took $m = 1$. In this approach the condition $\Omega_+ \cap \Omega_+^* \neq \emptyset$ is not needed. Note however, that one of the summations is with respect to the analytic variable and the other with respect to the co-analytic variable.

**Lemma 8.3.** *Let $\mathcal{Y}$ and $\mathcal{U}$ be Kreĭn spaces, and let $\mathfrak{D}$ be a $\mathcal{B}(\mathcal{U},\mathcal{Y})$-valued analytic function defined on an open subset $\Omega_+$ of $\mathbb{D}$. Define the two kernels $K_{\mathfrak{D}}$ and $K_{\widetilde{\mathfrak{D}}}$ as in* (8.2) *and* (8.3) *with $\mathfrak{D}^{\mathrm{tra}}$ replaced by $\mathfrak{D}$, and suppose that (at least) one of the following two conditions holds:*

1) *$K_{\mathfrak{D}}$ is positive definite on $\Omega_+ \times \Omega_+$ and $K_{\widetilde{\mathfrak{D}}}(z,z) \geq 0$ for all $z \in \Omega_+$,*
2) *$K_{\widetilde{\mathfrak{D}}}$ is positive definite on $\Omega_+ \times \Omega_+$ and $K_{\mathfrak{D}}(z,z) \geq 0$ for all $z \in \Omega_+$,*

*Then both these conditions hold, i.e., both $K_{\mathfrak{D}}$ and $K_{\widetilde{\mathfrak{D}}}$ are positive definite on $\Omega_+ \times \Omega_+$.*

*Proof.* Let

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{Y}}|_{\mathcal{W}_-} & P_{\mathcal{Y}}|_{\mathcal{W}_+} \\ P_{\mathcal{U}}|_{\mathcal{W}_-} & P_{\mathcal{U}}|_{\mathcal{W}_+} \end{bmatrix}, \tag{8.9}$$

$$\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{W}_-}|_{\mathcal{Y}} & P_{\mathcal{W}_-}|_{\mathcal{U}} \\ P_{\mathcal{W}_+}|_{\mathcal{Y}} & P_{\mathcal{W}_+}|_{\mathcal{U}} \end{bmatrix}. \tag{8.10}$$

where $\mathcal{W} = -\mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ is a fundamental decomposition of $\mathcal{W}$. Each of the two conditions 1) and 2) imply that both $K_{\mathfrak{D}}(z,z) \geq 0$ and $K_{\widetilde{\mathfrak{D}}}(z,z) \geq 0$ for all $z \in \Omega_+$. This is equivalent to the bi-contractivity of $\mathfrak{D}(z)$ with respect to the indefinite inner products in the Kreĭn spaces $\mathcal{U}$ and $\mathcal{Y}$. This guarantees that the Potapov–Ginzburg transform

$$S(z) = [\widetilde{\Theta}_{11}\mathfrak{D}(z) + \widetilde{\Theta}_{12}][\widetilde{\Theta}_{21}\mathfrak{D}(z) + \widetilde{\Theta}_{22}]^{-1}, \qquad z \in \Omega_+, \tag{8.11}$$

is well-defined for all $z \in \Omega_+$ (see, e.g., Lemma III.4.2, [DR90, Theorem 1.3.4], or [DR96, Theorem 2.8′]).

The two kernels $K_{\mathfrak{D}}$ and $K_S$ are connected to each other by the relation

$$K_S(z,\zeta) = [\widetilde{\Theta}_{21}\mathfrak{D}(z) + \widetilde{\Theta}_{22}]^{-*} K_{\mathfrak{D}}(z,\zeta) [\widetilde{\Theta}_{21}\mathfrak{D}(z) + \widetilde{\Theta}_{22}]^{-1}.$$

If 1) holds, then the positivity of $K_{\mathfrak{D}}$ on $\Omega_+ \times \Omega_+$ in the Kreĭn space $\mathcal{U}$ implies the positivity of the kernel $K_S$ on $\Omega_+ \times \Omega_+$ in the Hilbert space $\mathcal{W}_+$.

This guarantees that $S$ has an extension to a Schur function (a holomorphic function whose norm is bounded by one) defined on the full unit disk $\mathbb{D}$ (see [RR82]). We still denote the extended function by $S(z)$, $z \in \mathbb{D}$. It is well-known that for a Schur function both the kernels $K_S$ and $K_S^{\sim}$ are positive definite on $\mathbb{D}\times\mathbb{D}$ (see, e.g., [DR96, Theorem 5.9]). Since $K_{\mathfrak{D}}^{\sim}$ can be recovered from $K_{\mathfrak{D}}^{\sim}$ via the formula

$$K_{\mathfrak{D}}^{\sim}(z,\zeta) = [-\mathfrak{D}(z)\Theta_{21} + \Theta_{11}]^{-1} K_S^{\sim}(z,\zeta)[-\mathfrak{D}(z)\Theta_{21} + \Theta_{11}]^{-*} \tag{8.12}$$

we find that $K_{\mathfrak{D}}^{\sim}$ is positive definite on $\Omega_+ \times \Omega_+$ whenever 1) holds. The same argument applied to the function $\mathfrak{D}^{\sim}(z) = \mathfrak{D}(\overline{z})^*$ shows that the kernel $K_{\mathfrak{D}}$ is positive definite on $\Omega_+^* \times \Omega_+^*$ whenever 2) holds. □

As in Part III we define the *Potapov class* $P(\Omega;\mathcal{U},\mathcal{Y})$, where $\Omega$ is an open subset $\mathbb{D}$ and $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn spaces, to be the space of $\mathcal{B}(U;Y)$-valued holomorphic functions $\mathfrak{D}$ on $\Omega$ for which both the kernels $K_{\mathfrak{D}}$ and $K_{\mathfrak{D}}^{\sim}$ are positive definite on $\Omega \times \Omega$. We next present generalizations of Propositions III.4.8 and III.4.9, considering now any function $\mathfrak{D} \in P(\Omega;\mathcal{U},\mathcal{Y})$, without the restriction that $0 \in \Omega$ that we imposed in those theorems. This means that we no longer necessarily get i/s/o realizations like those given in Part III, but we still get realizations of the following type.

**Theorem 8.4.** *Let $\Omega$ be an open subset of $\mathbb{D}$, and let $\mathfrak{D} \in P(\Omega;\mathcal{U},\mathcal{Y})$. Then there exists a simple conservative s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ with Kreĭn signal space $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ such that $\Omega \subset \Omega(\Sigma;\mathcal{U},\mathcal{Y})$ and $\mathfrak{D}$ is the restriction to $\Omega$ of the generalized transmission matrix of $\Sigma$, corresponding to the decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$. The system $\Sigma$ is determined uniquely up to a unitary similarity transformation in the state space. The function $\mathfrak{D}$ has a unique extension (which we still denote by $\mathfrak{D}$) to a function in $P(\Omega_+;\mathcal{U},\mathcal{Y})$, where*

$$\Omega_+ := \Omega(\Sigma;\mathcal{Y},\mathcal{U}) \cap \mathbb{D},$$

*and also the kernel $\Delta_{\mathfrak{D}}$ defined as in* (8.4) *with $\mathfrak{D}^{\mathrm{tra}}$ replaced by $\mathfrak{D}$ is positive definite on* $(\Omega_+ \times \Omega_+) \times (\Omega_+ \times \Omega_+)$.

*Proof.* As in the proof of Lemma 8.3 we let $S$ be the Schur function which is the extension to $\mathbb{D}$ of the function given in (8.11). By Proposition II.6.2, there exists a simple conservative s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ such that $\mathcal{W} = -\mathcal{W}_-\,[\dotplus]\,\mathcal{W}_+$ is a fundamental decomposition of $\mathcal{W}$, and such that the corresponding scattering matrix $\mathfrak{D}^{\mathrm{sca}}$ of $\Sigma$ satisfies $\mathfrak{D}^{\mathrm{sca}}|_{\mathbb{D}} = S$. This system is determined uniquely by $S$ up to a unitary similarity transformation in the state space (see, e.g., [ADRdS97, Theorem 2.1.3]). By Theorem 5.4

with the scattering representation interpreted as a driving variable representation, the generalized transmission matrix of the system $\Sigma$ with respect to the decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ coincides with the given function $\mathfrak{D}$ on $\Omega$, and this transmission matrix is an extension of $\mathfrak{D}$ to $\Omega_+$ with the required properties. The uniqueness of the extension follows from the fact that the Schur function $S$ above is uniquely determined by $\mathfrak{D}$, and that any extension of $\mathfrak{D}$ for which the two kernels $K_{\mathfrak{D}}$ and $K_{\tilde{\mathfrak{D}}}$ are positive definite on $\Omega_+ \times \Omega_+$ has the property that on $\Omega_+$ the function $S$ is obtained from the extended $\mathfrak{D}$ via the formula (8.11), and hence $\mathfrak{D}$ is obtained from $S$ by (8.12). □

It follows from Theorem 8.4 that if $\mathfrak{D}$ is an arbitrary function in $P(\Omega;\mathcal{U},\mathcal{Y})$, where $\Omega$ is an open subset of $\mathbb{D}$, then $\mathfrak{D}$ has a unique extension to a function in $P(\Omega_+;\mathcal{U},\mathcal{Y})$, where $\Omega_+$ is the set defined in there. We claim that $\mathfrak{D}$ cannot be extended to any larger subset of $\mathbb{D}$ without loosing the positive definiteness of at least one of the two kernels $K_{\mathfrak{D}}$ and $K_{\tilde{\mathfrak{D}}}$. This is true, because if such an extension were possible, then it would still have to be true that on this larger set $\mathfrak{D}$ would still have to be given by (8.12); in particular, $-\mathfrak{D}(z)\Theta_{21} + \Theta_{11}$ must then have a bounded inverse on this larger set. But recall that $\mathbb{D} \subset \Lambda_{A^{\rm sca}}$, and therefore, by Theorem 5.4, for $z \in \mathbb{D}$, the operator $-\mathfrak{D}(z)\Theta_{21} + \Theta_{11}$ has a bounded inverse if and only if $z \in \Omega_+$. Thus, the set $\Omega_+$ in Theorem 8.4 is the *maximal subset of* $\mathbb{D}$ *to which* $\mathfrak{D}$ *has a (unique) extension in* $P(\Omega_+;\mathcal{U},\mathcal{Y})$. We call this the *natural domain* of $\mathfrak{D}$ in $\mathbb{D}$.

**Theorem 8.5.** *Let $\Omega$ be an open subset of $\mathbb{D}$, let $\mathfrak{D} \in P(\Omega;\mathcal{U},\mathcal{Y})$, and let $\Sigma$ be a s/s system of the type mentioned in Theorem 8.4. Let $\Sigma_\circ$ and $\Sigma_\bullet$ be the compressions of $\Sigma$ constructed in Theorem II.7.5. Then $\Sigma_\circ$ and $\Sigma_\bullet$ are minimal passive s/s systems with Kreĭn signal space $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$ such that*

$$\Omega \subset \Omega(\Sigma_\circ;\mathcal{U},\mathcal{Y}) \cap \mathbb{D} = \Omega(\Sigma_\bullet;\mathcal{U},\mathcal{Y}) \cap \mathbb{D} \; (= \Omega(\Sigma;\mathcal{U},\mathcal{Y}) \cap \mathbb{D}),$$

*and $\mathfrak{D}$ is the restriction to $\Omega$ of the generalized transmission matrices of $\Sigma_\circ$ and of $\Sigma_\bullet$, corresponding to the decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$.*

*Proof.* We once more introduce the same fundamental decomposition $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$ as in the proof of Theorem 5.4. By Theorem III.7.3, the three s/s systems $\Sigma$, $\Sigma_\circ$ and $\Sigma_\bullet$ have the same behavior, and the decomposition $\mathcal{W} = -\mathcal{W}_-\,[\dot{+}]\,\mathcal{W}_+$ gives us three scattering representations, one for each system. Since these three i/s/o systems are stable and externally equivalent, their i/o transfer functions coincide on $\mathbb{D}$. It then follows from Theorem 5.4 that also the three domains $\Omega(\Sigma_\circ;\mathcal{U},\mathcal{Y})\cap\mathbb{D}$, $\Omega(\Sigma_\bullet;\mathcal{U},\mathcal{Y})\cap\mathbb{D}$, and $\Omega(\Sigma;\mathcal{U},\mathcal{Y})\cap\mathbb{D}$ must coincide, and on this common domain the generalized transfer functions of these three systems coincide. □

**Proposition 8.6.** *Let $\mathfrak{D}$ be a analytic $\mathcal{B}(\mathcal{U},\mathcal{Y})$-valued function defined on a subset $\Omega_{\mathfrak{D}}^{+}$ of $D$ whose complement has no cluster points in $\mathbb{D}$. Define the kernels $K_{\mathfrak{D}}$, $K_{\mathfrak{D}}^{\sim}$, and $\Delta_{\mathfrak{D}}$ as in* (8.2)–(8.4) *with $\mathfrak{D}^{\mathrm{tra}}$ replaced by $\mathfrak{D}$. If*

$$\mathfrak{D}(z)^{*}\mathfrak{D}(z) \le 1_{\mathcal{U}} \text{ and } \mathfrak{D}(z)\mathfrak{D}(z)^{*} \le 1_{\mathcal{Y}} \text{ for all } z \in \Omega_{\mathfrak{D}}^{+}, \tag{8.13}$$

*then all the three kernels $K_{\mathfrak{D}}$, $K_{\mathfrak{D}}^{\sim}$, and $\Delta_{\mathfrak{D}}$ are positive definite on their appropriate domains.*

*Proof.* Under the condition (8.13) the transformation (8.11) is well-defined on $\Omega_{\mathfrak{D}}^{+}$, and the corresponding function $S(z)$ in (8.11) has a unique extension to a $\mathcal{B}(\mathcal{W}_{+};\mathcal{W}_{-})$-valued Schur function on $\mathbb{D}$, where we use the same notation as in the proof of Lemma 8.3. This implies that the two kernels $K_S$ and $K_S^{\sim}$ are positive definite on $\mathbb{D}\times\mathbb{D}$. Hence, via the inverse transform (8.12) we conclude that $\mathfrak{D} \in P(\Omega;\mathcal{U},\mathcal{Y})$. By Theorem 8.4, $\mathfrak{D}$ can be realized as the transmission matrix of a passive s/s system, and also the kernel $\Delta_{\mathfrak{D}}$ is positive definite. □

**Proposition 8.7.** *Let $\mathfrak{D}$ be a analytic $\mathcal{B}(\mathcal{U},\mathcal{Y})$-valued function defined on a subset $\Omega_{\mathfrak{D}}^{+}$ of $D$ whose complement has no cluster points in $\mathbb{D}$, and suppose that* $\operatorname{ind}_{-}\mathcal{U} = \operatorname{ind}_{-}\mathcal{Y} < \infty$. *If*

$$\mathfrak{D}(z)^{*}\mathfrak{D}(z) \le 1_{\mathcal{U}}, \qquad z \in \Omega_{\mathcal{D}}^{+}, \tag{8.14}$$

*where $\Omega_{\mathcal{D}}^{+}$ is the points of analyticity of $\mathfrak{D}$ in $\mathbb{D}$, then $\mathfrak{D}$ is the restriction to $\Omega_{\mathcal{D}}^{+}$ of the generalized transmission matrix corresponding to the decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of a simple conservative s/s system $\Sigma$, which is uniquely defined by $\mathfrak{D}$ up to unitary similarity. It is also the restriction to $\Omega_{\mathcal{D}}^{+}$ of the generalized transmission matrix corresponding to the same decomposition of a minimal passive s/s system $\Sigma'$, which is defined by $\mathfrak{D}$ up to pseudo-similarity.*

*Proof.* Since $\mathcal{U}$ and $\mathcal{Y}$ have the same negative dimension, condition (8.14) is equivalent to the dual condition

$$\mathfrak{D}(z)\mathfrak{D}(z)^{*} \le 1_{\mathcal{Y}}, \qquad z \in \Omega_{\mathcal{D}}^{+}$$

(see, e.g., Lemma III.4.2). By Proposition 8.6, $\mathfrak{D} \in P(\Omega_{\mathfrak{D}}^{+};\mathcal{U},\mathcal{Y})$. We can take the system $\Sigma$ to be the one given by Theorem 8.4, and we can take $\Sigma'$ to be either one of the two minimal passive systems $\Sigma_{\circ}$ and $\Sigma_{\bullet}$ in Theorem 8.5. □

**Remark 8.8.** V. Potapov studied in [Pot60] and in a number of additional publications [KP65], [EP73], and [KP74], the class of meomorphic matrix-valued functions $\mathfrak{D}$ defined on $\mathbb{D}$ of size $m \times m$ with a determinant which does not vanish identically and have $J$-contractive values with respect to a given signature matrix $J$. This is equivalent to (8.14) if we equip both the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ with the same indefinite inner product $[\xi, \xi'] = (\xi, J\xi')$, $\xi, \xi' \in \mathbb{C}^n$.

# 9 Affine Transmission and Impedance representations of Passive S/S Systems

In this section we shall take a closer look at affine transmission and impedance representations of a passive s/s system by using a scattering representation $\Sigma^{\rm sca} = \left(\left[\begin{smallmatrix} A^{\rm sca} & B^{\rm sca} \\ C^{\rm sca} & D^{\rm sca} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-\right)$ as a driving variable or an output nulling representation.

We begin with the transmission case. Let $\Sigma$ be a passive s/s system, let $\mathcal{W} = -\mathcal{Y}[\dotplus]\mathcal{U}$ be an orthogonal decomposition of $\mathcal{W}$, and let $\mathcal{Y} = -\mathcal{Y}_-[\dotplus]\mathcal{Y}_+$ and $\mathcal{U} = -\mathcal{U}_-[\dotplus]\mathcal{U}_+$ be fundamental decompositions of $\mathcal{Y}$ and $\mathcal{U}$, respectively. Let $\mathcal{W}_+ = \mathcal{Y}_-\,[\dotplus]\,\mathcal{U}_+$ and $\mathcal{W}_- = \mathcal{U}_-\,[\dotplus]\,\mathcal{Y}_+$. Then $\mathcal{W}_+$ and $\mathcal{W}_-$ are Hilbert spaces, and $\mathcal{W} = -\mathcal{W}_-\,[\dotplus]\,\mathcal{W}_+$ is a fundamental decomposition of $\mathcal{W}$. This is the same setting that we used in Remark III.4.6. In this setting the decompositions $\Theta$ and $\widetilde{\Theta}$ in (8.9) and (8.10) are given by (III.4.12) and (III.4.13). The corresponding decomposition of the corresponding scattering representation of $\Sigma$ and its is/so transfer function are given in (III.4.14) and (III.4.15). However, this time we do not assume that the decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ is admissible for $\Sigma$, i.e., we do not assume that (III.4.16) holds. Therefore, we have to replace formulas (III.4.17), (III.4.20) and (III.4.21) by the corresponding right and left affine formulas. The right affine representation (3.4) of the generating subspace $V$ becomes

$$V = \left\{ \begin{bmatrix} \begin{bmatrix} \dot{x} \\ y_- \\ y_+ \end{bmatrix} \\ \begin{bmatrix} x \\ u_- \\ u_+ \end{bmatrix} \end{bmatrix} \;\middle|\; \begin{matrix} \begin{bmatrix} \dot{x} \\ y_- \\ y_+ \end{bmatrix} = \left[\begin{array}{c|cc} A^{\rm sca} & B_1^{\rm sca} & B_2^{\rm sca} \\ \hline 0 & 1_{\mathcal{Y}_-} & 0 \\ C_2^{\rm sca} & D_{21}^{\rm sca} & D_{22}^{\rm sca} \end{array}\right] \begin{bmatrix} x \\ y_- \\ u_+ \end{bmatrix}, \\ \begin{bmatrix} x \\ u_- \\ u_+ \end{bmatrix} = \left[\begin{array}{c|cc} 1_{\mathcal{X}} & 0 & 0 \\ \hline C_1^{\rm sca} & D_{11}^{\rm sca} & D_{12}^{\rm sca} \\ 0 & 0 & 1_{\mathcal{U}_+} \end{array}\right] \begin{bmatrix} x \\ y_- \\ u_+ \end{bmatrix}, \end{matrix} \quad \begin{bmatrix} x \\ y_- \\ u_+ \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{Y}_- \\ \mathcal{U}_+ \end{bmatrix} \right\}. \tag{9.1}$$

and the corresponding left affine representation of $V$ is

$$V = \left\{ \begin{bmatrix} \begin{bmatrix} \dot{x} \\ y_- \\ y_+ \end{bmatrix} \\ \begin{bmatrix} x \\ u_- \\ u_+ \end{bmatrix} \end{bmatrix} \;\middle|\; \left[\begin{array}{c|cc} -1_{\mathcal{X}} & B_1^{\rm sca} & 0 \\ \hline 0 & D_{11}^{\rm sca} & 0 \\ 0 & D_{21}^{\rm sca} & -1_{\mathcal{U}_+} \end{array}\right] \begin{bmatrix} \dot{x} \\ y_- \\ y_+ \end{bmatrix} + \left[\begin{array}{c|cc} A^{\rm sca} & 0 & B_2^{\rm sca} \\ \hline C_1^{\rm sca} & -1_{\mathcal{U}_-} & D_{12}^{\rm sca} \\ C_2^{\rm sca} & 0 & D_{22}^{\rm sca} \end{array}\right] \begin{bmatrix} x \\ u_- \\ u_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}. \tag{9.2}$$

The (right Bezout) identity (3.12) becomes

$$
\begin{aligned}
&\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1_{\mathcal{Y}_-} & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} A^{\rm sca} & B_1^{\rm sca} & B_2^{\rm sca} \\ 0 & 1_{\mathcal{Y}_-} & 0 \\ C_2^{\rm sca} & D_{21}^{\rm sca} & D_{22}^{\rm sca} \end{bmatrix} \\
&\qquad + \begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_{\mathcal{U}_+} \end{bmatrix}
\begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 \\ C_1^{\rm sca} & D_{11}^{\rm sca} & D_{12}^{\rm sca} \\ 0 & 0 & 1_{\mathcal{U}_+} \end{bmatrix}
= 1_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y}_- \\ \mathcal{U}_+ \end{smallmatrix}\right]}, \qquad z \in \Lambda_{A^{\rm sca}}
\end{aligned} \tag{9.3}
$$

and the (left Bezout) identity (4.9) becomes

$$
\begin{aligned}
&\begin{bmatrix} -1_{\mathcal{X}} & B_1^{\rm sca} & 0 \\ 0 & D_{11}^{\rm sca} & 0 \\ 0 & D_{21}^{\rm sca} & -1_{\mathcal{Y}_+} \end{bmatrix}
\begin{bmatrix} -1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1_{\mathcal{Y}_+} \end{bmatrix} \\
&\qquad + \begin{bmatrix} A^{\rm sca} & 0 & B_1^{\rm sca} \\ C_1^{\rm sca} & -1_{\mathcal{U}_-} & D_{12}^{\rm sca} \\ C_2^{\rm sca} & 0 & D_{22}^{\rm sca} \end{bmatrix}
\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1_{\mathcal{U}_-} & 0 \\ 0 & 0 & 0 \end{bmatrix}
= 1_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U}_- \\ \mathcal{Y}_+ \end{smallmatrix}\right]},
\end{aligned} \tag{9.4}
$$

Formula (3.13) for the right affine is/so transfer function becomes

$$
\left[\begin{array}{cc} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'_{\mathcal{Y}}(z) & \mathfrak{D}'_{\mathcal{Y}}(z) \\ \hline 1_{\mathcal{X}} & 0 \\ \mathfrak{C}'_{\mathcal{U}}(z) & \mathfrak{D}'_{\mathcal{U}}(z) \end{array}\right]
= \left[\begin{array}{c}
\left[\begin{array}{c|cc} \mathfrak{A}^{\rm sca}(z) & \mathfrak{B}_1^{\rm sca}(z) & \mathfrak{B}_2^{\rm sca}(z) \\ \hline 0 & 1_{\mathcal{Y}_-} & 0 \\ \mathfrak{C}_2^{\rm sca}(z) & \mathfrak{D}_{21}^{\rm sca}(z) & \mathfrak{D}_{22}^{\rm sca}(z) \end{array}\right] \\ \hline
\left[\begin{array}{c|cc} 1_{\mathcal{X}} & 0 & 0 \\ \hline \mathfrak{C}_1^{\rm sca}(z) & \mathfrak{D}_{11}^{\rm sca}(z) & \mathfrak{D}_{12}^{\rm sca}(z) \\ 0 & 0 & 1_{\mathcal{U}_+} \end{array}\right]
\end{array}\right], \qquad z \in \Lambda_{A^{\rm sca}}. \tag{9.5}
$$

and formula (4.10) for the left affine is/so transfer function becomes

$$
\begin{aligned}
&\left[\begin{array}{cc|cc} -1_{\mathcal{X}} & \mathfrak{B}''_{\mathcal{Y}}(z) & \mathfrak{A}''(z) & \mathfrak{B}''_{\mathcal{U}}(z) \\ \hline 0 & \mathfrak{D}''_{\mathcal{Y}}(z) & \mathfrak{C}''(z) & \mathfrak{D}''_{\mathcal{U}}(z) \end{array}\right] \\
&\quad = \left[\begin{array}{c|c}
\left[\begin{array}{c|cc} -1_{\mathcal{X}} & \mathfrak{B}_1^{\rm sca}(z) & 0 \\ \hline 0 & \mathfrak{D}_{11}^{\rm sca}(z) & 0 \\ 0 & \mathfrak{D}_{21}^{\rm sca}(z) & -1_{\mathcal{Y}_+} \end{array}\right] &
\left[\begin{array}{c|cc} \mathfrak{A}^{\rm sca}(z) & 0 & \mathfrak{B}_2^{\rm sca}(z) \\ \hline \mathfrak{C}_1^{\rm sca}(z) & -1_{\mathcal{U}_-} & \mathfrak{D}_{12}^{\rm sca}(z) \\ \mathfrak{C}_2^{\rm sca}(z) & 0 & \mathfrak{D}_{22}^{\rm sca}(z) \end{array}\right]
\end{array}\right], \qquad z \in \Lambda_{A^{\rm sca}}.
\end{aligned} \tag{9.6}
$$

In particular, the right affine i/o transmission function is given by

$$
\begin{bmatrix} \mathfrak{D}'_{\mathcal{Y}}(z) \\ \mathfrak{D}'_{\mathcal{U}}(z) \end{bmatrix}
= \left[\begin{array}{cc} 1_{\mathcal{Y}_-} & 0 \\ \mathfrak{D}_{21}^{\rm sca}(z) & \mathfrak{D}_{22}^{\rm sca}(z) \\ \hline \mathfrak{D}_{11}^{\rm sca}(z) & \mathfrak{D}_{12}^{\rm sca}(z) \\ 0 & 1_{\mathcal{U}_+} \end{array}\right], \tag{9.7}
$$

and the left affine i/o transmission function is given by

$$
\begin{bmatrix} \mathfrak{D}''_{\mathcal{Y}}(z) & \mathfrak{D}''_{\mathcal{U}}(z) \end{bmatrix}
= \left[\begin{array}{cc|cc} \mathfrak{D}_{11}^{\rm sca}(z) & 0 & -1_{\mathcal{U}_-} & \mathfrak{D}_{12}^{\rm sca}(z) \\ \mathfrak{D}_{21}^{\rm sca}(z) & -1_{\mathcal{Y}_+} & 0 & \mathfrak{D}_{22}^{\rm sca}(z) \end{array}\right]. \tag{9.8}
$$

Since $\mathbb{D} \subset \Lambda_{A^{\text{sca}}}$, the restriction to $\mathbb{D}$ of the generalized transfer function is defined on the set

$$\Omega_+(\Sigma;\mathcal{U},\mathcal{Y}) = \left\{ z \in \mathbb{D} \;\middle|\; \mathfrak{D}_{11}^{\text{sca}}(z) \text{ has a bounded inverse} \right\}, \tag{9.9}$$

and at a point $z \in \Omega_+(\Sigma;\mathcal{U},\mathcal{Y})$ the is/so generalized transfer function is given by the same formulas (III.4.20) and (III.4.21) as in the case where the decomposition $\mathcal{W} = -\mathcal{Y} \,[\dot{+}]\, \mathcal{U}$ is admissible. These formulas are the same that we obtain from (1.11), (1.15), (5.8), and (5.20) taking into account the specific structure of the involved operators. In particular, the generalized transmission matrix is given by

$$\begin{aligned}
\begin{bmatrix} \mathfrak{D}_{11}^{\text{tra}}(z) & \mathfrak{D}_{12}^{\text{tra}}(z) \\ \mathfrak{D}_{21}^{\text{tra}}(z) & \mathfrak{D}_{22}^{\text{tra}}(z) \end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{Y}_-} & 0 \\ \mathfrak{D}_{21}^{\text{sca}}(z) & \mathfrak{D}_{22}^{\text{sca}}(z) \end{bmatrix} \begin{bmatrix} \mathfrak{D}_{11}^{\text{sca}}(z) & \mathfrak{D}_{12}^{\text{sca}}(z) \\ 0 & 1_{\mathcal{U}_+} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -\mathfrak{D}_{11}^{\text{sca}}(z) & 0 \\ -\mathfrak{D}_{21}^{\text{sca}}(z) & 1_{\mathcal{Y}_+} \end{bmatrix}^{-1} \begin{bmatrix} -1_{\mathcal{U}_-} & \mathfrak{D}_{12}^{\text{sca}}(z) \\ 0 & \mathfrak{D}_{22}^{\text{sca}}(z) \end{bmatrix} \\
&= \begin{bmatrix} (\mathfrak{D}_{11}^{\text{sca}}(z))^{-1} & -(\mathfrak{D}_{11}^{\text{sca}}(z))^{-1}\mathfrak{D}_{12}^{\text{sca}}(z) \\ \mathfrak{D}_{21}^{\text{sca}}(z)(\mathfrak{D}_{11}^{\text{sca}}(z))^{-1} & \mathfrak{D}_{22}^{\text{sca}}(z) - \mathfrak{D}_{21}^{\text{sca}}(z)(\mathfrak{D}_{11}^{\text{sca}}(z))^{-1}\mathfrak{D}_{12}^{\text{sca}}(z) \end{bmatrix}, \\
&\qquad z \in \Omega_+(\Sigma;\mathcal{U},\mathcal{Y}).
\end{aligned} \tag{9.10}$$

The difference compared to (III.4.20)–(III.4.21) is that we no longer require $0 \in \Omega_+(\Sigma;\mathcal{U},\mathcal{Y})$.

We now turn to the impedance case, still assuming $\Sigma$ be a passive s/s system. We decopose the singal space $\mathcal{W}$ into a Lagrangean decomposition $\mathcal{W} = \mathcal{F} \,[\dot{+}]\, \mathcal{E}$, and introduce the same notations as in Section III.5. In particular, $\Theta$ and $\widetilde{\Theta}$ satisfy (III..5.4), (III.5.7), and (III.5.8). The right affine representation (3.4) of the generating subspace $V$ becomes

$$V = \left\{ \begin{bmatrix} \begin{bmatrix} \dot{x} \\ f \end{bmatrix} \\ \begin{bmatrix} x \\ e \end{bmatrix} \end{bmatrix} \;\middle|\; \begin{aligned} \begin{bmatrix} \dot{x} \\ f \end{bmatrix} &= \begin{bmatrix} A^{\text{sca}} & B^{\text{sca}} \\ \Theta_{11}C^{\text{sca}} & \Theta_{11}(D^{\text{sca}} - \Phi) \end{bmatrix} \begin{bmatrix} x \\ w_+ \end{bmatrix}, \\ \begin{bmatrix} x \\ e \end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21}C^{\text{sca}} & \Theta_{21}(D^{\text{sca}} + \Phi) \end{bmatrix} \begin{bmatrix} x \\ w_+ \end{bmatrix}, \end{aligned} \quad \begin{bmatrix} x \\ w_+ \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix} \right\}. \tag{9.11}$$

and the corresponding left affine representation of $V$ is

$$V = \left\{ \begin{bmatrix} \left[\begin{smallmatrix} \dot{x} \\ f \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] \end{bmatrix} \;\middle|\; \begin{bmatrix} -1_{\mathcal{X}} & B^{\text{sca}}\widetilde{\Theta}_{21} \\ 0 & (D^{\text{sca}} + \Phi)\widetilde{\Theta}_{21} \end{bmatrix} \begin{bmatrix} \dot{x} \\ f \end{bmatrix} + \begin{bmatrix} A^{\text{sca}} & B^{\text{sca}}\widetilde{\Theta}_{22} \\ C^{\text{sca}} & (D^{\text{sca}} - \Phi)\widetilde{\Theta}_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}. \tag{9.12}$$

The (right Bezout) identity (3.12) becomes

$$\begin{aligned}
&\begin{bmatrix} 0 & 0 \\ 0 & \widetilde{\Theta}_{21} \end{bmatrix} \begin{bmatrix} A^{\text{sca}} & B^{\text{sca}} \\ \Theta_{11}C^{\text{sca}} & \Theta_{11}(D^{\text{sca}} - \Phi) \end{bmatrix} \\
&\qquad + \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & \widetilde{\Theta}_{22} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21}C^{\text{sca}} & \Theta_{21}(D^{\text{sca}} + \Phi) \end{bmatrix} = 1_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{smallmatrix}\right]}, \quad z \in \Lambda_{A^{\text{sca}}}
\end{aligned} \tag{9.13}$$

and the (left Bezout) identity (4.9) becomes

$$\begin{bmatrix} -1_{\mathcal{X}} & B^{\mathrm{sca}}\widetilde{\Theta}_{21} \\ 0 & (D^{\mathrm{sca}}+\Phi)\widetilde{\Theta}_{21} \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & 0 \\ 0 & -\Theta_{11} \end{bmatrix} + \begin{bmatrix} A^{\mathrm{sca}} & B^{\mathrm{sca}}\widetilde{\Theta}_{22} \\ C^{\mathrm{sca}} & (D^{\mathrm{sca}}-\Phi)\widetilde{\Theta}_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -\Theta_{21} \end{bmatrix} = 1_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{smallmatrix}\right]}, \tag{9.14}$$

Formula (3.13) for the right affine is/so impedance matrix becomes

$$\left[\begin{array}{cc} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'_{\mathcal{F}}(z) & \mathfrak{D}'_{\mathcal{E}}(z) \\ \hline 1_{\mathcal{X}} & 0 \\ \mathfrak{C}'_{\mathcal{F}}(z) & \mathfrak{D}'_{\mathcal{E}}(z) \end{array}\right] = \left[\begin{array}{c} \begin{bmatrix} \mathfrak{A}^{\mathrm{sca}}(z) & \mathfrak{B}^{\mathrm{sca}}(z) \\ \Theta_{11}\mathfrak{C}^{\mathrm{sca}}(z) & \Theta_{11}(\mathfrak{D}^{\mathrm{sca}}(z)-\Phi) \end{bmatrix} \\ \hline \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21}\mathfrak{C}^{\mathrm{sca}}(z) & \Theta_{21}(\mathfrak{D}^{\mathrm{sca}}(z)+\Phi) \end{bmatrix} \end{array}\right], \quad z \in \Lambda_{A^{\mathrm{sca}}}, \tag{9.15}$$

and formula (4.10) for the left affine is/so impedance matrix becomes

$$\left[\begin{array}{cc|cc} -1_{\mathcal{X}} & \mathfrak{B}''_{\mathcal{F}}(z) & \mathfrak{A}''(z) & \mathfrak{B}''_{\mathcal{E}}(z) \\ \hline 0 & \mathfrak{D}''_{\mathcal{F}}(z) & \mathfrak{C}''(z) & \mathfrak{D}''_{\mathcal{E}}(z) \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} -1_{\mathcal{X}} & \mathfrak{B}^{\mathrm{sca}}(z)\widetilde{\Theta}_{21} \\ 0 & (\mathfrak{D}^{\mathrm{sca}}(z)+\Phi)\widetilde{\Theta}_{21} \end{bmatrix} & \begin{bmatrix} \mathfrak{A}^{\mathrm{sca}}(z) & \mathfrak{B}^{\mathrm{sca}}(z)\widetilde{\Theta}_{22} \\ \mathfrak{C}^{\mathrm{sca}}(z) & (\mathfrak{D}^{\mathrm{sca}}(z)-\Phi)\widetilde{\Theta}_{22} \end{bmatrix} \end{array}\right], \quad z \in \Lambda_{A^{\mathrm{sca}}}. \tag{9.16}$$

In particular, the right affine i/o impedance matrix is given by

$$\begin{bmatrix} \mathfrak{D}'_{\mathcal{F}}(z) \\ \mathfrak{D}'_{\mathcal{E}}(z) \end{bmatrix} = \begin{bmatrix} \Theta_{11}(\mathfrak{D}^{\mathrm{sca}}(z)-\Phi) \\ \Theta_{21}(\mathfrak{D}^{\mathrm{sca}}(z)+\Phi) \end{bmatrix}, \tag{9.17}$$

and the left affine i/o impedance matrix is given by

$$\begin{bmatrix} \mathfrak{D}''_{\mathcal{F}}(z) & \mathfrak{D}''_{\mathcal{E}}(z) \end{bmatrix} = \begin{bmatrix} (\mathfrak{D}^{\mathrm{sca}}(z)+\Phi)\widetilde{\Theta}_{21} & (\mathfrak{D}^{\mathrm{sca}}(z)-\Phi)\widetilde{\Theta}_{22} \end{bmatrix}. \tag{9.18}$$

**Remark 9.1.** The restriction to $\mathbb{D}$ of the generalized transfer function corresponding to a Lagrangean decomposition $\mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ of a passive s/s system will be defined on the set

$$\Omega_+(\Sigma; \mathcal{E}, \mathcal{F}) = \left\{ z \in \mathbb{D} \;\middle|\; \mathfrak{D}^{\mathrm{sca}}(z) + \Phi \text{ has a bounded inverse} \right\}. \tag{9.19}$$

However, by Theorem III.5.10, either this set is the full disc $\mathbb{D}$, in which the decompostion $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ is admissible, or it is empty. Thus, we conclude that the notion of a generalized transfer functions that we introduced in Section 5 is not really applicable to the case of non-admissible Lagrangan decompositions of the signal space of a passive s/s system. For this situation an even more general notion of transfer function still remains to be developed. We shall return to this elsewhere.

# References


[ADRdS97] Daniel Alpay, Aad Dijksma, James Rovnyak, and Henrik de Snoo, *Schur functions, operator colligations, and reproducing kernel Hilbert spaces*, Operator Theory: Advances and Applications, vol. 96, Birkhäuser-Verlag, Basel Boston Berlin, 1997.

[AS05] Damir Z. Arov and Olof J. Staffans, *State/signal linear time-invariant systems theory. Part I: Discrete time systems*, The State Space Method, Generalizations and Applications (Basel Boston Berlin), Operator Theory: Advances and Applications, vol. 161, Birkhäuser-Verlag, 2005, pp. 115–177.

[AS06a] ———, *State/signal linear time-invariant systems theory. Part II: Passive discrete time systems*, Internat. J. Robust Nonlinear Control **16** (2006), 52 pages, Manuscript available at http://www.abo.fi/˜staffans/.

[AS06b] ———, *State/signal linear time-invariant systems theory. Part III: Transmission and impedance representations of discrete time systems*, To apperar in the volume dedicated to Tiberiu Constantinescu published by the Theta Foundation, 2006.

[DR90] Michael A. Dritschel and James Rovnyak, *Extension theorems for contraction operators on Kreĭn spaces*, Extension and interpolation of linear operators and matrix functions, Oper. Theory Adv. Appl., vol. 47, Birkhäuser, Basel, 1990, pp. 221–305.

[DR96] Michael A. Dritschel and James Rovnyak, *Operators on indefinite inner product spaces*, Lectures on operator theory and its applications (Waterloo, ON, 1994) (Providence, RI, USA) (Peter Lancaster, ed.), Fields Institute Monographs, vol. 3, American Mathematical Society, 1996, pp. 141–232.

[EP73] A. V. Efimov and V. P. Potapov, *$J$-expanding matrix-valued functions, and their role in the analytic theory of electrical circuits*, Uspehi Mat. Nauk **28** (1973), no. 1(169), 65–130. MR MR0394287 (52 #15090)

[KP65] I. V. Kovališina and V. P. Potapov, *Multiplicative structure of analytic real $J$-dilative matrix-functions*, Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **18** (1965), no. 6, 3–10. MR MR0214780 (35 #5629)

[KP74] ______, *An indefinite metric in the Nevanlinna-Pick problem*, Akad. Nauk Armjan. SSR Dokl. **59** (1974), no. 1, 17–22. MR MR0409825 (53 #13577)

[Opm05] Mark R. Opmeer, *Infinite-dimensional linear systems: A distributional approach*, Proc. London Math. Soc. **91** (2005), 738–760.

[Pot60] V. P. Potapov, *The multiplicative structure of $J$-contractive matrix functions*, Amer. Math. Soc. Transl. (2) **15** (1960), 131–243. MR MR0114915 (22 #5733)

[RR82] Marvin Rosenblum and James Rovnyak, *An operator-theoretic approach to theorems of the Pick–Nevanlinna and Loewner types, II*, Integral Equations Operator Theory **5** (1982), 870–887.

[Sta05] Olof J. Staffans, *Well-posed linear systems*, Cambridge University Press, Cambridge and New York, 2005.