# Symmetries in Special Classes of Passive State/Signal Systems

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**Abstract**

This work is devoted to the study of four types of symmetries in the class of possibly infinite-dimensional passive linear time-invariant state/signal systems in continuous time, namely the real, the reciprocal, the signature, and the transpose symmetry. We are, in particular, interested in the relationship between internal and external properties of systems which have one or several of these symmetries. Both the real, the reciprocal, and the transpose symmetries are well-known in the passive input/state/output theory. In that setting reality means that if the initial state and input are real, then the output of the system is real, and also the state of the system remains real for all time. The reciprocal and transpose symmetries is related to duality. The external characterisation of these symmetries in the input/state/output

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setting is a certain symmetry condition on the transfer function, and the internal characterisation of reciprocity says that the system should be unitarily similar to its adjoint.

It is not true for all classes of passive systems that external symmetry properties are automatically reflected in internal properties. This happens only for certain classes of systems that are uniquely determined up to unitary similarity by their external characteristics. One such class is the class of simple conservative systems, which is by now fairly well understood. We here introduce and study three other classes of passive state/signal system in continuous time, namely the classes of optimal, $*$-optimal, and passive balanced state/signal systems and study their symmetry properties. The optimal and $*$-optimal systems are passive and extremal in a certain sense, and the passive balanced systems are obtained by interpolation between the classes of minimal optimal and $*$-optimal systems. It is true for simple systems in all of these classes that external reality or signature symmetry implies internal reality or signature symmetry. The same statement remains true for the reciprocal and transpose symmetries for conservative systems and for passive balanced systems.

**Keywords**

Reality, reciprocity, available storage, required supply, optimal system, $*$-optimal system, conjugation, skew-conjugation, signature operator, skew-signature operator.

# Contents

# 1 Introduction

The roots of the passive s/s systems theory lie partially in operator theory, partially in circuit theory, and partially in passive i/s/o (input/state/output) systems theory. It is a well-known fact that the theories of passive and conservative scattering, transmission, and impedance i/s/o systems in continuous and discrete time are intimately connected with the theory on the harmonic analysis of operators in Hilbert spaces, see, e.g., [Liv73], [dBR66], [ADRdS97], and [Sta05]. In the so called *inverse problem* the goal is to a construct simple conservative, or an obersvable co-energy preserving, or a controllable energy-preserving i/s/o (input/state/output) realization of a given scattering, transmission, or impedance function. If the given data has some additional symmetries, then one expects this to be reflected in some extra symmetry properties of the constructed realizations; see, e.g., [Liv73, Chapter 5], [Wil72, Sections 8–9], and [ADRdS97, Section 3.5B].

A theory of passive linear time-invariant s/s (state/signal) systems in discrete and continuous time has recently been developed in a series of papers [AS05b, AS07b, AS07c, AS07d, AS09a, AS09b, AS10, Kur10, KS09, AKS11b, AKS11a, AKS11c]. Here we continue that development by introducing some additional classes of passive s/s systems which are uniquely determined by their external properties up to unitary similarity, namely the classes of minimal optimal, $*$-optimal, and balanced state/signal systems in continuous time. We also study their symmetry properties, and in particular, the connection between *external* and *internal* symmetry of systems belonging to the appropriate class of systems.

The symmetry results for s/s systems that we derive here have been motivated by, and they are closely connected to the corresponding symmetry results for i/s/o systems mentioned above. The principal connection is the following: By decomposing the signal space $\mathcal{W}$ of a passive s/s system into a direct sum $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ and interpreting $\mathcal{U}$ as an *input space* and $\mathcal{Y}$ as an *output space* one obtains different i/s/o representations of the given s/s system. Under suitable invariance conditions on the decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ with respect to some given s/s symmetery one may then from our results derive results about symmetries for i/s/o systems. To some extent it is also possible to proceed in the opposite direction. Because of lack of space we have not been able to here draw the full picture, but in Section 11 we point out one basic connection to i/s/o symmetry results of scattering type.

A linear continuous time invariant s/s (state/signal) system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has a Hilbert (state) space $\mathcal{X}$, a Kreĭn (signal) space $\mathcal{W}$, and a closed (generating) subspace $V$ of the (node) space $\mathfrak{K} = \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ that satisfies some additional conditions, among them the condition

$$\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0. \tag{1.1}$$

Condition (1.1) means that $V$ is the graph of some linear operator $G\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \to \mathcal{X}$ with domain $\operatorname{dom}(G) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$. Since $V$ is assumed to be closed, the operator $G$ is closed. By a *classical trajectory* of $\Sigma$ on the interval $I \subset \mathbb{R}$ we mean a pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(I;\mathcal{X}) \\ C(I;\mathcal{W}) \end{smallmatrix}\right]$ satisfying

$$\Sigma : \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in I, \tag{1.2}$$

or equivalently,

$$\Sigma : \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \operatorname{dom}(G) \text{ and } \dot{x}(t) = G \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \qquad t \in I. \tag{1.3}$$

By a *generalised trajectory* of $\Sigma$ on $I$ we mean a pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{\mathrm{loc}}(I;\mathcal{W}) \end{smallmatrix}\right]$ which is the limit in this space of a sequence $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ of classical trajectories of $\Sigma$ on $I$.

The notion of *passivity* of a s/s system $\Sigma$ is used to model s/s systems which "have no internal energy sources". More precisely, we interpret $\frac{1}{2}\|x(t)\|^2_{\mathcal{X}}$ as the *internal energy* of $\Sigma$, suppose that the power entering $\Sigma$ from the surroundings via the signal $w(t)$ is equal to $\frac{1}{2}[w(t), w(t)]_{\mathcal{W}}$, and require all classical trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$ on all intervals $I \subset \mathbb{R}$ to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|^2_{\mathcal{X}} \le [w(t), w(t)]_{\mathcal{W}}, \qquad t \in I. \tag{1.4}$$

Incidentally, this explains why we need to allow the inner product in $\mathcal{W}$ to be indefinite: If the inner product in $\mathcal{W}$ is positive, then no energy can leave the system through the signal, and if the inner product in $\mathcal{W}$ is negative, then no energy can enter the system.

By (1.2), a sufficient condition for (1.4) to hold is that

$$-(z, x)_{\mathcal{X}} - (z, x)_{\mathcal{X}} + [w, w]_{\mathcal{W}} \ge 0, \qquad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V. \tag{1.5}$$

This makes it natural to introduce the following (strictly indefinite) Kreĭn space inner product in the node space $\mathfrak{K}$:

$$\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix}\right]_{\mathfrak{K}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}. \tag{1.6}$$

Then (1.5) says that $V$ is a nonnegative subspace of $\mathfrak{K}$ with respect to the inner product (1.6), and (1.4) says that all classical trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$ on all intervals $I \subset \mathbb{R}$ should satisfy

$$\left[\left[\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}, \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}\right]\right]_{\mathfrak{K}} = -\frac{\mathrm{d}}{\mathrm{d}t}\,\|x(t)\|_{\mathcal{X}}^2 + [w(t), w(t)]_{\mathcal{W}} \geq 0, \qquad t \in I. \tag{1.7}$$

Thus, the generating subspace $V$ of a passive s/s system $\Sigma$ should at least be nonnegative in the node space $\mathfrak{K}$ with respect to the inner product (1.6). However, it turns out that it is natural to impose a somewhat stronger condition, namely that it should be *maximal nonnegative* in the sense that it is not contained in any other nonnegative subspace. Thus, we call a s/s system $\Sigma$ *passive* if its generating subspace $V$ is a *maximal nonnegative subspace of the Kreĭn* node space $\mathfrak{K}$ which satisfies (1.1).

By taking $I = \mathbb{R}^+ := [0, \infty)$ in (1.2), multiplying (1.2) by $\mathrm{e}^{-\lambda t}$ and integrating over $\mathbb{R}^+$ we find that the Laplace transform $\left[\begin{smallmatrix} \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{smallmatrix}\right]$ of a bounded trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$ on $\mathbb{R}^+$ satisfies

$$\begin{bmatrix} \lambda\widehat{x}(\lambda) - x(0) \\ \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{bmatrix} \in V, \qquad \Re\lambda > 0. \tag{1.8}$$

This can equivalently be written as $\left[\begin{smallmatrix} x(0) \\ \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda)$, where

$$\widehat{\mathfrak{E}}(\lambda) := \left\{ \begin{bmatrix} x_0 \\ x \\ w \end{bmatrix} \in \mathfrak{K} \;\middle|\; \begin{bmatrix} \lambda x - x_0 \\ x \\ w \end{bmatrix} \in V \right\}, \qquad \lambda \in \mathbb{C}. \tag{1.9}$$

The family $\widehat{\mathfrak{E}} := \{\widehat{\mathfrak{E}}(\lambda)\}_{\lambda\in\mathbb{C}}$ is called the *characteristic node bundle* of $\Sigma$, and each subspace $\widehat{\mathfrak{E}}(\lambda)$ is called the *fiber* of $\widehat{\mathfrak{E}}$ at $\lambda$. This bundle turns out to be *analytic* in $\mathbb{C}$ in the sense that the orientation of the fibers $\widehat{\mathfrak{E}}(\lambda)$ depens analytically on $\lambda$. From this analytic bundle we can obtain the *characteristic signal bundle* $\widehat{\mathfrak{F}} := \{\widehat{\mathfrak{F}}(\lambda)\}_{\lambda\in\mathbb{C}}$ by taking the initial state $x(0)$ to be zero and projecting each resulting fiber result onto $\mathcal{W}$, i.e.,

$$\widehat{\mathfrak{F}}(\lambda) := \left\{ w \in \mathcal{W} \;\middle|\; \begin{bmatrix} \lambda x \\ x \\ w \end{bmatrix} \in V \text{ for some } x \in \mathcal{X} \right\}, \qquad \lambda \in \mathbb{C}^+. \tag{1.10}$$

This bundle is analytic in $\mathbb{C}^+ := \{\lambda \in \mathbb{C} \mid \Re\lambda > 0\}$. As will be shown below, each fiber $\widehat{\mathfrak{F}}(\lambda)$ is a maximal nonnegative subspace of the signal space $\mathcal{W}$ for each $\lambda \in \mathbb{C}^+$.

After this short presentation of the class of passive s/s systems, let us continue by discussing the symmetry properties of such systems. The *external* symmetries that we are interested in can be expressed in terms of the characteristic signal bundle $\widehat{\mathfrak{F}}$ as follows. We let $\mathcal{J}_{\mathcal{W}}$, $\mathcal{C}_{\mathcal{W}}$, $\mathcal{I}_{\mathcal{W}}$, and $\mathcal{B}_{\mathcal{W}}$ be a singature operator, a conjugation, a skew-signature operator, and a skew-conjugation in $\mathcal{W}$, respectively (see Sections 2.4 and 2.3 for the definitions of a these classes of operators). *We call* $\Sigma$

(a) *externally* $\mathcal{C}_{\mathcal{W}}$*-real* if

$$\widehat{\mathfrak{F}}(\lambda) = \mathcal{C}_{\mathcal{W}}\widehat{\mathfrak{F}}(\overline{\lambda}), \qquad \lambda \in \mathbb{C}^+, \tag{1.11}$$

(b) *externally* $\mathcal{I}_{\mathcal{W}}$*-reciprocal* if

$$\widehat{\mathfrak{F}}(\lambda) = \mathcal{I}_{\mathcal{W}}\widehat{\mathfrak{F}}(\overline{\lambda})^{[\perp]}, \qquad \lambda \in \mathbb{C}^+, \tag{1.12}$$

(c) *externally* $\mathcal{J}_{\mathcal{X}}$*-signature invariant* if

$$\widehat{\mathfrak{F}}(\lambda) = \mathcal{J}_{\mathcal{W}}\widehat{\mathfrak{F}}(\lambda), \qquad \lambda \in \mathbb{C}^+, \tag{1.13}$$

(d) *externally* $\mathcal{B}_{\mathcal{W}}$*-transpose invariant* if

$$\widehat{\mathfrak{F}}(\lambda) = \mathcal{B}_{\mathcal{W}}\widehat{\mathfrak{F}}(\lambda)^{[\perp]}, \qquad \lambda \in \mathbb{C}^+. \tag{1.14}$$

In (1.12) and (1.14) the notation $\widehat{\mathfrak{E}}(\lambda)^{[\perp]}$ stands for the orthogonal companion to $\widehat{\mathfrak{E}}(\lambda)$ in $\mathcal{W}$, i.e., the set of vectors in $\mathcal{W}$ which are orthogonal to $\widehat{\mathfrak{F}}(\lambda)$ with respect to the Kreĭn space inner product in $\mathcal{W}$.

The above notions of four*external symmetries* of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ are related to the notions of their respective four (full) symmetries of $\Sigma$. To define these symmetries we introduce two additional operators in $\mathcal{X}$, a signature operator $\mathcal{J}_{\mathcal{X}}$ and a conjugation $\mathcal{C}_{\mathcal{X}}$. *We call* $\Sigma$

(a) $(\mathcal{C}_{\mathcal{X}}, \mathcal{J}_{\mathcal{X}})$*-real* if

$$V = \begin{bmatrix} \mathcal{C}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{C}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{C}_{\mathcal{W}} \end{bmatrix} V, \tag{1.15}$$

(b) $(\mathcal{J}_{\mathcal{X}}, \mathcal{I}_{\mathcal{W}})$*-reciprocal* if

$$V = \begin{bmatrix} -\mathcal{J}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{I}_{\mathcal{W}} \end{bmatrix} V^{[\perp]}, \tag{1.16}$$

(c) $(\mathcal{J}_{\mathcal{X}}, \mathcal{J}_{\mathcal{W}})$-*signature invariant* if

$$V = \begin{bmatrix} \mathcal{J}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{J}_{\mathcal{W}} \end{bmatrix} V, \tag{1.17}$$

(d) $(\mathcal{C}_{\mathcal{X}}, \mathcal{B}_{\mathcal{W}})$-*transpose invariant* if

$$V = \begin{bmatrix} -\mathcal{C}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{C}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{B}_{\mathcal{W}} \end{bmatrix} V^{[\perp]}, \tag{1.18}$$

It is not difficult to show that the block operators on the right-hand sides of the above equation are skew-unitary operators (which are either linear or conjugate-linear) in the node space $\mathfrak{K}$, and that if $V$ is the generating subspace of a passive s/s system, then each of the right-hand sides of (1.17), (1.15), (1.16), and (1.18) are also generating subspaces of passive s/s systems. The subspace $V^{\perp}$ appearing in (1.16) and (1.18) is related to the generating subspace of the adjoint s/s system $\Sigma_* = (V_*; \mathcal{X}, -\mathcal{W})$, where

$$V_* = \begin{bmatrix} -1_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{I}_{(\mathcal{W},-\mathcal{W})} \end{bmatrix} V^{[\perp]}, \tag{1.19}$$

and $\mathcal{I}_{(\mathcal{W},-\mathcal{W})}$ is the identity map from $\mathcal{W}$ to the anti-space $-\mathcal{W}$. Clearly, the two equations (1.16) and (1.18) can be rewritten in the forms

$$V = \begin{bmatrix} \mathcal{J}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{I}_{\mathcal{W}}\mathcal{I}_{(-\mathcal{W},\mathcal{W})} \end{bmatrix} V_* \tag{1.20}$$

and

$$V = \begin{bmatrix} \mathcal{C}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{C}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{B}_{\mathcal{W}}\mathcal{I}_{(-\mathcal{W},\mathcal{W})} \end{bmatrix} V_*, \tag{1.21}$$

respectively. Thus, reciprocity means that the system $\Sigma$ is *signature similar* to the system that one gets from the adjoint system $\Sigma_*$ by multiplying the dual signal by the unitary operator $\mathcal{I}_{\mathcal{W}}\mathcal{I}_{(-\mathcal{W},\mathcal{W})}$. Actually, as we shall see in Lemma 6.7 below, we can here alternatively replace "signature similar" by *unitarily similar*. Transpose invariance has a similar interpretation, as explained in more detail in Section 8.

As we shall show in Sections 5–8, it is always true that the full symmetries described above imply the corresponding external symmetries. For certain subclasses of systems the converse is also true in the sense that external symmetry implies the existence of a unique signature operator $\mathcal{J}_{\mathcal{X}}$ or conjugation $\mathcal{C}_{\mathcal{X}}$ in $\mathcal{X}$ such that the system is fully symmetric with respect to this operator in the state space $\mathcal{X}$ and the originally given operator in the signal space $\mathcal{W}$. In the case of the *signature* and *real symmetries* the converse claims hold for the classes of passive

a) simple conservative systems,

b) controllable energy preserving systems,

c) observable co-energy preserving systems,

d) minimal optimal systems,

e) minimal $*$-optimal systems,

f) minimal passive balanced systems.

All of these classes have the property that a passive s/s system in one of these classes is uniquely determined by its characteristic signal bundle up to a unitary similarity transformation in the state space. The first three classes a)–c) have been studied in [AKS11b], and here we introduce and study the three additional classes d)–f). In the case of the *reciprocal* and *transpose symmetries* we have to restrict the class further and require that it is closed under duality. This only leaves two of the above classes, namely the class a) of simple conservative systems, and the class f) of minimal passive balanced systems where external reciprocity implies full reciprocity.

For each one of the six classes a)–f) it is possible to construct a *canonical model* such that every passive s/s system in this class is unitarily similar to its model. The constructions of these models employ Hilbert spaces of type $\mathcal{H}(\mathcal{Z})$, where $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space. A short overview of such spaces is given in Section 2.2. These spaces were first presented in [AS09a], and they can be regarded as coordinate free versions of de Branges complementary spaces. Three canonical models of classes a)–c) were developed in [AKS11b], and they are reviewed in Section 3.6. In Chapter 4 we here develop three additional canonical models of type d)–f).

The discussion about symmetries given above is carried out in the frequency domain, because we feel that this setting is likely to be more familiar to most readers than a time domain setting. However, all the main results of this article are first presented in the time domain, and only in the very last chapter we show that the time domain results that we have obtained are equivalent to the frequency domain results that we present above.

To keep the size of this article within reasonabe limits we have not been able to include the answer of every question that naturally arises. We shall return to these questions elsewhere. In particular, we are thinking about, among others, the following additional results:

- introduction and study of the classes of minimal optimal, $*$-optimal, and balanced *discrete time* s/s systems, and their *canonical models*,
- the four basic *symmetries* for *discrete time* passive s/s systems,
- *internal symmetry implies external symmetry* in certain cases, both in continuous and discrete time,
- *input/state/output versions* of the four basic symmetries in continuous and discrete time,
- various *examples* of systems with symmetries,
- further studies of the *characteristic node and signal bundles.*

We end this section by presenting various notations and conventions that we use.

An (inner) direct sum decomposition of a Hilbert or Kreĭn space $\mathcal{W}$ into two closed subspaces $\mathcal{U}$ and $\mathcal{Y}$ will be denoted by $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, and the corresponding complementary projections onto $\mathcal{U}$ and $\mathcal{Y}$ will be denoted by $P_{\mathcal{U}}^{\mathcal{Y}}$ and $P_{\mathcal{Y}}^{\mathcal{U}}$. If, in addition, $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal to each other, then we write $\mathcal{W} = \mathcal{U} \oplus \mathcal{Y}$ in the case of a Hilbert space and $\mathcal{W} = \mathcal{U} \boxplus \mathcal{Y}$ in the case of a Kreĭn space. In the orthogonal case the subspaces $\mathcal{U}$ and $\mathcal{Y}$ become Hilbert or Kreĭn spaces when we let them inherit the inner product from $\mathcal{W}$, and we denote the (orthogonal) projections of $\mathcal{W}$ onto $\mathcal{U}$ and $\mathcal{Y}$ by $P_{\mathcal{U}}$ and $P_{\mathcal{Y}}$, respectively.

We denote the (external) direct sum of two Hilbert or Kreĭn spaces $\mathcal{U}$ and $\mathcal{Y}$ by $\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$. By this we mean the Cartesian product of $\mathcal{U}$ and $\mathcal{Y}$ equipped with the standard algebraic operations and standard product topology. We sometimes equip $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ with the induced Kreĭn space inner product (in the Kreĭn space notation)

$$\left[\begin{bmatrix}u_1\\ y_1\end{bmatrix}, \begin{bmatrix}u_2\\ y_2\end{bmatrix}\right]_{\mathcal{U}\boxplus\mathcal{Y}} = [u_1, u_2]_{\mathcal{U}} + [y_1, y_2]_{\mathcal{Y}}. \tag{1.22}$$

After identifying $\left[\begin{smallmatrix}\mathcal{U}\\ 0\end{smallmatrix}\right]$ with $\mathcal{U}$ and $\left[\begin{smallmatrix}0\\ \mathcal{Y}\end{smallmatrix}\right]$ with $\mathcal{Y}$ we can in this case identify $\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$ with orthogonal sum $\mathcal{U} \boxplus \mathcal{Y}$ of $\mathcal{U}$ and $\mathcal{Y}$. However, we shall often instead use a different Kreĭn space inner product in $\begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}$ of the type

$$\left[\begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}\right]_{\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]} = \left(\begin{bmatrix} u \\ y \end{bmatrix}, \mathcal{J}\begin{bmatrix} u \\ y \end{bmatrix}\right)_{\mathcal{U}\boxplus\mathcal{Y}},$$

where $\mathcal{J}$ is a given signature operator in $\mathcal{U} \boxplus \mathcal{Y}$. With respect to this inner product $\mathcal{U}$ and $\mathcal{Y}$ may or may not be orthogonal. Analogous notations are used for direct sums with three or more components.

**List of Notations.**

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| $\mathbb{R}, \mathbb{R}^+, \mathbb{R}^-$ | $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}^+ := [0, \infty)$, $\mathbb{R}^- = (-\infty, 0]$. |
| $\mathbb{C}, \mathbb{C}^+$ | $\mathbb{C}$ is the complex plane and $\mathbb{C}^+ = \{\lambda \in \mathbb{C} \mid \Re\lambda > 0\}$. |
| $\overline{\Omega}$ | The closure of $\Omega$. |
| $\mathcal{B}(\mathcal{U}; \mathcal{Y})$ | The space of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$. |
| $\overline{\mathcal{B}}(\mathcal{U}; \mathcal{Y})$ | The space of bounded conjugate-linear operators from $\mathcal{U}$ to $\mathcal{Y}$. |
| $\mathrm{dom}\,(A)\,, \mathrm{im}\,(A)\,, \ker\,(A)$: | The domain, range, and kernel of the operator $A$. |
| $A\vert_{\mathcal{Z}}$ | The restriction of the operator $A$ to $\mathcal{Z}$. |
| $(\cdot, \cdot)_{\mathcal{X}}$ | The inner product in the Hilbert space $\mathcal{X}$. |
| $[\cdot, \cdot]_{\mathcal{W}}$ | The inner product in the Kreĭn space $\mathcal{W}$. |
| $-\mathcal{K}$ | The anti-space of the Kreĭn space $\mathcal{K}$. This is the same topological vector space as $\mathcal{K}$, but it has a different inner product $[\cdot, \cdot]_{-\mathcal{K}} := -[\cdot, \cdot]_{\mathcal{K}}$. |
| $\tau^t$ | $(\tau^t w)(s) = w(s + t)$, $s$, $t \in \mathbb{R}$ (this is a left shift if $t > 0$). |
| $\tau^t_+$ | $(\tau^t_+ w)(s) = w(s + t)$, $s$, $t \in \mathbb{R}^+$ (this is a left shift if $t > 0$). |
| $\tau^t_-$ | $(\tau^t_- w)(s) = w(s + t)$ if $s + t \le 0$, $(\tau^t_- w)(s) = 0$ if $s + t > 0$. Here $s \in \mathbb{R}^-$, $t \in \mathbb{R}^+$. |
| $\tau^{*t}$ | $(\tau^{*t} w)(s) = (\tau^{-1} w)(s) = w(s - t)$, $s$, $t \in \mathbb{R}$ (this is a right shift if $t > 0$). |
| $\tau^{*t}_+$ | $(\tau^{*t}_+ w)(s) = w(s-t)$ if $s-t \ge 0$ and $(\tau^{*t}_+ w)(s) = 0$ if $s-t < 0$. Here $s$, $t \in \mathbb{R}^+$. |
| $\tau^{*t}_-$ | $(\tau^{*t}_- w)(s) = w(s - t)$ for all $s \in \mathbb{R}^-$, $t \in \mathbb{R}^+$. |
| $\pi_I, \pi_+, \pi_-$ | $(\pi_I w)(s) = w(s)$ for all $s \in I$. We abbreviate $\pi_- = \pi_{\mathbb{R}^-}$ and $\pi_+ = \pi_{\mathbb{R}^+}$. |
| $C(I; \mathcal{X}), BUC(I; \mathcal{X}), C^1(I; \mathcal{X})$: | The spaces of continuous, bounded uniformly continuous, or continuously differentiable functions, respectively, on $I$ with values in $\mathcal{X}$, with the standard norms. |

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|---|---|
| $L^2_{\mathrm{loc}}(I;\mathcal{W})$ | The space of functions from $I$ to $\mathcal{W}$ which belong locally to $L^2$. |
| $H^2(\mathbb{C}^+;\mathcal{X})$ | The space of holomorphic $\mathcal{W}$-valued functions on $\mathbb{C}^+$ with finite $H^2$-norm. |
| $K^2(\mathcal{W}), K^2_+(\mathcal{W}), K^2_-(\mathcal{W})$: | See (3.5). |
| $\widehat{K}^2_+(\mathcal{W})$ | See the discussion after (10.5). |
| $\mathfrak{K}$ | The Kreĭn node space $\mathfrak{K} = \mathcal{X} \times \mathcal{X} \times \mathcal{W}$ equipped with the inner product (1.6). |
| $1_{\mathcal{X}}$ | The identity operator in the topological vector space $\mathcal{X}$. |
| $\mathcal{I}_{(\mathcal{W},-\mathcal{W})}$ | The identity operator from the Kreĭn space $\mathcal{W}$ onto the anti-space $-\mathcal{W}$. |
| $\boldsymbol{\text{Я}}$ | The reflection operator $(\boldsymbol{\text{Я}}w)(t) = w(-t)$. If $w$ is defined on the interval $I \subset \mathbb{R}$, then $\boldsymbol{\text{Я}}w$ is defined on the reflected interval $\boldsymbol{\text{Я}}I = \{t \in \mathbb{R} \mid -t \in I\}$. |
| $\mathcal{H}(\mathcal{Z}), \mathcal{H}^0(\mathcal{Z})$ | See Section 2.1. |
| $\mathfrak{W}, \mathfrak{W}_+, \mathfrak{W}_-$ | A passive two-sided, future, or past behaviour, respectively, on the Kreĭn signal space $\mathcal{W}$. See Section 3.2. |
| $\mathcal{H}_\pm, \mathcal{H}(\mathfrak{W}_+), \mathcal{H}(\mathfrak{W}_-^{[\perp]})$: | See Section 3.4. |
| $\mathcal{H}^0_\pm, \mathcal{H}^0(\mathfrak{W}_+), \mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$: | See Section 3.4. |
| $\mathcal{K}_\pm, \mathcal{K}(\mathfrak{W}_+), \mathcal{K}(\mathfrak{W}_-^{[\perp]})$: | See Section 3.4. |
| $Q_+, Q_-, Q$ | See Section 3.4. |
| $\Gamma_{\mathfrak{W}}, \mathcal{D}(\mathfrak{W}), \mathcal{L}(\mathfrak{W})$: | See Section 3.4. |
| $\mathfrak{B}_\Sigma, \mathfrak{C}_\Sigma$ | The past/present and present/future maps of the passive system $\Sigma$. See Section 3.5. |
| $P_{\mathcal{U}}, P_{\mathcal{U}}^{\mathcal{Y}}$ | $P_{\mathcal{U}}$ is the orthogonal projection onto $\mathcal{U}$, and $P_{\mathcal{U}}^{\mathcal{Y}}$ is the projection onto $\mathcal{U}$ along $\mathcal{Y}$. |
| $\mathcal{X} \oplus \mathcal{Z}, \mathcal{X} \boxplus \mathcal{Z}$ | The orthogonal direct sum of the two subspaces $\mathcal{X}$ and $\mathcal{Z}$ of a Hilbert or Kreĭn space, respectively. |
| $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Z}\end{smallmatrix}\right]$ | The Cartesian product of the Kreĭn spaces $\mathcal{X}$ and $\mathcal{Z}$. The topology in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Z}\end{smallmatrix}\right]$ is the one induced by $\mathcal{X}$ and $\mathcal{Z}$, but $\left[\begin{smallmatrix}\mathcal{X}\\ 0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\ \mathcal{Z}\end{smallmatrix}\right]$ need not be orthogonal to each other with respect to the product of the inner products in $\mathcal{X}$ and $\mathcal{Z}$. |

**Remark 1.1.** If $A$ is a bounded linear (or conjugate-linear) operator in the Kreĭn space $\mathcal{W}$, then it induces a bounded linear (or conjugate-linear)

operator on the Kreĭn space $K^2(\mathcal{W})$, which we also denote by $A$, and which is defined point-wise by

$$(Aw)(t) = A(w(t)), \qquad t \in \mathbb{R}, \qquad w \in K^2(\mathcal{W}). \tag{1.23}$$

The operator $A$ on $K^2(\mathcal{W})$ defined in this way is shift-invariant, i.e., $\tau^t A = A\tau^t$, $t \in \mathbb{R}$, it commutes with the reflection operator $\mathbf{Я}$, i.e., $A\mathbf{Я} = \mathbf{Я}A$, and both $K^2_+(\mathcal{W})$ and $K^2_-(\mathcal{W})$ are invariant under $A$. If the original operator $A$ as a bounded inverse, or is unitary, or skew-unitary, or self-adjoint, or skew-adjoint, or a signature operator, or a conjugation, or a skew-signature operator, or a skew-conjugation, or an involution, then operator $A$ on $K^2(\mathcal{W})$ has the same property. Whenever $A$ is invertible we have, in addition, $AK^2_\pm(\mathcal{W}) = K^2_\pm(\mathcal{W})$.

# 2 Kreĭn Spaces

In this section we present the main notions on the geometry of Kreĭn spaces and related results that will be used in this article. We recall the definition of the special Hilbert spaces of type $\mathcal{H}(\mathcal{Z})$ introduced in [AS09a], where $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space. We also introduce two classes of involution operators in the Hilbert state space $\mathcal{X}$ and the Kreĭn signal space $\mathcal{W}$ which are needed in our study of the real and reciprocal symmetries.

## 2.1 Some properties of Kreĭn spaces

A Kreĭn space $\mathcal{W}$ is a vector space with an inner product $[\cdot,\cdot]_{\mathcal{W}}$ that satisfies all the standard properties required by a Hilbert space inner product, except that the condition $[w,w]_{\mathcal{W}} > 0$ for nonzero $w$ has been replaced by the condition that $\mathcal{W}$ can be decomposed into a direct sum

$$\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y} \tag{2.1}$$

such that the following conditions are satisfied:

(i) $\mathcal{U}$ is a Hilbert space with the inner product inherited from $\mathcal{W}$, i.e., $(u,u)_{\mathcal{U}} := [u,u]_{\mathcal{W}} > 0$ if $u \in \mathcal{U}$, $u \neq 0$, and $\mathcal{U}$ is complete with respect to the norm $\|u\|_{\mathcal{U}} = ((u,u)_{\mathcal{W}})^{1/2}$.

(ii) $-\mathcal{Y}$ is an anti-Hilbert space with the inner product inherited from $\mathcal{W}$, i.e., $[y,y]_{-\mathcal{Y}} := [y,y]_{\mathcal{W}} < 0$ if $y \in \mathcal{Y}$, $y \neq 0$, and $-\mathcal{Y}$ is complete with respect to the norm $\|y\|_{\mathcal{Y}} = (-[y,y]_{\mathcal{Y}})^{1/2}$.

(iii) $\mathcal{U}$ and $-\mathcal{Y}$ are orthogonal to each other with respect to the inner product $[\cdot,\cdot]_{\mathcal{W}}$, i.e., $[y,u]_{\mathcal{W}} = 0$ for all $u \in \mathcal{U}$ and all $y \in -\mathcal{Y}$.

A decomposition (2.1) with properties (i)–(iii) above is called a *fundamental decomposition*. Unless $\mathcal{W}$ itself is either a Hilbert space or an anti-Hilbert space, then it has infinitely many such decompositions. We denote the anti-space of $-\mathcal{Y}$ by $\mathcal{Y}$, i.e., $\mathcal{Y}$ is the Hilbert space which is algebraically the same as $-\mathcal{Y}$, but the inner product in $\mathcal{Y}$ is given by $(\cdot,\cdot)_{\mathcal{Y}} = -[\cdot,\cdot]_{-\mathcal{Y}} = -[\cdot,\cdot]_{\mathcal{W}}$.

Each fundamental decomposition (2.1) can be used to define a new Hilbert space inner product

$$\begin{gathered}(w,w')_{\mathcal{W}} = (w,w')_{\mathcal{U}\oplus\mathcal{Y}} = (u,u)_{\mathcal{U}} + (y,y')_{\mathcal{Y}} \\ w = u + y,\ u,u' \in \mathcal{U},\ y,y' \in \mathcal{Y}.\end{gathered} \tag{2.2}$$

An Hilbert space inner product in $\mathcal{W}$ obtained in this way is called *admissible*. The original inner product $[\cdot,\cdot]_{\mathcal{W}}$ satisfies

$$\begin{aligned} [w,w']_{\mathcal{W}} = [w,w']_{\mathcal{U}\boxplus-\mathcal{Y}} = (u,u)_{\mathcal{U}} - (y,y')_{\mathcal{Y}} \\ w = u+y,\ u,u' \in \mathcal{U},\ y,y' \in \mathcal{Y}. \end{aligned} \tag{2.3}$$

Although the inner produce (2.2) depends on the particular fundamental decomposition (2.1), the norms induced by the different Hilbert space inner products (2.2) are all equivalent to each other. These norms are called *admissible* norms in $\mathcal{W}$. The dimensions of the positive space $\mathcal{U}$ and the negative space $-\mathcal{Y}$ do not depend on the particular fundamental decomposition. These dimensions are called the positive and negative indices of $\mathcal{W}$, and they are denoted by $\operatorname{ind}_+\mathcal{W}$ and $\operatorname{ind}_-\mathcal{W}$.

The *orthogonal companion* $\mathcal{Z}^{[\perp]}$ of an arbitrary subset $\mathcal{Z} \subset \mathcal{W}$ with respect to the Kreĭn space inner product $[\cdot,\cdot]_{\mathcal{W}}$ consists of all vectors in $\mathcal{W}$ that are orthogonal to all vectors in $\mathcal{Z}$, i.e.,

$$\mathcal{Z}^{[\perp]} = \{w' \in \mathcal{W} \mid [w',w]_{\mathcal{W}} = 0 \text{ for all } w \in \mathcal{Z}\}.$$

This is always a closed subspace of $\mathcal{W}$, and $\mathcal{Z} = (\mathcal{Z}^{[\perp]})^{[\perp]}$ if and only if $\mathcal{Z}$ is a closed subspace. If $\mathcal{W}$ is a Hilbert space, then we write $\mathcal{Z}^{\perp}$ instead of $\mathcal{Z}^{[\perp]}$.

A vector $w \in \mathcal{W}$ is called positive, nonnegative, negative, nonpositive, or neutral if $[w,w]_{\mathcal{W}} > 0$, $[w,w]_{\mathcal{W}} \geq 0$, $[w,w]_{\mathcal{W}} < 0$, $[w,w]_{\mathcal{W}} \leq 0$, or $[w,w]_{\mathcal{W}} = 0$, respectively. A subspace $\mathcal{Z}$ of $\mathcal{W}$ is called positive, nonnegative, negative, nonpositive, or neutral if all nonzero vectors in $\mathcal{Z}$ are positive, nonnegative, negative, nonpositive, or neutral. It is clear that a subspace $\mathcal{Z}$ of $\mathcal{W}$ is neutral if and only if $\mathcal{Z} \subset \mathcal{Z}^{[\perp]}$. If instead $\mathcal{Z}^{[\perp]} \subset \mathcal{Z}$, then $\mathcal{Z}$ is called co-neutral, and if $\mathcal{Z} = \mathcal{Z}^{[\perp]}$, then $\mathcal{Z}$ is called a *Lagrangian* (or hypermaximal neutral) subspace of $\mathcal{W}$. A nonnegative subspace which is not strictly contained in any other nonnegative subspace is called *maximal nonnegative*, and the notion of a *maximal nonpositive subspace* is defined in an analogous way. Maximal nonnegative or nonpositive subspaces are always closed. Every nonnegative subspace is contained in some maximal nonnegative subspace, and every nonpositive subspace is contained in some maximal nonpositive subspace. This follows, for example, from the following proposition.

**Proposition 2.1.** *Let $\mathcal{W}$ be a Kreĭn space with fundamental decomposition* (2.1)*, and let $\mathcal{Z}$ be a subspace of $\mathcal{W}$. Then the following claims are true:*

(i) *The subspace $\mathcal{Z}$ is nonnegative if and only if it is the graph of a (unique) linear Hilbert space contraction $A_+ : \mathcal{U} \to \mathcal{Y}$ with domain* $\operatorname{dom}(A_+) \subset \mathcal{U}$*. In this case $\mathcal{Z}$ is maximal nonnegative if and only if* $\operatorname{dom}(A_+) = \mathcal{U}$*.*

(ii) *The subspace $\mathcal{Z}$ is nonpositive if and only if it is the graph of a (unique) linear contraction $A_-: \mathcal{Y} \mapsto \mathcal{U}$ with domain* $\operatorname{dom}(A_-) \subset \mathcal{Y}$. *In this case $\mathcal{Z}$ is maximal nonpositive if and only if* $\operatorname{dom}(A_-) = \mathcal{Y}$.

(iii) *The subspace $\mathcal{Z}$ is neutral if and only if it is the graph of an isometry $A_+: \mathcal{U} \to \mathcal{Y}$ with domain* $\operatorname{dom}(A_+) \subset \mathcal{U}$*, or equivalently, it is the graph of an isometry $A_-: \mathcal{Y} \to \mathcal{U}$ with domain* $\operatorname{dom}(A_-) \subset \mathcal{Y}$ *(here $A_- = A_+^{-1}$). The subspace $\mathcal{Z}$ is Lagrangian if and only if, in addition,* $\operatorname{dom}(A_+) = \mathcal{U}$ *and* $\operatorname{dom}(A_-) = \mathcal{Y}$.

(iv) *$\mathcal{Z}$ is maximal nonnegative if and only if $\mathcal{Z}$ is closed and $\mathcal{Z}^{[\perp]}$ is maximal nonpositive. More precisely, $\mathcal{Z}$ is the graph of a contraction $A_+ \in \mathcal{B}(\mathcal{U};\mathcal{Y})$ if and only if $\mathcal{Z}^{[\perp]}$ is the graph of $A_+^* \in \mathcal{B}(\mathcal{Y};\mathcal{U})$.*

(v) *$\mathcal{Z}$ is maximal nonnegative if and only if $\mathcal{Z}$ is closed and nonnegative and $\mathcal{Z}^{[\perp]}$ is nonpositive. In particular, $\mathcal{Z}$ is Lagrangian if and only if $\mathcal{Z}$ is both maximal nonnegative and maximal nonpositive.*

*Proof.* See [AI89, Section 1.8, pp. 48–64] or the following results in [Bog74]: Theorem 11.7 on p. 54, Theorems 4.2 and 4.4 on pp. 105–106, and Lemma 4.5 on p. 106. □

In particular, it follows from this proposition that $\mathcal{W}$ contains a Lagrangian subspace if and only if $\operatorname{ind}_+\mathcal{W} = \operatorname{ind}_-\mathcal{W}$.

The fundamental decompositions that we have considered above are a special case of *orthogonal decompositions* $\mathcal{W} = \mathcal{W}_1 \boxplus \mathcal{W}_2$ of $\mathcal{W}$, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are orthogonal with respect to $[\cdot,\cdot]_\mathcal{W}$, and both $\mathcal{W}_1$ and $\mathcal{W}_2$ are Kreĭn spaces with the inner products inherited from $\mathcal{W}$. Thus, if $w = w_1 + w_2$ with $w_1 \in \mathcal{W}_1$ and $w_2 \in \mathcal{W}_2$, then

$$[w,w]_\mathcal{W} = [w_1,w_1]_{\mathcal{W}_1} + [w_2,w_2]_{\mathcal{W}_2}. \tag{2.4}$$

This orthogonal decomposition is fundamental if and only if one of the two spaces is a Hilbert space and the other an anti-Hilbert space.

## 2.2 The Hilbert space $\mathcal{H}(\mathcal{Z})$

In [AS09a] a Hilbert space $\mathcal{H}(\mathcal{Z})$ was constructed, starting from an arbitrary maximal nonnegative subspace $\mathcal{Z}$ of a Kreĭn space. Below we give a short review of this construction. It will be used in the construction of canonical models for some special classes of passive s/s systems in Section 3.

Let $\mathcal{Z}$ be a maximal nonnegative subspace of the Kreĭn space $\mathcal{K}$, and let $\mathcal{K}/\mathcal{Z}$ be the quotient of $\mathcal{K}$ modulo $\mathcal{Z}$. We define $\mathcal{H}(\mathcal{Z})$ by

$$\mathcal{H}(\mathcal{Z}) = \{h \in \mathcal{K}/\mathcal{Z} \mid \sup\{-[x,x]_{\mathcal{K}} \mid x \in h\} < \infty\}. \tag{2.5}$$

It turns out that $\sup\{-[x,x]_{\mathcal{K}} \mid x \in h\} \geq 0$ for all $h \in \mathcal{H}(\mathcal{Z})$, that $\mathcal{H}(\mathcal{Z})$ is a subspace of $\mathcal{K}/\mathcal{Z}$, that $\mathcal{H}(\mathcal{Z})$ is a Hilbert space with the norm

$$\|h\|_{\mathcal{H}(\mathcal{Z})} = \big(\sup\{-[x,x]_{\mathcal{K}} \mid x \in h\}\big)^{1/2}, \qquad h \in \mathcal{H}(\mathcal{Z}), \tag{2.6}$$

and that $\mathcal{H}(\mathcal{Z})$ is continuously contained in $\mathcal{K}/\mathcal{Z}$ (where we use the standard quotient topology in $\mathcal{K}/\mathcal{Z}$, induced by some arbitrarily chosen admissible Hilbert space norm in $\mathcal{K}$). We denote the equivalence class $h \in \mathcal{K}/\mathcal{Z}$ that contains a particular vector $x \in \mathcal{K}$ by $h = x + \mathcal{Z}$. Thus, with this notation, (2.5) and (2.6) can be rewritten in the form

$$\mathcal{H}(\mathcal{Z}) = \{x + \mathcal{Z} \in \mathcal{K}/\mathcal{Z} \mid \|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} < \infty\}, \tag{2.7}$$

$$\|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = \sup\{-[x+z, x+z]_{\mathcal{K}} \mid z \in \mathcal{Z}\}, \qquad x \in \mathcal{K}. \tag{2.8}$$

A very important (and easily proved fact) is that if we define

$$\mathcal{H}^0(\mathcal{Z}) := \big\{z^\dagger + \mathcal{Z} \;\big|\; z^\dagger \in \mathcal{Z}^{[\perp]}\big\}, \tag{2.9}$$

then $\mathcal{H}^0(\mathcal{Z})$ is a subspace of $\mathcal{H}(\mathcal{Z})$. However, even more is true: $\mathcal{H}^0(\mathcal{Z})$ is a *dense subspace* of $\mathcal{H}(\mathcal{Z})$, and

$$[x + \mathcal{Z}, z^\dagger + \mathcal{Z}]_{\mathcal{H}(\mathcal{Z})} = -[x, z^\dagger]_{\mathcal{K}}, \quad x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z}), \quad z^\dagger \in \mathcal{Z}^{[\perp]}, \tag{2.10}$$

$$\|z^\dagger + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = -[z^\dagger, z^\dagger]_{\mathcal{K}}, \quad z^\dagger \in \mathcal{Z}^{[\perp]}. \tag{2.11}$$

Thus, $\mathcal{H}(\mathcal{Z})$ may be interpreted as a completion of $\mathcal{H}^0(\mathcal{Z})$. See [AS09a] for more details.

## 2.3 Conjugate-linear operators and conjugations

A continuous operator $A$ from one (complex) Kreĭn space $\mathcal{W}_1$ to another Kreĭn space $\mathcal{W}_2$ is called *real-linear* if

$$A(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 A w_1 + \lambda_2 A w_2, \quad \lambda_1, \lambda_2 \in \mathbb{R},\; w_1, w_2 \in \mathcal{W}_1, \tag{2.12}$$

it is called (complex) *linear* if

$$A(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 A w_1 + \lambda_2 A w_2, \quad \lambda_1, \lambda_2 \in \mathbb{C},\; w_1, w_2 \in \mathcal{W}_1, \tag{2.13}$$

and it is called (complex) *conjugate-linear* if

$$A(\lambda_1 w_1 + \lambda_2 w_2) = \overline{\lambda_1} A w_1 + \overline{\lambda_2} A w_2, \quad \lambda_1, \lambda_2 \in \mathbb{C},\ w_1, w_2 \in \mathcal{W}_1. \tag{2.14}$$

Note that both linear and conjugate-linear operators are real-linear. We recall that every complex Hilbert of Kreĭn $\mathcal{W}$ can be interpreted as a real Hilbert or Kreĭn space by restricting the scalars to be real and replacing the original complex inner product $[\cdot,\cdot]_{\mathcal{W}}$ by the real inner product $\Re[\cdot,\cdot]_{\mathcal{W}}$. The notion real-linearity defined above is equivalent to linearity in this real vector space.

We denote the set of all continuous conjugate-linear operators $\mathcal{W}_1 \to \mathcal{W}_2$ by $\overline{\mathcal{B}}(\mathcal{W}_1;\mathcal{W}_2)$, and by $\overline{\mathcal{B}}(\mathcal{W})$ if $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}$. This is a complete (complex) topological vector space whose topology is induced by a norm if we define scalar multiplication and addition point-wise by

$$(\lambda_1 A_1 + \lambda_2 A_2) w = \overline{\lambda_1} A_1 w + \overline{\lambda_2} w, \qquad w \in \mathcal{W}_1, \lambda_1,\ \lambda_2 \in \mathbb{C},$$

and it is a Banach space if $\mathcal{W}_1$ and $\mathcal{W}_2$ are Banach spaces (the norm of $A$ is then defined in the same way as in the case of a linear operator). The composition of two conjugate-linear operators is linear, and the composition of one linear and one conjugate-linear operator (in arbitrary order) is conjugate-linear. By the closed graph theorem, an operator $A \in \overline{\mathcal{B}}(\mathcal{W}_1;\mathcal{W}_2)$ is both injective and surjective if and only it has a conjugate-linear inverse $A^{-1} \in \overline{\mathcal{B}}(\mathcal{W}_2;\mathcal{W}_1)$.

**Definition 2.2.** The adjoint of a continuous real-linear operator $A\colon \mathcal{W}_1 \to \mathcal{W}_2$ is the unique real-linear operator $A^*$ which satisfies

$$\Re[Aw_1, w_2]_{\mathcal{W}_2} = \Re[w_1, A^* w_2]_{\mathcal{W}_1}, \qquad w_1 \in \mathcal{W}_1,\ w_2 \in \mathcal{W}_2. \tag{2.15}$$

Thus, this is the adjoint of $A$ when we replace the complex spaces $\mathcal{W}_1$ and $\mathcal{W}_2$ by the corresponding real spaces. Clearly $(A^*)^* = A$, and if $A$ is invertible, then $(A^{-1})^* = (A^*)^{-1}$. We denote $A^{-*} := (A^{-1})^* = (A^*)^{-1}$.

**Lemma 2.3.** *Let $A$ be a continuous real-linear bijection $\mathcal{W}_1 \to \mathcal{W}_2$. Then $(AV)^{[\perp]} = A^{-*} V^{[\perp]}$ for each $V \subset \mathcal{W}_1$. In particular, if $A^{-1} = \pm A^*$, then $(AV)^{[\perp]} = AV^{[\perp]}$.*

*Proof.* We have

$$\begin{aligned} w_2 &\in (AV)^{[\perp]} \\ &\Leftrightarrow \Re[w_2, Aw_1]_{\mathcal{W}_2} = 0 \qquad \forall w_1 \in V \\ &\Leftrightarrow \Re[A^* w_2, w_1]_{\mathcal{W}_2} = 0 \qquad \forall w_1 \in V \\ A^* w_2 &\in V^{[\perp]} \\ w_2 &\in A^{-*} V^{[\perp]}. \end{aligned}$$

This proves the first claim. The second claim follows from the first. ☐

**Definition 2.4.** An continuous real-linear operator $A\colon \mathcal{W}_1 \to \mathcal{W}_2$ is *isometric* if

$$[Aw, Aw]_{\mathcal{W}_2} = [w, w]_{\mathcal{W}_1}, \qquad w \in \mathcal{W}_1, \tag{2.16}$$

and it is *unitary* if, in addition, $A$ is bijective (so that it has a continuous everywhere defined inverse).

**Lemma 2.5.** *Let $A$ be a real-linear operator $A\colon \mathcal{W}_1 \to \mathcal{W}_2$.*

(i) *$A$ is isometric if and only if $A^*A = 1_{\mathcal{W}_1}$.*

(ii) *$A$ is unitary if and only if $A$ is invertible and $A^{-1} = A^*$.*

*Proof.* It suffices to prove (i), since (ii) is an immediate consequence of (i). If $A^*A = 1_{\mathcal{W}_1}$, then for all $w \in \mathcal{W}_1$,

$$[Aw, Aw]_{\mathcal{W}_2} = \Re[Aw, Aw]_{\mathcal{W}_2} = \Re[w, A^*Aw]_{\mathcal{W}_1} = \Re[w, w]_{\mathcal{W}_1} = [w, w]_{\mathcal{W}_1}.$$

Thus, $A$ is isometric. Conversely, suppose that $A$ is isometric. Then all $w \in \mathcal{W}_1$,

$$[w, w]_{\mathcal{W}_1} = [Aw, Aw]_{\mathcal{W}_2} = \Re[w, A^*Aw]_{\mathcal{W}_1}.$$

It follows from the polarisation formula that

$$\Re[w_1, w_2 - A^*Aw_2]_{\mathcal{W}_1} = 0, \qquad w_1,\ w_1 \in \mathcal{W}_1.$$

Replacing $w_1$ by $iw_1$ we find that $A^*Aw_2 = w_2$ for all $w_2 \in \mathcal{W}_1$, and hence $A^*A = 1_{\mathcal{W}_1}$. ☐

**Lemma 2.6.** *Let $A^*$ be the adjoint of a continuous real-linear operator $A\colon \mathcal{W}_1 \to \mathcal{W}_2$.*

(i) *$A$ is linear if and only if $A^*$ is linear. In this case*

$$[Aw_1, w_2]_{\mathcal{W}} = [w_1, A^*w_2]_{\mathcal{W}}, \qquad w_1 \in \mathcal{W}_1,\ w_2 \in \mathcal{W}_2. \tag{2.17}$$

(ii) *$A$ is conjugate-linear if and only if $A^*$ is conjugate-linear. In this case*

$$[Aw_1, w_2]_{\mathcal{W}} = \overline{[w_1, A^*w_2]}_{\mathcal{W}}, \qquad w_1 \in \mathcal{W}_1,\ w_2 \in \mathcal{W}_2. \tag{2.18}$$

*Proof.* The proofs of (i) and (ii) are analogous, so it suffices to prove (ii). If we in (2.15) replace $w_1$ by $iw_1$ and use the conjugate-linearity of $A$ we get

$$-\Im[Aw_1, w_2]_{\mathcal{W}_2} = \Im[w_1, A^*w_2]_{\mathcal{W}_1}, \qquad w_1 \in \mathcal{W}_1,\ w_2 \in \mathcal{W}_2.$$

Thus (2.18) holds. That $A^*$ is conjugate-linear follows from (2.18). ☐

**Lemma 2.7.** *Let $A$ be a continuous bijection $\mathcal{W}_1 \to \mathcal{W}_2$, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are Kreĭn spaces.*

(i) *The following conditions are equivalent:*

*(a) $A$ is linear and unitary;*

*(b) $A$ satisfies*

$$[Aw_1, Aw_2]_{\mathcal{W}_2} = [w_1, w_2]_{\mathcal{W}_1}, \qquad w_1, w_2 \in \mathcal{W}_1. \tag{2.19}$$

(ii) *The following conditions are equivalent:*

*(a) $A$ is conjugate-linear and unitary;*

*(b) $A$ satisfies*

$$[Aw_1, Aw_2]_{\mathcal{W}_2} = \overline{[w_1, w_2]_{\mathcal{W}_1}}, \qquad w_1, w_2 \in \mathcal{W}_1. \tag{2.20}$$

*Proof.* The proof is essentially the same in cases (i) and (ii), so it suffices to prove, for example, (ii).

That (a) implies (b) follows from (2.16) and the polarisation formula. Conversely, if (b) holds, then by fixing $w_2$ and letting $w_1$ vary in (2.20) we find that $A$ is conjugate-linear, and by taking $w_1 = w_2$ in (2.20) we find that $A$ is unitary. □

**Definition 2.8.** Let $A \colon \mathcal{W} \to \mathcal{W}$ be a continuous real-linear operator.

(i) $A$ is *self-adjoint* if $A^* = A$.

(ii) $A$ is *skew-adjoint* if $A^* = -A$.

(iii) $A$ is a *involution* if $A^2 = 1_{\mathcal{W}}$.

**Definition 2.9.** Let $\mathcal{W}$ be a Kreĭn space.

(i) By a *signature operator* $\mathcal{J}$ in $\mathcal{W}$ we mean a *linear self-adjoint involution* in $\mathcal{W}$, i.e., $\mathcal{J}$ is linear and invertible and

$$\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}. \tag{2.21}$$

(ii) by a *conjugation* $\mathcal{C}$ in $\mathcal{W}$ we mean a *conjugate-linear self-adjoint involution* in $\mathcal{W}$, i.e., $\mathcal{C}$ is conjugate-linear and invertible and (2.21) holds with $\mathcal{J}$ replaced by $\mathcal{C}$.

**Lemma 2.10.** *Let $A\colon \mathcal{W} \to \mathcal{W}$ be a continuous real-linear operator. Then the following conditions are equivalent:*

(i) *$A$ is a self-adjoint involution.*

(ii) *$A$ is a unitary involution.*

(iii) *$A$ is both self-adjoint and unitary.*

*Proof.* If $A$ is a self-adjoint involution, then $A^*A = AA^* = A^2 = 1_{\mathcal{W}}$, and hence $A$ is unitary. If $A$ is a unitary involution, then $A^*A = AA^* = 1_{\mathcal{W}}$, and hence $A^* = A^{-1} = A$. Thus $A$ is self-adjoint. Finally, if $A$ is both self-adjoint and unitary, then $A^* = A$ and $A^*A = AA^* = 1_{\mathcal{W}}$, and hence $A^2 = 1_{\mathcal{W}}$, which means that $A$ is an involution. □

**Lemma 2.11.** *If $\mathcal{J}$ is a signature operator or a conjugation in a Kreĭn space $\mathcal{W}$, then $(\mathcal{J}V)^{[\perp]} = \mathcal{J}V^{[\perp]}$ for all subsets $V$ of $\mathcal{W}$.*

*Proof.* This follows from Lemmas 2.3 and 2.10. □

**Definition 2.12.** Let $\mathcal{C}$ be a conjugation in the Kreĭn space $\mathcal{W}$.

(i) A subspace $\mathcal{Z}$ of $\mathcal{W}$ is said to be *$\mathcal{C}$-invariant* if $\mathcal{C}\mathcal{Z} = \mathcal{Z}$,

(ii) An operator $A$ mapping a Kreĭn space $\mathcal{W}_1$ with a conjugation $\mathcal{C}_1$ into a Kreĭn space $\mathcal{W}_2$ with a conjugation $\mathcal{C}_2$ is called *$(\mathcal{C}_1, \mathcal{C}_2)$-real* (or simply $\mathcal{C}$-real if $\mathcal{W}_1 = \mathcal{W}_2$ and $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$) if $\mathcal{C}_2 Aw = A\mathcal{C}_1 w$ for all $w_1 \in \mathcal{W}_1$.

In part (i) one can replace the condition $\mathcal{C}\mathcal{Z} = \mathcal{Z}$ by the formally weaker condition $\mathcal{C}\mathcal{Z} \subset \mathcal{Z}$, since the latter condition implies that $\mathcal{Z} = \mathcal{C}^2\mathcal{Z} \subset \mathcal{C}\mathcal{Z}$.

**Lemma 2.13.** *If $A \in \mathcal{B}(\mathcal{W}_1; \mathcal{W}_2)$ is $(\mathcal{C}_1, \mathcal{C}_2)$-real, then* $\ker(A)$ *and* $(\ker(A))^{[\perp]}$ *are $\mathcal{C}_1$-invariant and* $\operatorname{im}(A)$, $\overline{\operatorname{im}(A)}$, *and* $(\operatorname{im}(A))^{[\perp]}$ *are $\mathcal{C}_2$-invariant.*

*Proof.* That $\ker(A)$ is $\mathcal{C}_1$-invariant and $\operatorname{im}(A)$ is $\mathcal{C}_2$-invariant follows from the intertwinement condition $A\mathcal{C}_1 = \mathcal{C}_2 A$. The $\mathcal{C}_2$-invariance of $\operatorname{im}(A)$ implies that also $\overline{\operatorname{im}(A)}$ is $\mathcal{C}_2$-invariant. Finally, the invariance of $(\ker(A))^{[\perp]}$ and $(\operatorname{im}(A))^{[\perp]}$ follows from Lemma 2.3. □

## 2.4 Skew-unitary operators and skew-adjoint involutions

In the sequel we shall also need the notion of an *skew-unitary* linear operator between two Kreĭn spaces.

**Definition 2.14.** An continuous real-linear operator $A\colon \mathcal{W}_1 \to \mathcal{W}_2$ is *skew-isometric* if

$$[Aw, Aw]_{\mathcal{W}_2} = -[w, w]_{\mathcal{W}_1}, \qquad w \in \mathcal{W}_1. \tag{2.22}$$

and it is *skew-unitary* if, in addition, $A$ is bijective (so that it has a continuous everywhere defined inverse).

Clearly, the existence of a non-trivial skew-unitary operator $\mathcal{W}_1 \to \mathcal{W}_2$ implies both $\mathcal{W}_1$ and $\mathcal{W}_2$ cannot possibly be Hilbert or anti-Hilbert spaces. A typical example of a linear skew-unitary operator between two Kreĭn spaces is the identity operator $\mathcal{I}_{(\mathcal{W},-\mathcal{W})}$ defined on a Kreĭn space $\mathcal{W}$ with values in the anti-space $-\mathcal{W}$.

**Lemma 2.15.** *Let $A$ be a real-linear operator $A\colon \mathcal{W}_1 \to \mathcal{W}_2$.*

(i) *$A$ is skew-isometric if and only if $A^*A = -1_{\mathcal{W}_1}$.*

(ii) *$A$ is skew-unitary if and only if $A$ is invertible and $A^{-1} = -A^*$.*

*Proof.* The proof is essentially the same as the proof of Lemma 2.5. □

**Lemma 2.16.** *Let $A$ be a continuous real-linear bijection $\mathcal{W}_1 \to \mathcal{W}_2$, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are Kreĭn spaces.*

(i) *The following conditions are equivalent:*

  *(a) $A$ is linear and skew-unitary;*

  *(b) $A$ satisfies*

$$[Aw_1, Aw_2]_{\mathcal{W}_2} = -[w_1, w_2]_{\mathcal{W}_1}, \qquad w_1, w_2 \in \mathcal{W}_1. \tag{2.23}$$

(ii) *The following conditions are equivalent:*

  *(a) $A$ is conjugate-linear and skew-unitary;*

  *(b) $A$ satisfies*

$$[Aw_1, Aw_2]_{\mathcal{W}_2} = -\overline{[w_1, w_2]}_{\mathcal{W}_1}, \qquad w_1, w_2 \in \mathcal{W}_1. \tag{2.24}$$

*Proof.* The proof is essentially the same as the proof of Lemma 2.7. □

**Definition 2.17.** Let $\mathcal{W}$ be a Kreĭn space.

(i) By an *skew-signature operator* in $\mathcal{W}$ we mean a *linear skew-adjoint involution* in $\mathcal{W}$, i.e., a linear operator $\mathcal{I}$ in $\mathcal{W}$ satisfying

$$\mathcal{I} = -\mathcal{I}^* = \mathcal{I}^{-1}. \tag{2.25}$$

(ii) By an *skew-conjugation* in $\mathcal{W}$ we mean a *conjugate-linear skew-adjoint involution* in $\mathcal{W}$, i.e., a conjugate-linear operator $\mathcal{B}$ in $\mathcal{W}$ satisfying (2.25) with $\mathcal{I}$ replaced by $\mathcal{B}$.

**Lemma 2.18.** *Let $A\colon \mathcal{W} \to \mathcal{W}$ be a continuous real-linear operator. Then the following conditions are equivalent:*

(i) *$A$ is a skew-adjoint involution.*

(ii) *$A$ is a skew-unitary involution.*

(iii) *$A$ is both skew-adjoint and skew-unitary.*

*Proof.* The proof is essentially the same as the proof of Lemma 2.10. □

**Lemma 2.19.** *If $\mathcal{I}$ is an skew-signature operator or a skew-conjugation in a Kreĭn space $\mathcal{W}$, then $(\mathcal{I}V)^{[\perp]} = \mathcal{I}V^{[\perp]}$ for all subsets $V$ of $\mathcal{W}$.*

*Proof.* This follows from Lemmas 2.3 and 2.18. □

# 3 Passive State/Signal Systems

In the introduction we already gave a short description of the notion of a passive s/s system. Here we shall present some additional notions and results that will be needed in this article. The reader is referred to [AKS11b] for details and proofs.

## 3.1 Basic definitions and properties

**Definition 3.1.** Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ a Kreĭn space, and let $I$ be one of the intervals $I = \mathbb{R}^+$, $I = \mathbb{R}$, or $I = \mathbb{R}^-$.

(i) By a *passive s/s node* in continuous time we mean a triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$ where $V$ is a maximal nonnegative subspace satisfying (1.1) of the Kreĭn *node space* $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ equipped with the inner product (1.6).

(ii) A *classical trajectory* generated by a subspace $V$ of $\mathfrak{K}$ on an interval $I$ is a pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \begin{bmatrix} C^1(I;\mathcal{X}) \\ C(I;\mathcal{W}) \end{bmatrix}$ satisfying (1.2).

(iii) A *(generalised) trajectory* generated by a subspace $V$ of $\mathfrak{K}$ on an interval $I$ is a pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \begin{bmatrix} C(I;\mathcal{X}) \\ L^2_{\mathrm{loc}}(I;\mathcal{W}) \end{bmatrix}$ which can be approximated by a sequence of classical trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ in such a way that $x_n \to x$ in $\mathcal{X}$ locally uniformly on $I$, and $w_n \to w$ in $L^2_{\mathrm{loc}}(I;\mathcal{W})$.

(iv) The passive s/s node $\Sigma$ together with its families of classical and generalised trajectories is called a *passive s/s system*, and it is denoted by the same symbol $\Sigma$ as the node.

(v) By a *past*, *two-sided* (or full), or *future* trajectory of $\Sigma$ we mean a trajectory of $\Sigma$ on $\mathbb{R}^-$, $\mathbb{R}$, or $\mathbb{R}^+$, respectively.

(vi) A (generalised) trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ on an interval $I$ is *externally generated* if the following condition holds: If $I$ has a finite left end-point $t_0$, then we require that $x(t_0) = 0$, and if the left end-point of $I$ is $-\infty$, then we require that $\lim_{t\to-\infty} x(t) = 0$ and that $w \in L^2((-\infty, T]; \mathcal{W})$ for every finite $T \in I$.

(vii) A (generalised) trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is *stable* if $x$ is bounded on $I$ and $w \in L^2(I; \mathcal{W})$.

**Definition 3.2.** Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system.

(i) The *reachable subspace* $\mathfrak{R}_\Sigma$ of $\Sigma$ is the closure of the set

$$\left\{ x_0 \in \mathcal{X} \;\middle|\; \begin{array}{l} x_0 = x(0) \text{ for some (stable) past} \\ \text{trajectory of } \Sigma \text{ with compact support} \end{array} \right\}.$$

(ii) $\Sigma$ is *controllable* if $\mathfrak{R}_\Sigma = \mathcal{X}$.

(iii) By an *unobservable future trajectory* of $\Sigma$ we mean a future trajectory of $\Sigma$ of the type $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ (i.e., the signal part is identically zero).

(iv) The *unobservable* subspace $\mathfrak{U}_\Sigma$ of $\Sigma$ consists of all the initial states $x(0)$ of all unobservable future trajectories of $\Sigma$.

(v) $\Sigma$ is *observable* if $\mathfrak{U}_\Sigma = \{0\}$.

(vi) $\Sigma$ is *simple* if $\mathfrak{U}_\Sigma \cap \mathfrak{R}_\Sigma^\perp = 0$, or equivalently, if $\mathfrak{R}_\Sigma \vee \mathfrak{U}_\Sigma^\perp = \mathcal{X}$.

(vii) $\Sigma$ is *minimal* if it is both controllable and observable.

As the following lemma shows, the boundedness condition on $x$ in Definition 3.1(vii) is often redundant.

**Lemma 3.3.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system.*
*If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a (generalised) trajectory of $\Sigma$ on $I = \mathbb{R}^+$, then*

$$\|x(t)\|_{\mathcal{X}}^2 \le \|x(0)\|_{\mathcal{X}}^2 + \int_0^t [w(s), w(s)]_{\mathcal{W}} \, \mathrm{d}s, \qquad t \in \mathbb{R}^+, \tag{3.1}$$

*and if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is externally generated trajectory on an interval $I$ with left endpoint $-\infty$, then*

$$\|x(t)\|_{\mathcal{X}}^2 \le \int_{-\infty}^t [w(s), w(s)]_{\mathcal{W}} \, \mathrm{d}s, \qquad t \in I. \tag{3.2}$$

*Thus, in both cases $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is stable if and only if $w \in L^2(I; \mathcal{W})$.*

*Proof.* See [AKS11b, Lemma 3.2]. □

**Lemma 3.4.** *A generalised trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of a passive s/s system $\Sigma$ on some interval $I$ is classical if and only if $x \in C^1(I; \mathcal{X})$ and $w \in C^1(I; \mathcal{W})$.*

*Proof.* See [AKS11b, Proposition 3.7]. □

**Definition 3.5.** Let $\Sigma_1 = (V_1; \mathcal{X}_1; \mathcal{W})$ and $\Sigma_2 = (V_2; \mathcal{X}_2; \mathcal{W})$ be two passive s/s systems (with the same signal space $\mathcal{W}$).

(i) A bounded linear operator $E\colon \mathcal{X}_1 \to \mathcal{X}_2$ *intertwines* $\Sigma_1$ and $\Sigma_2$ if the formula

$$(x_1, w) \mapsto (Ex_1, w) \tag{3.3}$$

defines a map from the set of all stable future trajectories $\left[\begin{smallmatrix} x_1 \\ w \end{smallmatrix}\right]$ of $\Sigma_1$ onto the set of all stable future trajectories $\left[\begin{smallmatrix} x_2 \\ w \end{smallmatrix}\right]$ of $\Sigma_2$ satisfying $x_2(0) \in \operatorname{im}(E)$.

(ii) $\Sigma_1$ and $\Sigma_2$ are *boundedly intertwined* if there exists an operator $E \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$. The operator $E$ is called an intertwining operator between $\Sigma_1$ and $\Sigma_2$.

(iii) $\Sigma_1$ and $\Sigma_2$ are *contractively intertwined* if there exists a contraction $E \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$.

(iv) $\Sigma_1$ and $\Sigma_2$ are *similar* if there exists a boundedly invertible operator $E \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$. The operator $E$ is called a similarity operator between $\Sigma_1$ and $\Sigma_2$.

(v) $\Sigma_1$ and $\Sigma_2$ are *unitarily similar* if there exists a unitary operator $E \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$.

**Definition 3.6.**

(i) The s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called a *restriction* of the s/s system $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ if $\mathcal{X}$ is a closed subspace of $\mathcal{X}_1$ and the embedding operator $\mathcal{X} \hookrightarrow \mathcal{X}_1$ intertwines $\Sigma$ and $\Sigma_1$.

(ii) The s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called an *orthogonal projection* of the s/s system $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ if $\mathcal{X}$ is a closed subspace of $\mathcal{X}_1$ and the projection operator $P_{\mathcal{X}}$ intertwines $\Sigma_1$ and $\Sigma$.

**Definition 3.7.** Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system.

(i) $\Sigma$ is *energy preserving* if $V$ is neutral, i.e., if $V \subset V^{[\perp]}$.

(ii) $\Sigma$ is *co-energy preserving* if $V$ is co-neutral, i.e., if $V^{[\perp]} \subset V$.

(iii) $\Sigma$ is *conservative* if $V$ is Lagrangian, i.e., if $V = V^{[\perp]}$.

## 3.2 Passive behaviours and their passive realizations

It follows from (3.1) and (3.2) that if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated trajectory of a passive s/s system $\Sigma$ on one of the intervals $I = \mathbb{R}^+$, $I = \mathbb{R}$, or $I = \mathbb{R}^-$ with $w \in L^2(I; \mathcal{W})$, then

$$\int_I [w(s), w(s)]_{\mathcal{W}} \, \mathrm{d}s \geq 0.$$

This can be interpreted as a nonnegativity condition in the Kreĭn space $K^2(I; \mathcal{W})$, which is defined as follows. For nontrivial interval $I \subset \mathbb{R}$ we define the Kreĭn space $K^2(I; \mathcal{W})$ to be the space which coincides with $L^2(I; \mathcal{W})$ as a topological vector space, equipped with the inner product

$$[w_1, w_2]_{K^2(I;\mathcal{W})} := \int_I [w_1(s), w_2(s)]_{\mathcal{W}} \, \mathrm{d}s, \tag{3.4}$$

and we denote

$$K^2(\mathcal{W}) := K^2(\mathbb{R}; \mathcal{W}), \quad K^2_\pm(\mathcal{W}) := K^2(\mathbb{R}^\pm; \mathcal{W}). \tag{3.5}$$

This is a Kreĭn space, and if $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ is a fundamental decomposition of $\mathcal{W}$, then $K^2(I; \mathcal{W}) = L^2(I; \mathcal{U}) \boxplus -L^2(I; \mathcal{Y})$ is a fundamental decomposition of $K^2(I; \mathcal{W})$.

**Definition 3.8.** Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system.

(i) The *future behaviour* $\mathfrak{W}^\Sigma_+$ of $\Sigma$ is the set

$$\mathfrak{W}^\Sigma_+ := \left\{ w \in K^2_+(\mathcal{W}) \;\middle|\; \begin{array}{l} w \text{ is the signal part of a externally generated} \\ \text{stable future trajectory } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \text{ of } \Sigma. \end{array} \right\}$$

(ii) The *two-sided behaviour* $\mathfrak{W}^\Sigma$ of $\Sigma$ is the set

$$\mathfrak{W}^\Sigma := \left\{ w \in K^2(\mathcal{W}) \;\middle|\; \begin{array}{l} w \text{ is the signal part of a externally generated} \\ \text{stable two-sided trajectory } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \text{ of } \Sigma. \end{array} \right\}$$

(iii) The *past behaviour* $\mathfrak{W}^\Sigma_-$ of $\Sigma$ is the set

$$\mathfrak{W}^\Sigma_- := \left\{ w \in K^2_-(\mathcal{W}) \;\middle|\; \begin{array}{l} w \text{ is the signal part of a externally generated} \\ \text{stable past trajectory } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \text{ of } \Sigma. \end{array} \right\}$$

Thus, $\mathfrak{W}_+^\Sigma$, $\mathfrak{W}^\Sigma$, and $\mathfrak{W}_-^\Sigma$ are nonnegative subspaces of $K_+^2(\mathcal{W})$, $K^2(\mathcal{W})$, and $K_-^2(\mathcal{W})$, respectively. As the following lemma shows, they also have some additional characteristic properties.

**Lemma 3.9.** *The past, two-sided, and future behaviours $\mathfrak{W}_-^\Sigma$, $\mathfrak{W}^\Sigma$, and $\mathfrak{W}_+^\Sigma$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ have the following properties:*

(i) *$\mathfrak{W}_\pm^\Sigma$ are right-shift invariant and $\mathfrak{W}^\Sigma$ is bilaterally shift-invariant, i.e.,*

$$\begin{aligned} \tau_\pm^{*t}\mathfrak{W}_\pm^\Sigma &\subset \mathfrak{W}_\pm^\Sigma, \qquad && t \in \mathbb{R}^+, \\ \tau^t\mathfrak{W}^\Sigma &= \mathfrak{W}^\Sigma, \qquad && t \in \mathbb{R}. \end{aligned} \tag{3.6}$$

(ii) *$\mathfrak{W}_\pm^\Sigma$ can be recovered from $\mathfrak{W}^\Sigma$ by the formulas*

$$\begin{aligned} \mathfrak{W}_-^\Sigma &= \pi_-\mathfrak{W}^\Sigma := \{w_- \in K_-^2(\mathcal{W}) \mid w_- = \pi_- w \text{ for some } w \in \mathfrak{W}^\Sigma\}, \\ \mathfrak{W}_+^\Sigma &= \mathfrak{W}^\Sigma \cap K_+^2(\mathcal{W}) := \{w \in \mathfrak{W}^\Sigma \mid w(t) = 0 \text{ for } t < 0\}. \end{aligned} \tag{3.7}$$

(iii) *$\mathfrak{W}_\pm^\Sigma$ is a maximal nonnegative subspace of $K_\pm^2(\mathcal{W})$ and $\mathfrak{W}^\Sigma$ is a maximal nonnegative subspace of $K^2(\mathcal{W})$.*

*Proof.* This is [AKS11b, Lemma 3.12]. □

See the list of notations at the end of Section 1 for the definition of the restriction operator $\pi_-$.

**Lemma 3.10.** *Let $\mathfrak{W}$ be a maximal nonnegative subspace $\mathfrak{W}$ of $K^2(\mathcal{W})$, and define $\mathfrak{W}_-$ and $\mathfrak{W}_+$ by*

$$\mathfrak{W}_- := \pi_-\mathfrak{W}, \qquad \mathfrak{W}_+ := \mathfrak{W} \cap K_+^2(\mathcal{W}), \tag{3.8}$$

*Then the following conditions are equivalent:*

(i) *$\mathfrak{W}_-$ is a maximal nonnegative subspace of $K_-^2(\mathcal{W})$.*

(ii) *$\mathfrak{W}_+$ is a maximal nonnegative subspace of $K_+^2(\mathcal{W})$.*

(iii) *For* some *fundamental decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ the following implication is valid: If $w \in \mathfrak{W}$ and $\pi_- P_\mathcal{U} w = 0$, then $\pi_- P_\mathcal{Y} w = 0$.*

(iv) *For* every *fundamental decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ the following implication is valid: If $w \in \mathfrak{W}$ and $\pi_- P_\mathcal{U} w = 0$, then $\pi_- P_\mathcal{Y} w = 0$.*

*Proof.* This is [AKS11b, 3.13]. □

Motivated by Lemmas 3.9 and 3.10 we make the following definition:

**Definition 3.11.** Let $\mathcal{W}$ be a Kreĭn space.

(i) A maximal nonnegative right-shift invariant subspace of $K^2_-(\mathcal{W})$ is called a *passive past behaviour* on the (signal) space $\mathcal{W}$.

(ii) A maximal nonnegative right-shift invariant subspace $\mathfrak{W}_+$ of $K^2_+(\mathcal{W})$ is called a *passive future behaviour* on the (signal) space $\mathcal{W}$.

(iii) A maximal nonnegative bilaterally shift invariant subspace $\mathfrak{W}$ of $K^2(\mathcal{W})$ which satisfies the equivalent conditions (i)–(iv) listed in Lemma 3.10 is called a *passive two-sided behaviour* on the Kreĭn (signal) space $\mathcal{W}$.

The following lemma complements Lemmas 3.9 and 3.10.

**Lemma 3.12.** *Let $\mathcal{W}$ be a Kreĭn space.*

(i) *If $\mathfrak{W}_-$ is a passive past behaviour on $\mathcal{W}$, and if we define $\mathfrak{W}$ by*

$$\mathfrak{W} = \bigcap_{t\in\mathbb{R}^+} \bigl\{w \in K^2(\mathcal{W}) \bigm| \pi_-\tau^t w \in \mathfrak{W}_-\bigr\}, \tag{3.9}$$

*then $\mathfrak{W}$ is a passive two-sided behaviour on $\mathcal{W}$ and $\mathfrak{W}_- = \pi_-\mathfrak{W}$.*

(ii) *If $\mathfrak{W}_+$ is a passive future behaviour on $\mathcal{W}$, and if we define $\mathfrak{W}$ by*

$$\mathfrak{W} = \bigvee_{t\in\mathbb{R}^+} \tau^t\mathfrak{W}_+, \tag{3.10}$$

*then $\mathfrak{W}$ is a passive two-sided behaviour on $\mathcal{W}$, and $\mathfrak{W}_+ = \mathfrak{W}\cap K^2_+(\mathcal{W})$.*

(iii) *Let $\mathfrak{W}$ be a passive two-sided behaviour on the Kreĭn signal space $\mathcal{W}$, and define $\mathfrak{W}_-$ and $\mathfrak{W}_+$ by* (3.8)*. Then $\mathfrak{W}_-$ is a passive past behaviour on $\mathcal{W}$, $\mathfrak{W}_+$ is a passive future behaviour on $\mathcal{W}$, and $\mathfrak{W}$ can be recovered from $\mathfrak{W}_+$ and from $\mathfrak{W}_-$ by means of formulas* (3.9) *and* (3.10)*.*

*Proof.* This is [AKS11b, Lemma 3.18]. □

From Lemmas 3.9 and 3.12 we conclude that the future, two-sided, and past behaviours of a passive s/s system $\Sigma$ are *passive* future, two-sided, and past behaviours, respectively.

**Definition 3.13.** A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called a *realization* of a passive future behaviour $\mathfrak{W}_+$, or a passive two-sided behaviour $\mathfrak{W}$, or a passive past behaviour $\mathfrak{W}_-$, if the corresponding behaviour of $\Sigma$ coincides with the given behaviour $\mathfrak{W}_+$, $\mathfrak{W}$, or $\mathfrak{W}_-$, respectively.

**Theorem 3.14.** *Every passive future behaviour $\mathfrak{W}_+$, passive two-sided behaviour $\mathfrak{W}$, or passive past behaviour $\mathfrak{W}_-$ has a passive s/s realization $\Sigma$ in each of the following three classes of passive s/s systems:*

*a) $\Sigma$ is simple and conservative;*

*b) $\Sigma$ is controllable and energy preserving;*

*c) $\Sigma$ is observable and co-energy preserving.*

*Moreover, within each class the realization $\Sigma$ is determined uniquely by the given behaviour up to unitary similarity in the sense of Definition 3.5(v).*

*Proof.* This follows from Theorems 8.1, 9.1, and 10.1 and Corollaries 8.7, 9.8, and 10.7 in [AKS11b]. □

In this article we shall expand the above list by adding the classes d), e), and f) mentioned in the introduction.

**Definition 3.15.** Two passive s/s systems $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ (with the same signal space) are *externally equivalent* if they realize the same past, two-sided, and future behaviours.

**Lemma 3.16.** *If two systems $\Sigma_1$ and $\Sigma_2$ are boundedly intertwined, then they are externally equivalent.*

*Proof.* This follows from Definitions 3.5, 3.8, and 3.15. □

**Theorem 3.17.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with reachable subspace $\mathfrak{R}$ and unobservable subspace $\mathfrak{U}$.*

(i) *Define*

$$V_{\mathfrak{R}} = V \cap \begin{bmatrix} \mathcal{X} \\ \mathfrak{R} \\ \mathcal{W} \end{bmatrix}, \tag{3.11}$$

*then $V_{\mathfrak{R}} = V \cap \begin{bmatrix} \mathfrak{R} \\ \mathfrak{R} \\ \mathcal{W} \end{bmatrix}$ and $\Sigma_{\mathfrak{R}} = (V_{\mathfrak{R}}, \mathfrak{R}, \mathcal{W})$ is a passive s/s system, and it is the restriction of $\Sigma$ to $\mathfrak{R}$. The system $\Sigma_{\mathfrak{R}}$ is always controllable, and it is minimal if $\Sigma$ is observable.*

(ii) *Define*

$$V_{\mathfrak{U}^\perp} = \begin{bmatrix} P_{\mathfrak{U}^\perp} & 0 & 0 \\ 0 & P_{\mathfrak{U}^\perp} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V, \tag{3.12}$$

*then $\Sigma_{\mathfrak{U}^\perp} = (V_{\mathfrak{U}^\perp}; \mathfrak{U}^\perp, \mathcal{W})$ is a passive s/s system, and it is the orthogonal projection of $\Sigma$ to $\mathfrak{U}^\perp$. The system $\Sigma_{\mathfrak{U}^\perp}$ is always observable, and it is minimal if $\Sigma$ is controllable.*

*Proof.* The discrete time version of this theorem can be derived from [AS07b, Theorems 7.3 and 7.7], and the proof of the continuous time result is analogous to the proof of the discrete time result (cf. [AKS11b, Remark 3.17]). □

Theorem 3.17 can alternatively be derived from the corresponding i/s/o result by means of a scattering i/s/o representation of $\Sigma$.

**Remark 3.18.** A passive s/s system is non-minimal if and only if at least one of the two transformations described in Theorem 3.17 can be applied to replace $\Sigma$ by a "smaller" externally equivalent system.

## 3.3 The adjoints of passive systems and behaviours

**Lemma 3.19.** *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a passive s/s system, and define $V_*$ by* (1.19)*, where $\mathcal{I}_{(\mathcal{W},-\mathcal{W})}$ is the identity map from $\mathcal{W}$ to the anti-space $-\mathcal{W}$. Then $\Sigma_* = (V_*,\mathcal{X},-\mathcal{W})$ is a passive s/s system.*

*Proof.* By Proposition 2.1, $V^{[\perp]}$ is a maximal nonpositive subspace of the node space $\mathfrak{K}$. It is easy to see that this implies that $V_*$ is maximal nonnegative. It follows from [Kur10, Corollary 4.8], condition 1.1 holds with $V$ replaced by $V_*$. Thus, $V_*$ generates a passive s/s system $\Sigma_* = (V_*,\mathcal{X},-\mathcal{W})$. □

**Definition 3.20.** The system $\Sigma_*$ in Lemma 3.19 is called the *adjoint* of the s/s system $\Sigma$.

**Lemma 3.21.** *If a bounded operator $E$ intertwines two passive s/s systems $\Sigma_1$ and $\Sigma_2$, then $E^*$ intertwines the dual systems $\Sigma_{2*}$ and $\Sigma_{1*}$ of $\Sigma_2$ and $\Sigma_1$, respectively.*

*Proof.* This follows from Definition 3.5 and [AKS11b, Remark 4.2 and Theorem 4.5]. □

**Lemma 3.22.** *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a passive system with adjoint $\Sigma_* = (V_*;\mathcal{X},-\mathcal{W})$.*

(i) *The adjoint of $\Sigma_*$ is $\Sigma$.*

(ii) *$\Sigma$ is energy preserving if and only if $\Sigma_*$ is co-energy preserving.*

(iii) *$\Sigma$ is co-energy preserving if and only if $\Sigma_*$ is energy preserving.*

(iv) *$\Sigma$ is conservative if and only if $\Sigma$ is conservative.*

*Proof.* All of these claims are easy consequences of (1.19). □

See the list of notations at the end of Section 1 for the definition of the reflection operator $\mathbf{Я}$.

**Lemma 3.23.** *Let $\mathfrak{W}_+$, $\mathfrak{W}$, and $\mathfrak{W}_-$ be passive future, two-sided, and past behaviours on $\mathcal{W}$. Then also*

$$\mathfrak{W}_{*+} = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\mathbf{Я}\mathfrak{W}_-^{[\perp]},\ \mathfrak{W}_* = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\mathbf{Я}\mathfrak{W},\ \mathfrak{W}_{*-} = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\mathbf{Я}\mathfrak{W}_+^{[\perp]} \quad (3.13)$$

*are passive future, two-sided, and past behaviours, respectively, on the anti-space $-\mathcal{W}$. If $\mathfrak{W}_+$, $\mathfrak{W}$, and $\mathfrak{W}_-$ correspond to each other in the sense that they satisfy* (3.8)*,* (3.9)*, and* (3.10)*, then $\mathfrak{W}_{*+}$, $\mathfrak{W}_*$, and $\mathfrak{W}_{*-}$ correspond to each other in the same sense.*

*Proof.* See [AKS11b, Lemma 2.3, Lemma 4.11 and Remark 4.12]. □

**Definition 3.24.** The passive behaviours $\mathfrak{W}_{*+}$, $\mathfrak{W}_*$, and $\mathfrak{W}_{*-}$ in Lemma 3.23 are called the *adjoints* of the behaviours $\mathfrak{W}_-$, $\mathfrak{W}$, and $\mathfrak{W}_+$, respectively.

**Lemma 3.25.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with future, two-sided, and past behaviours $\mathfrak{W}_+$, $\mathfrak{W}$, and $\mathfrak{W}_-$. Then the future, two-sided, and past behaviours of the adjoint system $\Sigma_* = (V_*, \mathcal{X}, -\mathcal{W})$ are the adjoints of $\mathfrak{W}_-$, $\mathfrak{W}$, and $\mathfrak{W}_+$, respectively, in the sense of Definition 3.24.*

*Proof.* See [AKS11b, Remark 4.12 and Proposition 4.16]. □

## 3.4 The Hilbert Spaces $\mathcal{H}(\mathfrak{W}_+)$, $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and $\mathcal{D}(\mathfrak{W})$

Three special canonical passive s/s realizations of the classes a)–c) in Theorem 3.14 were constructed in [AKS11b]. These canonical realizations and their state spaces play an important role especially in the study of the real symmetry, and for this reason we recall the most important facts about these state spaces. Two of these are spaces of the type $\mathcal{H}(\mathcal{Z})$ described in Section 2.1.

Let $\mathfrak{W}_+$ and $\mathfrak{W}_-$ be a passive future and past behaviour, respectively, on the signal space $\mathcal{W}$. The Hilbert space $\mathcal{H}(\mathcal{Z})$ where $\mathcal{Z} = \mathfrak{W}_+$ and the underlying Kreĭn space $\mathcal{K}$ is equal to $\mathcal{K} = K_+^2(\mathcal{W})$ will be denoted by $\mathcal{H}_+ := \mathcal{H}(\mathfrak{W}_+)$, and the Hilbert space $\mathcal{H}(\mathcal{Z})$ where $\mathcal{Z} = \mathfrak{W}_-^{[\perp]}$ and the underlying Kreĭn space $\mathcal{K}$ is equal to $\mathcal{K} = -K_-^2(\mathcal{W})$ will be denoted by $\mathcal{H}_- := \mathcal{H}(\mathfrak{W}_-^{[\perp]})$. Thus, in particular, the set

$$\mathcal{H}^0(\mathfrak{W}_+) := \left\{ u^\dagger + \mathfrak{W}_+ \;\middle|\; u^\dagger \in \mathfrak{W}_+^{[\perp]} \right\}$$

is a dense subspace of $\mathcal{H}_+$, and the set

$$\mathcal{H}^0(\mathfrak{W}_-^{[\perp]}) := \left\{ w_- + \mathfrak{W}_-^{[\perp]} \;\middle|\; w_- \in \mathfrak{W}_- \right\}$$

is a dense subspace of $\mathcal{H}_-$. We denote

$$\begin{aligned}\mathcal{K}_+ &:= \mathcal{K}(\mathfrak{W}_+) := \{u \in K^2_+(\mathcal{W}) \mid u + \mathfrak{W}_+ \in \mathcal{H}_+\},\\ \mathcal{K}_- &:= \mathcal{K}(\mathfrak{W}_-^{[\perp]}) := \{w_- \in K^2_-(\mathcal{W}) \mid w_- + \mathfrak{W}_-^{[\perp]} \in \mathcal{H}_-\},\\ Q_+ w_+ &:= w_+ + \mathfrak{W}_+, \qquad w_+ \in \mathcal{K}_+,\\ Q_- w_- &:= w_- + \mathfrak{W}_-^{[\perp]}, \qquad w_- \in \mathcal{K}_-.\end{aligned}$$

Thus, $Q_+$ and $Q_-$ are the restrictions of the quotient maps $K^2_+(\mathcal{W}) \mapsto K^2_+(\mathcal{W})/\mathfrak{W}_+$ and $K^2_-(\mathcal{W}) \mapsto K^2_-(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ to $\mathcal{K}_+$ and $\mathcal{K}_-$, respectively. With these notations,

$$\begin{aligned}(w_+^\dagger + \mathfrak{W}_+, w_+ + \mathfrak{W}_+)_{\mathcal{H}_+} &= -[w_+^\dagger, w_+]_{K^2_+(\mathcal{W})}, \qquad w_+^\dagger \in \mathfrak{W}_+^{[\perp]}, \qquad w_+ \in \mathcal{K}_+,\\ (w_- + \mathfrak{W}_-^{[\perp]}, w_-^\dagger + \mathfrak{W}_-^{[\perp]})_{\mathfrak{W}_-} &= [w_-, w_-^\dagger]_{K^2_-(\mathcal{W})}, \qquad w_- \in \mathfrak{W}_-, \qquad w_-^\dagger \in \mathcal{K}_-.\end{aligned}$$

Let $\mathfrak{W}$ be a passive two-sided behaviour on $\mathcal{W}$ with the corresponding passive past behaviour $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and passive future behaviour $\mathfrak{W}_+ = \mathfrak{W} \cap K^2_+(\mathcal{W})$. By definition, the past/future map $\Gamma_\mathfrak{W}$ is the unique contraction in $\mathcal{B}(\mathcal{H}_-;\mathcal{H}_+)$ whose restriction to the subspace $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$ given by

$$\Gamma_\mathfrak{W}(\pi_- w + \mathfrak{W}_-^{[\perp]}) = \pi_+ w + \mathfrak{W}_+, \qquad w \in \mathfrak{W}. \tag{3.14}$$

See [AKS11b, Lemma 5.7] for details.

For each passive two-sided behaviour $\mathfrak{W}$ on $\mathcal{W}$ we define the operator $A_\mathfrak{W}$ by

$$A_\mathfrak{W} := \begin{bmatrix} 1_{\mathcal{H}_+} & \Gamma_\mathfrak{W} \\ \Gamma_\mathfrak{W}^* & 1_{\mathcal{H}_-}\end{bmatrix}. \tag{3.15}$$

This is a nonnegative bounded linear operator on $\mathcal{H}_+ \oplus \mathcal{H}_-$, and we define $\mathcal{D}(\mathfrak{W})$ to be the range of $A_\mathfrak{W}^{1/2}$, with the range norm, i.e.,

$$\left\| \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \right\|_{\mathcal{D}(\mathfrak{W})} = \left\| (A_\mathfrak{W}^{1/2})^{[-1]} \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \right\|_{\mathcal{H}_+ \oplus \mathcal{H}_-},$$

where $(A_\mathfrak{W}^{1/2})^{[-1]}$ is the pseudo-inverse of $A_\mathfrak{W}^{1/2}$, i.e., $\left[\begin{smallmatrix} x'_+ \\ x'_- \end{smallmatrix}\right] := (A_\mathfrak{W}^{1/2})^{[-1]} \left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right]$ is the unique vector in $\overline{\operatorname{im}(A_\mathfrak{W})} = \overline{\operatorname{im}\left(A_\mathfrak{W}^{1/2}\right)}$ which satisfies $\left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right] = A_\mathfrak{W}^{1/2} \left[\begin{smallmatrix} x'_+ \\ x'_- \end{smallmatrix}\right]$. With respect to this inner product in the range space the operator $A_\mathfrak{W}^{1/2}{}_{|\overline{\operatorname{im}(A_\mathfrak{W})}}$ is a unitary operator mapping $\overline{\operatorname{im}(A_\mathfrak{W})}$ onto $\mathcal{D}(\mathfrak{W})$. We denote

$$\begin{aligned}\mathcal{L}(\mathfrak{W}) &:= \{w \in K^2(\mathcal{W}) \mid w + (\mathfrak{W}_+ + \mathfrak{W}_-^{[\perp]}) \in \mathcal{D}(\mathfrak{W})\},\\ Qw &:= w + (\mathfrak{W}_+ + \mathfrak{W}_-^{[\perp]}), \qquad w \in \mathcal{L}(\mathfrak{W}).\end{aligned}$$

Thus, $Q$ is the restrictions of the quotient map $K^2(\mathcal{W}) \mapsto K^2_+/(\mathfrak{W}_+ + \mathfrak{W}_-^{[\perp]})$. See [AKS11b, Section 5.3] for details.

## 3.5 The past/present and present/future maps $\mathfrak{B}_\Sigma$ and $\mathfrak{C}_\Sigma$

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with past and future behaviours $\mathfrak{W}_-$ and $\mathfrak{W}_+$. With the notations introduced in Section 3.4 we have the following result:

**Lemma 3.26.** *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with future behaviour $\mathfrak{W}_+$. If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma$, then*

$$w \in \mathcal{K}(\mathfrak{W}_+) \text{ and } \|Q_+ w\|_{\mathcal{H}_+} \leq \|x(0)\|_{\mathcal{X}}. \tag{3.16}$$

*Proof.* This is [AKS11b, Lemma 6.1]. □

**Lemma 3.27.** *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with future behaviour $\mathfrak{W}_+$. Then the formula*

$$\mathfrak{C}_\Sigma x_0 = \left\{ Q_+ w \;\middle|\; \begin{array}{l} w \text{ is the signal part of some stable future} \\ \text{trajectory } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \text{ of } \Sigma \text{ with } x(0) = x_0 \end{array} \right\} \tag{3.17}$$

*defines a linear contraction $\mathfrak{C}_\Sigma \colon \mathcal{X} \to \mathcal{H}_+$.*

*Proof.* This is [AKS11b, Lemma 6.2]. □

**Definition 3.28.** The contraction $\mathfrak{C}_\Sigma$ defined in Lemma 3.27 is called the *present/future map* of $\Sigma$.

**Lemma 3.29.** *If two passive s/s systems $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ and $\Sigma_2 = (V_2; \mathcal{X}_2, \mathcal{W})$ are intertwined by a bounded operator $E$, then their present/future maps satisfy $\mathfrak{C}_{\Sigma_1} = \mathfrak{C}_{\Sigma_2} E$.*

*Proof.* This follows from Definitions 3.5 and 3.28. □

**Lemma 3.30.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with present/future map $\mathfrak{C}_\Sigma$.*

(i) *The unobservable subspace $\mathfrak{U}_\Sigma$ is equal to the null space of its present/future map $\mathfrak{C}_\Sigma$. Thus, $\Sigma$ is observable if and only if $\mathfrak{C}_\Sigma$ is injective.*

(ii) *If $\Sigma$ is co-energy preserving, then $\mathfrak{C}_\Sigma$ is co-isometric.*

(iii) *$\Sigma$ is observable and co-energy preserving if and only if $\mathfrak{C}_\Sigma$ is unitary.*

*Proof.* See Lemmas 6.6 and 6.19 and Corollary 8.8 in [AKS11b]. □

**Theorem 3.31.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with present/future map $\mathfrak{C}_\Sigma$, and let $\Sigma = (V_1; \mathcal{X}_1, \mathcal{W})$ be an observable co-energy preserving s/s system with present/future map $\mathfrak{C}_{\Sigma_1}$ which is externally equivalent to $\Sigma$. Then $\Sigma$ and $\Sigma_1$ are contractively intertwined by $\mathfrak{C}_{\Sigma_1}^{-1}\mathfrak{C}_\Sigma$. In particular, any two externally equivalent observable and co-energy preserving s/s systems are unitarily similar to each other.*

*Proof.* This follows from Theorems 8.4 and 8.5 in [AKS11b]. □

**Lemma 3.32.** *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with past behaviour $\mathfrak{W}_-$. Then there exist a unique linear contraction $\mathfrak{B}_\Sigma \colon \mathcal{H}_- \to \mathcal{X}$ whose restriction to $\mathcal{H}_-^0$ is given by*

$$\mathfrak{B}_\Sigma Q_- w = x(0), \quad w \in \mathfrak{W}_-, \tag{3.18}$$

*where $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is the unique stable externally generated past trajectory of $\Sigma$ whose signal part is $w$.*

*Proof.* See [AKS11b, Lemmas 3.11 and 6.9]. □

**Definition 3.33.** The contraction $\mathfrak{B}_\Sigma$ defined in Lemma 3.32 is called the *past/present map* of $\Sigma$.

**Lemma 3.34.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with past/present map $\mathfrak{B}_\Sigma$.*

(i) *The reachable subspace $\mathfrak{R}_\Sigma$ is equal to the closure of the range of $\mathfrak{B}_\Sigma$. Thus, $\Sigma$ is controllable if and only if $\mathfrak{B}_\Sigma$ has dense range.*

(ii) *If $\Sigma$ is energy preserving system, then $\mathfrak{B}_\Sigma$ is an isometry.*

(iii) *$\Sigma$ is controllable and energy preserving if and only if $\mathfrak{B}_\Sigma$ is unitary.*

*Proof.* See Lemmas 6.13 and 6.15 and Corollary 9.8 in [AKS11b]. □

**Theorem 3.35.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with past/present map $\mathfrak{B}_\Sigma$, and let $\Sigma = (V_1; \mathcal{X}_1, \mathcal{W})$ be a controllable energy preserving s/s system with past/present map $\mathfrak{B}_{\Sigma_1}$ which is externally equivalent to $\Sigma$. Then $\Sigma_1$ and $\Sigma$ are contractively intertwined by $\mathfrak{B}_\Sigma\mathfrak{B}_{\Sigma_1}^{-1}$. In particular, any two externally equivalent controllable and energy preserving s/s systems are unitarily similar to each other.*

*Proof.* This follows from Theorems 9.5 and 9.6 in [AKS11b]. □

**Lemma 3.36.** *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with past behaviour $\mathfrak{W}_-$, future behaviour $\mathfrak{W}_+$, two-sided behaviour $\mathfrak{W}$, past/present map $\mathfrak{B}_\Sigma$, and present/future map $\mathfrak{C}_\Sigma$.*

(i) *A pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable past trajectory of $\Sigma$ if and only if*

$$w \in \mathfrak{W}_- \text{ and } x(t) = \mathfrak{B}_\Sigma Q_- \pi_- \tau^t w, \quad t \in \mathbb{R}^-. \tag{3.19}$$

(ii) *A pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable two-sided trajectory of $\Sigma$ if and only if*

$$w \in \mathfrak{W} \text{ and } x(t) = \mathfrak{B}_\Sigma Q_- \pi_- \tau^t w, \quad t \in \mathbb{R}. \tag{3.20}$$

*In this case*

$$\mathfrak{C}_\Sigma x(t) = Q_+ \pi_+ \tau^t w, \quad t \in \mathbb{R}. \tag{3.21}$$

(iii) *A pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable future trajectory of $\Sigma$ if and only if*

$$w \in \mathfrak{W}_+ \text{ and } x(t) = \mathfrak{B}_\Sigma Q_- \pi_- \tau^t w, \quad t \in \mathbb{R}^+. \tag{3.22}$$

*In this case*

$$\mathfrak{C}_\Sigma x(t) = Q_+ \pi_+ \tau^t w, \quad t \in \mathbb{R}^+. \tag{3.23}$$

*Proof.* This is [AKS11b, Lemma 6.11]. □

**Definition 3.37.** Let $\mathfrak{W}$ be the two-sided behaviour of a passive s/s system $\Sigma$. Then the past/future map $\Gamma_{\mathfrak{W}}$ defined by means of (3.14) is also called the *past/future map of* $\Sigma$, and it is alternatively denoted by $\Gamma_\Sigma$.

**Lemma 3.38.** *The past/future map $\Gamma_\Sigma$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ factors into the product*

$$\Gamma_\Sigma = \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma \tag{3.24}$$

*of the past/present map $\mathfrak{B}_\Sigma$ and the present/future map $\mathfrak{C}_\Sigma$ of $\Sigma$.*

*Proof.* See [AKS11b, Lemma 7.2]. □

**Lemma 3.39.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with past/present map $\mathfrak{B}_\Sigma$, present/future map $\mathfrak{C}_\Sigma$, and past/future map $\Gamma_\Sigma$,*

(i) *If $\Sigma$ is observable, then* $\ker(\mathfrak{C}_\Sigma) = \ker(\Gamma_\Sigma)$.

(ii) *If $\Sigma$ is controllable, then* $\operatorname{im}(\mathfrak{B}_\Sigma)$ *is a dense subspace of* $\operatorname{im}(\Gamma_\Sigma)$.

*Proof.* This follows from Lemmas 3.30, 3.34, and 3.38. □

**Lemma 3.40.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with past/present map $\mathfrak{B}_\Sigma$, present/future map $\mathfrak{C}_\Sigma$, and past/future map $\Gamma_\Sigma$, and let $\Sigma_* = (V_*; \mathcal{X}, -\mathcal{W})$ be the adjoint of $\Sigma$ with past/present map $\mathfrak{B}_{\Sigma_*}$, present/future map $\mathfrak{C}_{\Sigma_*}$, and past/future map $\Gamma_{\Sigma_*}$. Then*

$$\mathfrak{B}_{\Sigma_*} = \mathfrak{C}_\Sigma^* \boldsymbol{\mathfrak{R}} \mathcal{I}_{(\mathcal{W},-\mathcal{W}))}, \quad \mathfrak{C}_{\Sigma_*} = \mathcal{I}_{(\mathcal{W},-\mathcal{W}))} \boldsymbol{\mathfrak{R}} \mathfrak{B}_\Sigma^*, \quad \Gamma_{\Sigma_*} = \mathcal{I}_{(\mathcal{W},-\mathcal{W}))} \boldsymbol{\mathfrak{R}} \Gamma_\Sigma^* \boldsymbol{\mathfrak{R}} \mathcal{I}_{(-\mathcal{W},\mathcal{W}))}.$$

*Proof.* See Remark 4.2, Lemma 6.18, and Lemma 7.6 in [AKS11b]. □

**Lemma 3.41.** *If two passive s/s systems $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ and $\Sigma_2 = (V_2; \mathcal{X}_2, \mathcal{W})$ are intertwined by a bounded operator $E$, then their past/present maps satisfy $\mathfrak{B}_{\Sigma_2} = E\mathfrak{B}_{\Sigma_1}$.*

*Proof.* This follows from Lemmas 3.21, 3.29, and 3.40. □

**Lemma 3.42.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with reachable subspace $\mathfrak{R}_\Sigma$ and unobservable subspace $\mathfrak{U}_\Sigma$. Then the reachable and unobservable subspaces of the adjoint system $\Sigma_* = (V_*; \mathcal{X}, -\mathcal{W})$ are equal to $\mathfrak{R}_{\Sigma_*} = \mathfrak{U}_\Sigma^\perp$ and $\mathfrak{U}_{\Sigma_*} = \mathfrak{R}_\Sigma^\perp$, respectively.*

*Proof.* This follows from Lemmas 3.30 and 3.40. □

## 3.6 Canonical models of passive state/signal systems

Throughout this subsection $\mathfrak{W}_+$, $\mathfrak{W}$, and $\mathfrak{W}_-$ are passive, future, two-sided, and past behaviours on a Kreĭn space which are related to each other by (3.8), (3.9), and (3.10), and $\Gamma_{\mathfrak{W}}$ stands for the corresponding past/future map.

**Theorem 3.43.** *Let $\mathfrak{W}_+$ be a passive future behaviour on the Kreĭn space $\mathcal{W}$ with the corresponding two-sided passive behaviour $\mathfrak{W}$. With the notations introduced in Section 3.4, define*

$$V_{\rm oce}^{\mathfrak{W}_+} = \left\{ \begin{bmatrix} Q_+\dot{w}_+ \\ Q_+w_+ \\ w_+(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_+ \\ \mathcal{H}_+ \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{array}{l} w_+ \in \mathcal{K}_+ \text{ is locally absolutely} \\ \text{continuous with } \dot{w}_+ \in K_+^2(\mathcal{W}), \text{ and} \\ \lim\limits_{t\to 0+} \dfrac{1}{t} Q_+(\tau_+^t w_+ - w_+) \text{ exists in } \mathcal{H}_+. \end{array} \right\} \tag{3.25}$$

*Then $\Sigma_{\rm oce}^{\mathfrak{W}_+} = \left(V_{\rm oce}^{\mathfrak{W}_+}; \mathcal{H}_+, \mathcal{W}\right)$ is a passive observable co-energy preserving s/s system with future behaviour $\mathfrak{W}_+$. The past/present map of $\Sigma_{\rm oce}^{\mathfrak{W}_+}$ is the past/future map $\Gamma_{\mathfrak{W}}$ of $\mathfrak{W}$, and the present/future map of $\Sigma_{\rm oce}^{\mathfrak{W}_+}$ is the identity on $\mathcal{H}_+$.*

*Proof.* See [AKS11b, Theorem 8.1]. □

**Theorem 3.44.**

(i) *Two externally equivalent observable passive s/s systems* $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ *and* $\Sigma_2 = (V_2; \mathcal{X}_2, \mathcal{W})$ *are unitarily similar if and only if their present/future maps satisfy* $\mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^* = \mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^*$.

(ii) *Two externally equivalent controllable passive s/s systems* $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ *and* $\Sigma_2 = (V_2; \mathcal{X}_2, \mathcal{W})$ *are unitarily similar if and only if their past/present maps satisfy* $\mathfrak{B}_{\Sigma_1}^*\mathfrak{B}_{\Sigma_1} = \mathfrak{B}_{\Sigma_2}^*\mathfrak{B}_{\Sigma_2}$.

*Proof.* The necessity of the two conditions $\mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^* = \mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^*$ and $\mathfrak{B}_{\Sigma_1}^*\mathfrak{B}_{\Sigma_1} = \mathfrak{B}_{\Sigma_1}^*\mathfrak{C}_{\Sigma_1}$ for unitary similarity follows from Lemmas 3.29 and 3.41.

In order to prove the sufficiency of the condition $\mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^* = \mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^*$ we assume that this condition holds and let $\mathfrak{W}$ be the common two-sided behaviour of $\Sigma_1$ and $\Sigma_2$, and let $\Sigma_{\text{oce}}^{\mathfrak{W}_+} = (V_{\text{oce}}^{\mathfrak{W}_+}; \mathcal{H}_+, \mathcal{W})$ be the observable co-energy preserving system in Theorem 3.43. By Theorem 3.31, for $i = 1, 2$, the operator $\mathfrak{C}_{\Sigma_i}$ intertwines the system $\Sigma_i$ and $\Sigma_{\text{cep}}^{\mathfrak{W}_+}$ (recall that the present/future map of $\Sigma_{\text{cep}}^{\mathfrak{W}_+}$ is the identity on $\mathcal{H}_+$). Explicitly, this means $\left[\begin{smallmatrix} x_i \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_i$, $i = 1, 2$, if and only if $\left[\begin{smallmatrix} \mathfrak{C}_{\Sigma_i} x_i \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_{\text{mo}}^{\mathfrak{W}}$ whose initial state is contained in $\operatorname{im}(\mathfrak{C}_{\Sigma_i})$. By assumption, $\mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^* = \mathfrak{C}_{\Sigma_2}\mathfrak{C}_{\Sigma_2}^*$ and therefore

$$\operatorname{im}(\mathfrak{C}_{\Sigma_1}) = \operatorname{im}\left((\mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^*)^{1/2}\right) = \operatorname{im}\left((\mathfrak{C}_{\Sigma_2}\mathfrak{C}_{\Sigma_2}^*)^{1/2}\right) = \operatorname{im}(\mathfrak{C}_{\Sigma_2}).$$

In particular, the operator $\mathcal{V} := \mathfrak{C}_{\Sigma_2}^{-1}\mathfrak{C}_{\Sigma_1}$ is well-defined. It follows from, for example, the polar decompositions of $\mathfrak{C}_{\Sigma_1}$ and $\mathfrak{C}_{\Sigma_2}$ (see [Kat80, pp. 334–335]) that $\mathcal{V}$ is a unitary operator $\mathcal{X}_1 \to \mathcal{X}_2$. Moreover, $\left[\begin{smallmatrix} x_1 \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_1$ if and only if $\left[\begin{smallmatrix} Ex_1 \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_2$. Thus, $\Sigma_1$ and $\Sigma_2$ are unitarily similar with similarity operator $E$.

Claim (ii) follows from Claim (i) applied to the adjoint system $\Sigma_*$. □

**Theorem 3.45.** *Let* $\mathfrak{W}_-$ *be a passive past behaviour on the Kreĭn space* $\mathcal{W}$*, and let* $\mathfrak{W}$ *be the corresponding two-sided passive behaviour. With the notations introduced in Section 3.4, define*

$$V_{\text{cep}}^{\mathfrak{W}_-} = \left\{ \begin{bmatrix} Q_-\pi_-\dot{w} \\ Q_-\pi_-w \\ w(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_- \\ \mathcal{H}_- \\ \mathcal{W} \end{bmatrix} \,\middle|\, \begin{array}{l} w \in \operatorname{im}\left(\left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right]\right) \textit{ is locally absolutely} \\ \textit{continuous with } \dot{w} \in K^2(\mathcal{W}), \textit{ and} \\ \lim_{t\to 0+} \dfrac{1}{t} Q_-\pi_-(\tau^t w - w) \textit{ exists in } \mathcal{H}_-. \end{array} \right\} \tag{3.26}$$

*Then* $\Sigma^{\mathfrak{W}_-}_{\mathrm{cep}} = \big(V^{\mathfrak{W}_-}_{\mathrm{cep}}; \mathcal{H}_-, \mathcal{W}\big)$ *is a passive controllable energy preserving s/s system with past behaviour* $\mathfrak{W}_-$*. The past/present map of* $\Sigma^{\mathfrak{W}_-}_{\mathrm{cep}}$ *is the identity on* $\mathcal{H}_-$ *and the present/future map of* $\Sigma^{\mathfrak{W}_-}_{\mathrm{cep}}$ *is the past/future map* $\Gamma_{\mathfrak{W}}$ *of* $\mathfrak{W}$*.*

*Proof.* See [AKS11b, Theorem 9.1]. □

**Theorem 3.46.** *The operator* $\Gamma_{\mathfrak{W}}$ *intertwines the two s/s systems* $\Sigma^{\mathfrak{W}_-}_{\mathrm{cep}}$ *and* $\Sigma^{\mathfrak{W}_+}_{\mathrm{oce}}$*.*

*Proof.* This follows from Theorems 3.43 and and 3.45 and combined with Theorem 3.31 or Theorem 3.35. □

**Theorem 3.47.** *Let* $\mathfrak{W}$ *be a passive two-sided behaviour on the Kreĭn space* $\mathcal{W}$*. With the notations introduced in Section 3.4, define*

$$V^{\mathfrak{W}}_{\mathrm{sc}} = \left\{ \begin{bmatrix} Q\dot{w} \\ Qw \\ w(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{D}(\mathfrak{W}) \\ \mathcal{D}(\mathfrak{W}) \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{array}{l} w \in \mathcal{L}(\mathfrak{W}) \text{ is locally absolutely} \\ \text{continuous with } \dot{w} \in K^2(\mathcal{W}), \text{ and} \\ \lim_{t\to 0} \dfrac{1}{t} Q(\tau^t w - w) \text{ exists in } \mathcal{D}(\mathfrak{W}). \end{array} \right\} \tag{3.27}$$

*Then the following claims are true:*

(i) $\Sigma^{\mathfrak{W}}_{\mathrm{sc}} = \big(V^{\mathfrak{W}}_{\mathrm{sc}}; \mathcal{D}(\mathfrak{W}), \mathcal{W}\big)$ *is a simple conservative s/s system with two-sided behaviour* $\mathfrak{W}$*. The past/present map of* $\Sigma^{\mathfrak{W}}_{\mathrm{sc}}$ *is* $\mathfrak{B}_{\Sigma^{\mathfrak{W}}_{\mathrm{sc}}} = \left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right]$ *with* $(\mathfrak{B}^{\mathfrak{W}}_{\Sigma})^* = \Pi_{-|\mathcal{D}(\mathfrak{W})}$*, the present/future map of* $\Sigma^{\mathfrak{W}}_{\mathrm{sc}}$ *is* $\mathfrak{C}_{\Sigma^{\mathfrak{W}}_{\mathrm{sc}}} = \Pi_{+|\mathcal{D}(\mathfrak{W})}$ *with* $\mathfrak{C}^*_{\Sigma^{\mathfrak{W}}_{\mathrm{sc}}} = \left[\begin{smallmatrix} 1_{\mathcal{H}_+} \\ \Gamma^*_{\mathfrak{W}} \end{smallmatrix}\right]$*.*

(ii) *Every simple conservative s/s system* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *with two-sided behaviour* $\mathfrak{W}$ *is unitarily similar to* $\Sigma^{\mathfrak{W}}_{\mathrm{sc}}$*. The unitary similarity operator is the so called* two-sided state/signal map

$$\mathfrak{C}^{\mathrm{bil}}_{\Sigma} := \begin{bmatrix} \mathfrak{C}_{\Sigma} \\ \mathfrak{B}^*_{\Sigma} \end{bmatrix} \tag{3.28}$$

*where* $\mathfrak{B}_{\Sigma}$ *and* $\mathfrak{C}_{\Sigma}$ *are the past/present and present/future maps of* $\Sigma$*.*

*Proof.* See [AKS11b, Theorems 10.1, 10.2, and 10.5]. □

**Corollary 3.48.** *Any two externally equivalent simple conservative s/s systems are unitarily similar to each other.*

*Proof.* This follows from part (ii) of Theorem 3.47. □

In view of Theorems 3.31, 3.35, 3.43, 3.45, and 3.47 the passive systems $\Sigma^{\mathfrak{W}_+}_{\mathrm{oce}}$, $\Sigma^{\mathfrak{W}_-}_{\mathrm{cep}}$, and $\Sigma^{\mathfrak{W}}_{\mathrm{sc}}$ are called *canonical models* of passive s/s systems within one of the classes a)–c) listed in Theorem 3.14.

## 3.7 Simple passive s/s systems

**Example 3.49.** A conservative s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ may be similar to itself with a nontrivial unitary similarity operator $\mathcal{V}_\mathcal{X}$. This can be seen as follows. Take $\mathcal{W} = \{0\}$, so that the signal part of the system is missing, and let $V = \left[\begin{smallmatrix} A \\ 1_\mathcal{X} \end{smallmatrix}\right] \mathcal{X}$ for some skew-adjoint operator $A \in \mathcal{B}(\mathcal{X})$. Then $\Sigma = (V; \mathcal{X}, \{0\})$ is a conservative s/s system. Choose some arbitrary operator $\mathcal{V}_\mathcal{X} \neq 1_\mathcal{X}$ (for example, $\mathcal{V}_\mathcal{X} = -1_\mathcal{X}$), such that $\mathcal{V}_\mathcal{X} A = A\mathcal{V}_\mathcal{X}$. Then $A = \mathcal{V}_\mathcal{X} A \mathcal{V}_\mathcal{X}^{-1}$, so that $A$ is similar to itself with similarity operator $\mathcal{V}_\mathcal{X}$, and

$$\begin{aligned}\begin{bmatrix} \mathcal{V}_\mathcal{X} & 0 \\ 0 & \mathcal{V}_\mathcal{X} \end{bmatrix} V &= \begin{bmatrix} \mathcal{V}_\mathcal{X} & 0 \\ 0 & \mathcal{V}_\mathcal{X} \end{bmatrix} \begin{bmatrix} A \\ 1_\mathcal{X} \end{bmatrix} \mathcal{X} \\ &= \begin{bmatrix} \mathcal{V}_\mathcal{X} A \\ \mathcal{V}_\mathcal{X} \end{bmatrix} \mathcal{X} = \begin{bmatrix} \mathcal{V}_\mathcal{X} A \mathcal{V}_\mathcal{X}^{-1} \\ 1_\mathcal{X} \end{bmatrix} \mathcal{X} = \begin{bmatrix} A \\ 1_\mathcal{X} \end{bmatrix} \mathcal{X} = V.\end{aligned}$$

Thus, $\Sigma$ is unitarily similar to itself with the non-trivial similarity operator $\mathcal{V}_\mathcal{X}$.

The above example was based on the fact that the s/s system in this example is not simple. As we show below, for a simple conservative system this cannot happen.

**Lemma 3.50.** *Let $V$ be the generating subspace of a simple passive system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, and suppose that*

$$^{(VSimV)} V = \begin{bmatrix} \mathcal{V}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{V}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{V}_\mathcal{W} \end{bmatrix} V \tag{3.29}$$

*for some unitary operators $\mathcal{V}_\mathcal{X} \colon \mathcal{X} \to \mathcal{X}$ and $\mathcal{V}_\mathcal{W} \colon \mathcal{W} \to \mathcal{W}$, where either both $\mathcal{V}_\mathcal{X}$ and $\mathcal{V}_\mathcal{W}$ are linear or both $\mathcal{V}_\mathcal{X}$ and $\mathcal{V}_\mathcal{W}$ are conjugate-linear. Then the following claims are true.*

(i) *If $\mathcal{V}_\mathcal{X}$ is linear and $\mathcal{V}_\mathcal{W} = 1_\mathcal{W}$, then $\mathcal{V}_\mathcal{X} = 1_\mathcal{X}$.*

(ii) *If $\mathcal{V}_\mathcal{X}$ is linear and $\mathcal{V}_\mathcal{W}$ is a signature operator, then $\mathcal{V}_\mathcal{X}$ is a signature operator.*

(iii) *If $\mathcal{V}_\mathcal{X}$ is conjugate-linear and $\mathcal{V}_\mathcal{W}$ is a conjugation, then $\mathcal{V}_\mathcal{X}$ is a conjugation.*

*Proof of (i).* It follows from (3.29) that if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an arbitrary trajectory of $\Sigma$, then $\mathcal{V}_\mathcal{X} x = x$. Consequently, $\mathcal{V}_\mathcal{X} x = x$ for all $x$ in the reachable subspace $\mathfrak{R}$. Since $\mathcal{V}_\mathcal{X}$ is unitary, also $\mathcal{V}_\mathcal{X}^* x = \mathcal{V}_\mathcal{X}^* \mathcal{V}_\mathcal{X} x = x$ for all $x \in \mathfrak{R}$.

If we repeat the same argument with the original system replaced by the dual system, then we find that $\mathcal{V}_{\mathcal{X}}$ (and $\mathcal{V}_{\mathcal{X}}^*$) also is the identity on $\mathfrak{U}^\perp$, where $\mathfrak{U}^\perp$ is the reachable subspace of the adjoint system. By the simplicity assumption, the span of $\mathfrak{R}$ and $\mathfrak{U}^\perp$ is dense in $\mathcal{X}$, and hence $\mathcal{V}_{\mathcal{X}} = 1_{\mathcal{X}}$.

*Proof of (ii).* It follows from (3.29) that

$$V = \begin{bmatrix} \mathcal{V}_{\mathcal{X}}^{-1} & 0 & 0 \\ 0 & \mathcal{V}_{\mathcal{X}}^{-1} & 0 \\ 0 & 0 & \mathcal{V}_{\mathcal{W}}^{-1} \end{bmatrix} V = \begin{bmatrix} \mathcal{V}_{\mathcal{X}}^{*} & 0 & 0 \\ 0 & \mathcal{V}_{\mathcal{X}}^{*} & 0 \\ 0 & 0 & \mathcal{V}_{\mathcal{W}} \end{bmatrix} V$$
$$= \begin{bmatrix} \mathcal{V}_{\mathcal{X}}^{*} & 0 & 0 \\ 0 & \mathcal{V}_{\mathcal{X}}^{*} & 0 \\ 0 & 0 & \mathcal{V}_{\mathcal{W}} \end{bmatrix} \begin{bmatrix} \mathcal{V}_{\mathcal{X}}^{-1} & 0 & 0 \\ 0 & \mathcal{V}_{\mathcal{X}}^{-1} & 0 \\ 0 & 0 & \mathcal{V}_{\mathcal{W}}^{-1} \end{bmatrix} V = \begin{bmatrix} \mathcal{V}_{\mathcal{X}}^{*}\mathcal{V}_{\mathcal{X}}^{-1} & 0 & 0 \\ 0 & \mathcal{V}_{\mathcal{X}}^{*}\mathcal{V}_{\mathcal{X}}^{-1} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V.$$

By part (i), $\mathcal{V}_{\mathcal{X}}^*\mathcal{V}_{\mathcal{X}}^{-1} = 1_{\mathcal{X}}$, and thus $\mathcal{V}_{\mathcal{X}}$ is a signature operator.

*Proof of (iii).* This proof is essentially the same as the proof of (ii). Observe that $\mathcal{V}_{\mathcal{X}}^*\mathcal{V}_{\mathcal{X}}^{-1}$ is linear also in the case where $\mathcal{V}_{\mathcal{X}}$ is conjugate-linear. □

**Lemma 3.51.** *Let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1)$ and $\Sigma_2 = (V_2; \mathcal{X}_2, \mathcal{W}_2)$ be two simple passive s/s systems whose generating subspaces satisfy*

$$\text{(V2SimV1)}\quad V_2 = \begin{bmatrix} \mathcal{V}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{V}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{V}_{\mathcal{W}} \end{bmatrix} V_1 \tag{3.30}$$

*for some unitary operators $\mathcal{V}_{\mathcal{X}} \colon \mathcal{X}_1 \to \mathcal{X}_2$ and $\mathcal{V}_{\mathcal{W}} \colon \mathcal{W}_1 \to \mathcal{W}_2$, where either both $\mathcal{V}_{\mathcal{X}}$ and $\mathcal{V}_{\mathcal{W}}$ are linear or both $\mathcal{V}_{\mathcal{X}}$ and $\mathcal{V}_{\mathcal{W}}$ are conjugate-linear. Then the operator $\mathcal{V}_{\mathcal{X}}$ is uniquely determined by $V_1$, $V_2$, and $\mathcal{V}_{\mathcal{W}}$.*

*Proof.* Suppose that (3.30) is true for two different unitary operators $\mathcal{V}_{\mathcal{X}}$ and $\widetilde{\mathcal{V}}_{\mathcal{X}}$, but with the same operator $\mathcal{V}_{\mathcal{W}}$. It follows from (3.30) that

$$V_1 = \begin{bmatrix} \widetilde{\mathcal{V}}_{\mathcal{X}}^{-1} & 0 & 0 \\ 0 & \widetilde{\mathcal{V}}_{\mathcal{X}}^{-1} & 0 \\ 0 & 0 & \mathcal{V}_{\mathcal{W}}^{-1} \end{bmatrix} V_2 = \begin{bmatrix} \widetilde{\mathcal{V}}_{\mathcal{X}}^{-1} & 0 & 0 \\ 0 & \widetilde{\mathcal{V}}_{\mathcal{X}}^{-1} & 0 \\ 0 & 0 & \mathcal{V}_{\mathcal{W}}^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{V}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{V}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{V}_{\mathcal{W}} \end{bmatrix} V_1$$
$$= \begin{bmatrix} \widetilde{\mathcal{V}}_{\mathcal{X}}^{-1}\mathcal{V}_{\mathcal{X}} & 0 & 0 \\ 0 & \widetilde{\mathcal{V}}_{\mathcal{X}}^{-1}\mathcal{V}_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_1.$$

By part (i), $\widetilde{\mathcal{V}}_{\mathcal{X}}^{-1}\mathcal{V}_{\mathcal{X}} = 1_{\mathcal{X}}$, and thus $\mathcal{V}_{\mathcal{X}} = \widetilde{\mathcal{V}}_{\mathcal{X}}$. □

# 4 Optimal, $*$-Optimal, and Passive Balanced Systems

In this section we study two extremal minimal passive realizations of a passive behaviour, namely *minimal optimal* and *minimal $*$-optimal* passive s/s systems. The corresponding extremal minimal passive realizations for i/s/o systems with scattering supply rate have been studied in, e.g., [Aro79, Nud92, Sta05, AS07b, AKP06] in discrete time and in [AN96, Sta05] in continuous time. A system in either of these classes is determined uniquely by its behaviours (future, two-sided, or past) up to a unitary similarity transformation in the state space.

By doing a half-way interpolation between a minimal optimal and a minimal $*$-optimal system we get another type of systems, namely the *passive balanced* s/s systems. Systems in this class are also determined uniquely by their behaviours up to unitary similarity. The corresponding i/s/o systems have been studied in [Sta05] in continuous time and in [AS07a] in discrete time.

## 4.1 Optimal and $*$-optimal passive s/s systems

**Definition 4.1.** Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system.

(i) $\Sigma$ is called *optimal* if it satisfies the following condition: If $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ is a passive s/s system with the same past behaviour as $\Sigma$, if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_1 \\ w \end{smallmatrix}\right]$ are two externally generated past trajectories of $\Sigma$ and $\Sigma_1$, respectively, with the same signal part $w$, then $\|x(0)\|_{\mathcal{X}} \le \|x_1(0)\|_{\mathcal{X}_1}$.

(ii) $\Sigma$ is called *$*$-optimal* if the (causal) adjoint $\Sigma_*$ of $\Sigma$ is optimal.

**Lemma 4.2.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with reachable subspace $\mathfrak{R}$ and unobservable subspace $\mathfrak{U}$.*

(i) *$\Sigma$ is optimal if and only if its restriction to $\mathfrak{R}$ is optimal.*

(ii) *If $\Sigma$ is optimal, then $\mathfrak{R} \subset \mathfrak{U}^\perp$. In particular, an optimal system is minimal if and only if it is controllable.*

(iii) *$\Sigma$ is $*$-optimal if and only if its orthogonal projection onto $\mathfrak{U}^\perp$ is $*$-optimal.*

(iv) *If $\Sigma$ is $*$ optimal, then $\mathfrak{U}^\perp \subset \mathfrak{R}$. In particular, a $*$-optimal system is minimal if and only if it is observable.*

*Proof.* It suffices to prove claims (i) and (ii), since (iii) and (iv) then follows by duality.
*Proof of (i).* Let us denote the restricted system in Claim (i) by $\Sigma_{\mathfrak{R}}$. Then $\Sigma$ and $\Sigma_{\mathfrak{R}}$ have the same stable past trajectories, and consequently $\Sigma$ is optimal if and only if $\Sigma_{\mathfrak{R}}$ is optimal.
*Proof of (ii)* Denote the orthogonal projection of $\Sigma$ onto $\mathfrak{U}^{\perp}$ by $\Sigma_{\mathfrak{U}^{\perp}}$, and choose the system $\Sigma_1$ in Definition 4.1 to be $\Sigma_{\mathfrak{U}^{\perp}}$. Then, by the optimality of $\Sigma$ and the fact that $\left[\begin{smallmatrix} P_{\mathfrak{U}^{\perp}}x \\ w \end{smallmatrix}\right]$ is the past externally generated trajectory of $\Sigma_{\mathfrak{U}^{\perp}}$ corresponding to the externally generated past trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$, we find that $\|x(0)\| \le \|P_{\mathfrak{U}^{\perp}}x(0)\|$ for all externally generated past trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$. This will be true if and only if the restriction of $P_{\mathfrak{U}^{\perp}}$ to $\mathfrak{R}$ is the identity, or equivalently, if and only if $\mathfrak{R} \subset \mathfrak{U}^{\perp}$.

*Proofs of (iii) and (iv).* Claims (iii) and (iv) follow from (i) and (ii) combined with Definition 4.1 and Lemma 3.42. □

**Theorem 4.3.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with two-sided behaviour $\mathfrak{W}$, past/future map $\Gamma_{\mathfrak{W}}$, past/present map $\mathfrak{B}_{\Sigma}$, present/future map $\mathfrak{C}_{\Sigma}$, and reachable subspace $\mathfrak{R}$. Denote the restriction of $\Sigma$ onto $\mathfrak{R}$ by $\Sigma_{\mathfrak{R}}$. Then*

$$\Gamma_{\mathfrak{W}}^*\Gamma_{\mathfrak{W}} \le \mathfrak{B}_{\Sigma}^*\mathfrak{B}_{\Sigma}, \quad \Gamma_{\mathfrak{W}}\Gamma_{\mathfrak{W}}^* \le \mathfrak{C}_{\Sigma}\mathfrak{C}_{\Sigma}^*, \tag{4.1}$$

*and the following conditions are equivalent.*

(i) $\Sigma$ *is optimal,*

(ii) *If $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ is a passive s/s system with the same two-sided behaviour $\mathfrak{W}$ and past/present map $\mathfrak{B}_{\Sigma_1}$, then $\mathfrak{B}_{\Sigma}^*\mathfrak{B}_{\Sigma} \le \mathfrak{B}_{\Sigma_1}^*\mathfrak{B}_{\Sigma_1}$.*

(iii) $\mathfrak{B}_{\Sigma}^*\mathfrak{B}_{\Sigma} = \Gamma_{\mathfrak{W}}^*\Gamma_{\mathfrak{W}}$,

(iv) *$\Sigma_{\mathfrak{R}}$ is minimal and if $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ is a passive controllable s/s system with the same two-sided behaviour $\mathfrak{W}$ and present/future map $\mathfrak{C}_{\Sigma_1}$, then $\mathfrak{C}_{\Sigma}P_{\mathfrak{R}}\mathfrak{C}_{\Sigma}^* \ge \mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^*$.*

(v) *$\Sigma_{\mathfrak{R}}$ is minimal and $\mathfrak{C}_{\Sigma}P_{\mathfrak{R}}\mathfrak{C}_{\Sigma}^* = P_{\overline{\mathrm{im}(\Gamma_{\mathfrak{W}})}}$.*

(vi) *$\mathfrak{C}_{\Sigma}|_{\mathfrak{R}}$ maps $\mathfrak{R}$ unitarily onto $\overline{\mathrm{im}\,(\Gamma_{\mathfrak{W}})}$.*

(vii) *$\Sigma_{\mathfrak{R}}$ is unitarily similar to the restriction onto its reachable subspace $\overline{\mathrm{im}\,(\Gamma_{\mathfrak{W}})}$ of the canonical model $\Sigma_{\mathrm{oce}}^{\mathfrak{W}}$ of an observable and co-energy preserving s/s system with two-sided behaviour $\mathfrak{W}$.*

*If these equivalent conditions holds, then the unitary similarity operator in (vii) is equal to $\mathfrak{C}_{\Sigma}|_{\mathfrak{R}}$ with inverse $(\mathfrak{C}_{\Sigma}|_{\mathfrak{R}})^{-1} = P_{\mathfrak{R}}\mathfrak{C}_{\Sigma|\overline{\mathrm{im}(\Gamma_{\mathfrak{W}})}}^*$.*

*Proof.* By Lemmas 6.2 and 7.2 in [AKS11b], $\mathfrak{B}_\Sigma$ and $\mathfrak{C}_\Sigma$ are contractions, and $\Gamma_{\mathfrak{W}} = \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma$. Consequently,

$$\Gamma_{\mathfrak{W}}^* \Gamma_{\mathfrak{W}} = \mathfrak{B}_\Sigma^* \mathfrak{C}_\Sigma^* \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma \le \mathfrak{B}_\Sigma^* \mathfrak{B}_\Sigma, \qquad \Gamma_{\mathfrak{W}} \Gamma_{\mathfrak{W}}^* = \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma \mathfrak{B}_\Sigma^* \mathfrak{C}_\Sigma^* \le \mathfrak{C}_\Sigma \mathfrak{C}_\Sigma^*.$$

This proves (4.1).

*(i) ⇔ (ii):* Condition (i) is equivalent to the statement that $\|\mathfrak{B}_\Sigma Q_- w\| \le \|\mathfrak{B}_{\Sigma_1} Q_- w\|$ for all $w \in \mathfrak{W}_-$. Since $Q_- \mathfrak{W}_-$ is a dense subspace of $\mathcal{H}_-$, this means that (i) and (ii) are equivalent.

*(iii) ⇔ (vi):* The inequality $\mathfrak{B}_\Sigma^* \mathfrak{C}_\Sigma^* \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma \le \mathfrak{B}_\Sigma^* \mathfrak{B}_\Sigma$ becomes an equality if and only if $\mathfrak{C}_\Sigma|_{\mathfrak{R}}$ is isometric on the range of $\mathfrak{B}_\Sigma$, or equivalently, on $\mathfrak{R}$, since the range of $\mathfrak{B}_\Sigma$ is a dense subspace of $\mathfrak{R}$. For the same reason $\operatorname{im}(\Gamma_{\mathfrak{W}}) = \operatorname{im}(\mathfrak{C}_\Sigma \mathfrak{B}_\Sigma)$ is a dense subspace of $\operatorname{im}(\mathfrak{C}_\Sigma|_{\mathfrak{R}})$.

*(v) ⇔ (vi):* $\Sigma_{\mathfrak{R}}$ is minimal if and only if $\mathfrak{C}_{\Sigma_{\mathfrak{R}}} = \mathfrak{C}_\Sigma|_{\mathfrak{R}}$ is injective. The operator $\mathfrak{C}_{\Sigma_{\mathfrak{R}}} \mathfrak{C}_{\Sigma_{\mathfrak{R}}}^* = \mathfrak{C}_\Sigma P_{\mathfrak{R}} \mathfrak{C}_\Sigma^*$ is a self-adjoint contraction on $\mathcal{H}_+$ whose range is contained in $\operatorname{im}(\mathfrak{C}_\Sigma|_{\mathfrak{R}}) \subset \overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}$, and it is equal to $P_{\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}}$ if and only if (vi) holds.

*(ii) ⇒ (iii):* Take system $\Sigma_1$ in Definition 4.1 to be the canonical model $\Sigma_{\text{oce}}^{\mathfrak{W}} = (V_{\text{oce}}^{\mathfrak{W}}; \mathcal{H}_+; \mathcal{W})$ of a controllable passive co-energy preserving s/s system with two-sided behaviour $\mathfrak{W}$. The past/present map of this system is equal to $\Gamma_{\mathfrak{W}}$, and hence by condition (ii), $\mathfrak{B}_\Sigma^* \mathfrak{B}_\Sigma \le \Gamma_{\mathfrak{W}}^* \Gamma_{\mathfrak{W}}$. On the other hand, by (4.1), $\mathfrak{B}_\Sigma^* \mathfrak{B}_\Sigma \ge \Gamma_{\mathfrak{W}}^* \Gamma_{\mathfrak{W}}$. Thus (ii) ⇒ (iii).

*(iii) ⇒ (ii):* This follows from (4.1).

*(iv) ⇒ (v):* We choose the system $\Sigma_1$ in (iv) to be the system in (vii). The present/future map of this system is $P_{\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}}$, and hence (iv) implies that $\mathfrak{C}_\Sigma P_{\mathfrak{R}} \mathfrak{C}_\Sigma^* \ge P_{\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}}$. On the other hand, $\mathfrak{C}_\Sigma P_{\mathfrak{R}} \mathfrak{C}_\Sigma^*$ is a self-adjoint contraction, whose range is contained in $\operatorname{im}(\mathfrak{C}_\Sigma|\mathfrak{R}) \subset \overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}$, and therefore $\mathfrak{C}_\Sigma P_{\mathfrak{R}} \mathfrak{C}_\Sigma^* \le P_{\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}}$. Thus $\mathfrak{C}_\Sigma P_{\mathfrak{R}} \mathfrak{C}_\Sigma^* = P_{\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}}$.

*(vi) ⇒ (vii):* This follows from Theorem 3.44.

*(vii) ⇒ (iv):* If (vii) holds, then $\Sigma_{\mathfrak{R}}$ is minimal and $\mathfrak{C}_\Sigma|_{\mathfrak{R}}$ is unitarily similar to $P_{\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}}$. Consequently, $\mathfrak{C}_\Sigma P_{\mathfrak{R}} \mathfrak{C}_\Sigma^* = P_{\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}}$. The operator $\mathfrak{C}_{\Sigma_1}$ is a contraction whose range is contained in $\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}$ (because $\Sigma_1$ is controllable), and hence (iv) holds.

□

**Proposition 4.4.**

(i) *Every observable passive and co-energy preserving s/s system is optimal.*

(ii) *Every controllable passive and energy preserving s/s system is ∗-optimal.*

*Proof.* It suffices to prove Part (i), since Part (ii) then follows by duality.

Suppose that $\Sigma$ is observable and co-energy preserving. By Lemma 3.30, $\mathfrak{C}_\Sigma$ is unitary. Thus,

$$\mathfrak{B}_\Sigma^*\mathfrak{B}_\Sigma = \mathfrak{B}_\Sigma^*\mathfrak{C}_\Sigma^*\mathfrak{C}_\Sigma\mathfrak{B}_\Sigma = \Gamma_{\mathfrak{W}}^*\Gamma_{\mathfrak{W}}.$$

By Theorem 4.3, $\Sigma$ is optimal. □

**Theorem 4.5.** *Let $\Sigma = (V, \mathcal{X}, \mathcal{W})$ be a controllable passive s/s system. Then the following conditions are equivalent:*

(i) $\Sigma$ *is optimal,*

(ii) $\mathfrak{C}_\Sigma$ *is an isometry,*

(iii) $\mathfrak{C}_\Sigma|_{\mathfrak{R}}$ *maps* $\mathcal{X}$ *unitarily onto* $\overline{\mathrm{im}\,(\Gamma_{\mathfrak{W}})}$.

(iv) *For all* $x_0 \in \mathcal{X}$,

$$\overset{\textit{(AvailStor)}}{\|x_0\|_{\mathcal{X}}^2} = \|\mathfrak{C}_\Sigma x_0\|_{\mathcal{H}_+} = \sup_{w\in\mathfrak{C}_\Sigma x_0} -[w,w]_{K_+^2(\mathcal{W})}. \tag{4.2}$$

*Moreover, such a system is automatically minimal.*

*Proof.* The equivalence of (i), (ii), and (iii) follows from the equivalence of (i) and (vi) in Theorem 4.3. The equivalence of (iii) and (iv) follows from the fact that the right-hand side of (4.2) is equal to $\|\mathfrak{C}_\Sigma x_0\|_{\mathcal{H}_+}^2$, by the definition of the norm in $\mathcal{H}_+$. The minimality follows from Lemma 4.2. □

**Theorem 4.6.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a controllable passive s/s system with present/future map $\mathfrak{C}_\Sigma$, and let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be a minimal optimal s/s system with present/future map $\mathfrak{C}_{\Sigma_1}$ which is externally equivalent to $\Sigma$. Then $\mathrm{im}\,(\mathfrak{C}_\Sigma) \subset \mathrm{im}\,(\mathfrak{C}_{\Sigma_1})$, and $\Sigma$ and $\Sigma_1$ are contractively intertwined by $\mathfrak{C}_{\Sigma_1}^{-1}\mathfrak{C}_\Sigma$. In particular, any two externally equivalent minimal optimal s/s systems are unitarily similar to each other.*

*Proof.* Let $\mathfrak{W}_+$ be the common future behaviour of $\Sigma$ and $\Sigma_1$, and let $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+} = \big(V_{\mathrm{oce}}^{\mathfrak{W}_+}; \mathcal{H}_+, \mathcal{W}\big)$ be the co-energy preserving observable system in Theorem 3.43. By Theorem 3.31, $\Sigma$ and $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$ are intertwined by $\mathfrak{C}_\Sigma$, whereas $\Sigma$ and $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$ are intertwined by $\mathfrak{C}_{\Sigma_1}$ (recall that the present/future map of $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$ is the identity on $\mathcal{H}_+$). Explicitly, this means $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma$ if and only if $\left[\begin{smallmatrix} \mathfrak{C}_\Sigma x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$ whose initial state is contained in $\mathrm{im}\,(\mathfrak{C}_\Sigma)$, and that $\left[\begin{smallmatrix} x_1 \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_1$ if and only if $\left[\begin{smallmatrix} \mathfrak{C}_{\Sigma_1} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$ whose initial state is contained in $\operatorname{im}(\mathfrak{C}_{\Sigma_1})$. By Theorem 4.5, $\operatorname{im}(\mathfrak{C}_{\Sigma_1}) = \overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}$, and by Lemma 3.39, $\operatorname{im}(\mathfrak{C}_{\Sigma_1}) \subset \operatorname{im}(\Gamma_{\mathfrak{W}})$. Since $\mathfrak{C}_{\Sigma_1}$ is a unitary map of $\mathcal{X}_1$ onto $\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}$, we can define $E$ by $E = \mathfrak{C}_{\Sigma_1}^{-1}\mathfrak{C}_{\Sigma}$. Then $E$ is a contraction from $\mathcal{X}$ to $\mathcal{X}_1$, and $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma$ if and only if $\left[\begin{smallmatrix} Ex \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma_1$ whose initial state is contained in $\operatorname{im}(E)$. Consequently $E$ intertwines $\Sigma$ and $\Sigma_1$. □

**Theorem 4.7.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with two-sided behaviour $\mathfrak{W}$, past/future map $\Gamma_{\mathfrak{W}}$, past/present map $\mathfrak{B}_{\Sigma}$, present/future map $\mathfrak{C}_{\Sigma}$, and unobservable subspace $\mathfrak{U}$. Denote the orthogonal projection of $\Sigma$ onto $\mathfrak{U}^{\perp}$ by $\Sigma_{\mathfrak{U}^{\perp}}$. Then the following conditions are equivalent.*

(i) *$\Sigma$ is $*$-optimal,*

(ii) *If $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ is a passive s/s system with the same two-sided behaviour $\mathfrak{W}$ and present/future map $\mathfrak{C}_{\Sigma_1}$, then $\mathfrak{C}_{\Sigma}\mathfrak{C}_{\Sigma}^* \le \mathfrak{C}_{\Sigma_1}\mathfrak{C}_{\Sigma_1}^*$.*

(iii) $\mathfrak{C}_{\Sigma}\mathfrak{C}_{\Sigma}^* = \Gamma_{\mathfrak{W}}\Gamma_{\mathfrak{W}}^*$,

(iv) *$\Sigma_{\mathfrak{U}^{\perp}}$ is minimal and if $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ is a passive observable s/s system with the same two-sided behaviour $\mathfrak{W}$ and past/present map $\mathfrak{B}_{\Sigma_1}$, then $\mathfrak{B}_{\Sigma}^* P_{\mathfrak{U}^{\perp}}\mathfrak{B}_{\Sigma} \ge \mathfrak{B}_{\Sigma_1}^*\mathfrak{B}_{\Sigma_1}$.*

(v) *$\Sigma_{\mathfrak{U}^{\perp}}$ is minimal and $\mathfrak{B}_{\Sigma}^* P_{\mathfrak{U}^{\perp}}\mathfrak{B}_{\Sigma} = P_{(\ker(\Gamma_{\mathfrak{W}}))^{\perp}}$.*

(vi) *$\mathfrak{B}_{\Sigma|\mathfrak{U}^{\perp}}^*$ maps $\mathfrak{U}^{\perp}$ unitarily onto $(\ker(\Gamma_{\mathfrak{W}}))^{\perp}$.*

(vii) *$\Sigma_{\mathfrak{U}^{\perp}}$ is unitarily similar to the orthogonal projection onto $(\ker(\Gamma_{\mathfrak{W}}))^{\perp}$ of the canonical model $\Sigma_{\mathrm{cep}}^{\mathfrak{W}}$ of a controllable and energy preserving s/s system with two-sided behaviour $\mathfrak{W}$.*

*If these equivalent conditions holds, then the unitary similarity operator in (vii) is equal to $\mathfrak{B}_{\Sigma|\mathfrak{U}^{\perp}}^*$ with inverse $(\mathfrak{B}_{\Sigma|\mathfrak{U}^{\perp}}^*)^{-1} = P_{\mathfrak{U}^{\perp}}\mathfrak{B}_{\Sigma|(\ker(\Gamma_{\mathfrak{W}}))^{\perp}}$.*

*Proof.* This follows from Theorem 4.3 by duality, taking into account Lemmas 3.40 and 3.42. □

**Theorem 4.8.** *Let $\Sigma = (V, \mathcal{X}, \mathcal{W})$ be an observable passive system. Then the following conditions are equivalent:*

(i) *$\Sigma$ is $*$-optimal.*

(ii) *$\mathfrak{B}_{\Sigma}$ is a co-isometry.*

(iii) *$\mathfrak{B}_{\Sigma}$ maps $(\ker(\Gamma_{\mathfrak{W}}))^{\perp}$ unitarily onto $\mathcal{X}$.*

(iv) *For all* $x_0 \in \mathrm{im}\,(\mathfrak{B}_\Sigma)$,

$$\|x_0\|^2_{\mathcal{X}} \overset{\text{(ReqSupply)}}{=} \inf_{\substack{w_-\in\mathfrak{W}_-\\ x_0=\mathfrak{B}_\Sigma Q_- w_-}} [w_-, w_-]_{K^2_-(\mathcal{W})}. \tag{4.3}$$

(v) *If* $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ *is a passive observable s/s system with the same two-sided behaviour* $\mathfrak{W}$*, if* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} x_1 \\ w \end{smallmatrix}\right]$ *are two externally generated past trajectories of* $\Sigma$ *and* $\Sigma_1$*, respectively, with the same signal part* $w$*, then* $\|x(0)\|_{\mathcal{X}} \geq \|x_1(0)\|_{\mathcal{X}_1}$.

*Moreover, such a system is automatically minimal.*

*Proof.* The equivalence of (i), (ii) and (iii) follows from the equivalence of (i) and (vi) in Theorem 4.7. The equivalence of (iii) and (iv) follows from the fact that the set $\{Q_- w_- \mid w_- \in \mathfrak{W}_-\}$ is dense in $\mathcal{H}_-$, plus the definition of the norm in $\mathcal{H}_-$. Finally, the equivalence of (i) and (v) follows from the equivalence of (i) and (iv) in Theorem 4.7. Also the minimality follows from Theorem 4.7. □

**Remark 4.9.** The identities (4.2) and (4.3) mean that the square of the norms of in the states of a minimal optimal system and minimal $*$-optimal system coincide, in the terminology of [Wil72], with the *available storage* and *required supply*, respectively, of a minimal system with two-sided behaviour $\mathfrak{W}$.

**Theorem 4.10.** *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be an observable passive s/s system with past/present map* $\mathfrak{B}_\Sigma$*, let* $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ *be a minimal* $*$*-optimal s/s system with past/present map* $\mathfrak{B}_{\Sigma_1}$ *which is externally equivalent to* $\Sigma$*, and denote the common past/future map of* $\Sigma$ *and* $\Sigma_1$ *by* $\Gamma_{\mathfrak{W}}$*. Then* $\Sigma_1$ *and* $\Sigma$ *are contractively intertwined by* $\mathfrak{B}_\Sigma(\mathfrak{B}_{\Sigma_1|(\ker(\Gamma_{\mathfrak{W}}))^\perp})^{-1}$*. In particular, any two externally equivalent controllable and energy preserving s/s systems are unitarily similar to each other.*

*Proof.* This follows from Theorem 4.6 by duality. □

**Remark 4.11.** Since, by Lemma [AKS11b, 5.5], the subspace

$$\mathcal{H}^0_0(\mathfrak{W}^{[\perp]}_-) := \{Q_- w_- \mid w_- \in \mathfrak{W}_- \text{ has compact support}\}$$

is dense in $\mathcal{H}_-$, it is possible to further restrict the signal $w_-$ in Definition 4.1 and in (4.3) so that it has compact support. This implies that our definition of optimality is the natural s/s counterpart of the definition of optimality given in [AN96] in a scattering i/s/o setting (there the argument is based on trajectories defined on $\mathbb{R}^+$ instead of $\mathbb{R}^-$). However, our definition of $*$-optimality is more general than the corresponding definition of $*$-optimality in [AN96], since the $*$-optimal systems in [AN96] are required to be observable, and hence minimal.

**Definition 4.12.**

(i) By the *canonical model* $\Sigma^{\mathfrak{W}}_{\mathrm{mo}} = (V^{\mathfrak{W}}_{\mathrm{mo}}; \overline{\mathrm{im}\,(\Gamma_{\mathfrak{W}})}, \mathcal{W})$ *of a minimal optimal s/s system with two-sided behaviour* $\mathfrak{W}$ we mean the restriction of the observable co-energy preserving model $\Sigma^{\mathfrak{W}+}_{\mathrm{oce}}$ onto its reachable subspace $\overline{\mathrm{im}\,(\Gamma_{\mathfrak{W}})}$.

(ii) By the *canonical model* $\Sigma^{\mathfrak{W}}_{\mathrm{m*o}} = (V^{\mathfrak{W}}_{\mathrm{m*o}}; (\ker(\Gamma_{\mathfrak{W}}))^{\perp}, \mathcal{W})$ *of a minimal $*$-optimal s/s system with two-sided behaviour* $\mathfrak{W}$ we mean the orthogonal projection of the controllable energy preserving model $\Sigma^{\mathfrak{W}-}_{\mathrm{cep}}$ onto the orthogonal complement $(\ker(\Gamma_{\mathfrak{W}}))^{\perp}$ of its unobservable subspace $\ker(\Gamma_{\mathfrak{W}})$.

**Lemma 4.13.**

(i) *The past/present map* $\mathfrak{B}^{\mathfrak{W}}_{\mathrm{mo}}$ *of* $\Sigma^{\mathfrak{W}}_{\mathrm{mo}}$ *is* $\Gamma_{\mathfrak{W}}$ *with the original range space* $\mathcal{H}_+$ *of* $\Gamma_{\mathfrak{W}}$ *replaced by* $\overline{\mathrm{im}\,(\Gamma_{\mathfrak{W}})}$, *and the present/future map* $\mathfrak{C}^{\mathfrak{W}}_{\mathrm{mo}}$ *of* $\Sigma^{\mathfrak{W}}_{\mathrm{mo}}$ *is* $1_{\mathcal{H}_+|\overline{\mathrm{im}(\Gamma_{\mathfrak{W}})}}$. *The adjoints of these operators are* $(\mathfrak{B}^{\mathfrak{W}}_{\mathrm{mo}})^* = \Gamma^*_{\mathfrak{W}|\overline{\mathrm{im}(\Gamma_{\mathfrak{W}})}}$ *and* $(\mathfrak{C}^{\mathfrak{W}}_{\mathrm{mo}})^* = P_{\overline{\mathrm{im}(\Gamma_{\mathfrak{W}})}}$.

(ii) *The past/present map* $\mathfrak{B}^{\mathfrak{W}}_{\mathrm{m*o}}$ *of* $\Sigma^{\mathfrak{W}}_{\mathrm{m*o}}$ *is* $P_{(\ker(\Gamma_{\mathfrak{W}}))^{\perp}}$, *and the present/future map* $\mathfrak{C}^{\mathfrak{W}}_{\mathrm{m*o}}$ *of* $\Sigma^{\mathfrak{W}}_{\mathrm{m*o}}$ *is* $\Gamma_{\mathfrak{W}|(\ker(\Gamma_{\mathfrak{W}}))^{\perp}}$. *The adjoint of* $\mathfrak{B}^{\mathfrak{W}}_{\mathrm{m*o}}$ *is* $\Gamma^*_{\mathfrak{W}}$ *with the original range space* $\mathcal{H}_-$ *of* $\Gamma^*_{\mathfrak{W}}$ *replaced by* $(\ker(\Gamma_{\mathfrak{W}}))^{\perp}$, *and the* $(\mathfrak{C}^{\mathfrak{W}}_{\mathrm{m*o}})^* = 1_{\mathcal{H}_-|(\ker(\Gamma_{\mathfrak{W}}))^{\perp}}$.

*Proof.* These claims follow from the Theorems 3.43 and 3.45. □

**Theorem 4.14.** *The operator* $\Gamma_{\mathfrak{W}}$, *interpreted as an operator defined on* $(\ker\Gamma_{\mathfrak{W}})^{\perp}$ *with values in* $\overline{\mathrm{im}\,(\Gamma_{\mathfrak{W}})}$, *intertwines the two s/s systems* $\Sigma^{\mathfrak{W}}_{\mathrm{m*o}}$ *and* $\Sigma^{\mathfrak{W}}_{\mathrm{mo}}$.

*Proof.* This follows from Theorems 3.43 and and 3.45 and combined with Theorem 3.31 or Theorem 3.35. □

## 4.2 Passive balanced state/signal systems

There is another class of passive s/s system, the class of so called passive *balanced* s/s systems, which we have not yet looked at, but which will be important in our discussion of the reciprocal symmetry of a s/s system. The corresponding i/s/o counterparts are found in, e.g., [Wil72] (for finite-dimensional impedance systems) and [Sta05, Section 11.8] (for infinite-dimensional scattering systems). (There also exists another type of balanced i/s/o systems that we shall not discuss here, namely *Hankel balanced*. For various version of Hankel balanced i/s/o systems, see e.g., [You86], [OMS90], [OW93], [OW96], and [Sta05, Section 9.5].)

**Definition 4.15.** A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with past/present map $\mathfrak{B}_\Sigma$ and present/future map $\mathfrak{C}_\Sigma$ is *passive balanced* if $\mathfrak{B}_\Sigma \mathfrak{B}_\Sigma^* = \mathfrak{C}_\Sigma^* \mathfrak{C}_\Sigma$.

**Lemma 4.16.** *A passive s/s system $\Sigma$ is balanced if and only if its adjoint $\Sigma_*$ is balanced, in which case*

$$\mathfrak{B}_{\Sigma_*} \mathfrak{B}_{\Sigma_*}^* = \mathfrak{B}_\Sigma \mathfrak{B}_\Sigma^*, \qquad \mathfrak{C}_{\Sigma_*}^* \mathfrak{C}_{\Sigma_*} = \mathfrak{C}_\Sigma^* \mathfrak{C}_\Sigma,$$

*Proof.* This follows from Definition 4.15 and Lemma 3.40. □

**Lemma 4.17.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive balanced s/s system with two-sided behaviour $\mathfrak{W}$, past/present map $\mathfrak{B}_\Sigma$, present/future map $\mathfrak{C}_\Sigma$, reachable subspace $\mathfrak{R}$, and unobservable subspace $\mathfrak{U}$. Then the following claims hold:*

(i) *$\mathcal{X} = \mathfrak{R} \oplus \mathcal{U}$, and consequently, $\Sigma$ is minimal if and only if it is controllable, or equivalently, if and only if it is observable.*

(ii) *The restriction of $\Sigma$ onto $\mathfrak{R}$ is a minimal passive balanced realization of $\mathfrak{W}$. This restriction coincides with the orthogonal projection of $\Sigma$ onto $\mathfrak{U}^\perp$.*

(iii) *$\mathfrak{B}_\Sigma^* \mathfrak{B}_\Sigma = (\Gamma_\mathfrak{W}^* \Gamma_\mathfrak{W})^{1/2}$ and $\mathfrak{C}_\Sigma \mathfrak{C}_\Sigma^* = (\Gamma_\mathfrak{W} \Gamma_\mathfrak{W}^*)^{1/2}$.*

(iv) *If $\Sigma$ is minimal, then $\Sigma$ is uniquely determined by $\mathfrak{W}$ up to unitary similarity. More precisely, if $\Sigma_1$ and $\Sigma_2$ are two minimal balanced externally equivalent s/s systems, then $\Sigma_1$ and $\Sigma_2$ are unitarily similar with similarity operator $\mathfrak{C}_{\Sigma_2}^{-1} \mathfrak{C}_{\Sigma_1}$.*

*Proof of (i).* It follows from Definition 4.15 that

$$\mathfrak{U} = \ker(\mathfrak{C}_\Sigma) = \ker(\mathfrak{B}_\Sigma^*) = \operatorname{im}(\mathfrak{B}_\Sigma)^\perp.$$

Thus, $\mathfrak{R} = \overline{\operatorname{im}(\mathfrak{B}_\Sigma)} = \mathfrak{U}^\perp$, and so $\mathcal{X} = \mathfrak{R} \oplus \mathfrak{U}$.

*Proof of (ii).* Denote the restriction of $\Sigma$ to $\mathfrak{R}$ by $\Sigma_{\mathfrak{R}}$. By (i), $\Sigma_{\mathfrak{R}}$ is minimal. Moreover, $\mathfrak{B}_{\Sigma_{\mathfrak{R}}}$ is equal to $\mathfrak{B}_\Sigma$ interpreted as an operator with values in $\mathfrak{R}$, $\mathfrak{B}^*_{\Sigma_{\mathfrak{R}}} = \mathfrak{B}^*_\Sigma|_{\mathfrak{R}}$, $\mathfrak{C}_{\Sigma_{\mathfrak{R}}} = \mathfrak{C}_\Sigma|_{\mathfrak{R}}$, and $\mathfrak{C}^*_{\Sigma_{\mathfrak{R}}}$ is equal to $\mathfrak{C}_\Sigma$ interpreted as an operator with values in $\overline{\mathrm{im}\,(\mathfrak{C}^*_\Sigma)} = \mathfrak{R}$. Thus,

$$\mathfrak{B}_{\Sigma_{\mathfrak{R}}}\mathfrak{B}^*_{\Sigma_{\mathfrak{R}}} = \mathfrak{B}_\Sigma\mathfrak{B}^*_\Sigma|_{\mathfrak{R}} = \mathfrak{C}^*_\Sigma\mathfrak{C}_\Sigma|_{\mathfrak{R}} = \mathfrak{C}^*_{\Sigma_{\mathfrak{R}}}\mathfrak{C}_{\Sigma_{\mathfrak{R}}},$$

This proves that $\Sigma_{\mathfrak{R}}$ is balanced passive.

*Proof of (iii).* We have

$$\Gamma^*_{\mathfrak{W}}\Gamma_{\mathfrak{W}} = \mathfrak{B}^*_\Sigma\mathfrak{C}^*_\Sigma\mathfrak{C}_\Sigma\mathfrak{B}_\Sigma = \mathfrak{B}^*_\Sigma\mathfrak{B}_\Sigma\mathfrak{B}^*_\Sigma\mathfrak{B}_\Sigma = (\mathfrak{B}^*_\Sigma\mathfrak{B}_\Sigma)^2.$$

Since $\mathfrak{B}^*_\Sigma\mathfrak{B}_\Sigma$ is nonnegative, this implies that $\mathfrak{B}^*_\Sigma\mathfrak{B}_\Sigma = (\Gamma^*_{\mathfrak{W}}\Gamma_{\mathfrak{W}})^{1/2}$. An analogous computation shows that $\mathfrak{C}_\Sigma\mathfrak{C}^*_\Sigma = (\Gamma_{\mathfrak{W}}\Gamma^*_{\mathfrak{W}})^{1/2}$.

*Proof of (iv).* This follows from (iii) and Theorem 3.44 □

The main question that still remains to be answered concerns the existence of a minimal balanced s/s realization of a given passive two-sided behaviour. In order to prepare for a positive answer to this question we first map the canonical $*$-optimal model $\Sigma^{\mathfrak{W}}_{\mathrm{m*o}}$ with state space $(\ker\,(\Gamma_{\mathfrak{W}}))^\perp$ onto another canonical $*$-optimal model whose state space is equal to $\mathrm{im}\,(\Gamma_{\mathfrak{W}})$ with the range norm of $\Gamma_{\mathfrak{W}}$.

**Lemma 4.18.** *Let $\mathfrak{W}$ be a passive two-sided behaviour on $\mathcal{W}$ with past/future map $\Gamma_{\mathfrak{W}}$, and let $\Sigma^{\mathfrak{W}}_{\mathrm{mo}} = (V^{\mathfrak{W}}_{\mathrm{mo}}; \overline{\mathrm{im}\,(\Gamma_{\mathfrak{W}})}, \mathcal{W})$ and $\Sigma^{\mathfrak{W}}_{\mathrm{m*o}} = (V^{\mathfrak{W}}_{\mathrm{m*o}}; (\ker\,(\Gamma_{\mathfrak{W}}))^\perp, \mathcal{W})$ be the canonical models of a minimal optimal and minimal $*$-optimal s/s system with two-sided behaviour $\mathfrak{W}$. Let $\mathcal{X}_\circ := \overline{\mathrm{im}\,(\Gamma_{\mathfrak{W}})}$ with the norm inherited from $\mathcal{H}_+$, denote $V_\circ = V^{\mathfrak{W}}_{\mathrm{mo}}$, and denote $|\Gamma^*_{\mathfrak{W}}| := (\Gamma_{\mathfrak{W}}\Gamma^*_{\mathfrak{W}})^{1/2}$.*

(i) *Denote $\mathcal{X}_\bullet := \mathrm{im}\,(\Gamma_{\mathfrak{W}})$, and equip $\mathcal{X}_\bullet$ with the range norm*

$$\|\Gamma_{\mathfrak{W}}y\|_{\mathcal{X}_\bullet} = \|y\|_{\mathcal{H}_-}, \qquad y \in (\ker\,(\Gamma_{\mathfrak{W}}))^\perp.$$

*Then $\mathcal{X}_\bullet$ is a Hilbert space which is contractively and densely embedded in $\mathcal{X}_\circ$, and $\Gamma_{\mathfrak{W}|(\ker(\Gamma_{\mathfrak{W}}))^\perp}$, regarded as an operator with values in $\mathcal{X}_\bullet$, is a unitary map from $(\ker\,(\Gamma_{\mathfrak{W}}))^\perp$ onto $\mathcal{X}_\bullet$. The adjoint of the embedding map $\mathcal{X}_\bullet \hookrightarrow \mathcal{X}_\circ$ is the restriction to $\mathcal{X}_\circ$ of the operator $|\Gamma^*_{\mathfrak{W}}|^2 = \Gamma_{\mathfrak{W}}\Gamma^*_{\mathfrak{W}}$.*

(ii) *Define*

$$V_\bullet = \begin{bmatrix} |\Gamma^*_{\mathfrak{W}}| & 0 & 0 \\ 0 & |\Gamma^*_{\mathfrak{W}}| & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_\circ.$$

*Then $\Sigma_\bullet = (V_\bullet; \mathcal{X}_\bullet; \mathcal{W})$ is a minimal $*$-optimal realization of $\mathfrak{W}$, which is unitarily similar to $\Sigma^{\mathfrak{W}}_{\mathrm{m*o}}$ with similarity operator $(\Gamma_{\mathfrak{W}|(\ker(\Gamma_{\mathfrak{W}}))^\perp})^{-1}$.*

(iii) *The past/present map* $\mathfrak{B}_{\Sigma_\bullet}$ *of* $\Sigma_\bullet$ *is equal to* $\Gamma_{\mathfrak{W}}$*, regarded as an operator with values in* $\mathcal{X}_\bullet$*, and the present/future map* $\mathfrak{C}_{\Sigma_\bullet}$ *of* $\Sigma_\bullet$ *is the embedding operator* $\mathcal{X}_\bullet \hookrightarrow \mathcal{H}_+$*. The adjoint of* $\mathfrak{B}_{\Sigma_\bullet}$ *is* $\mathfrak{B}^*_{\Sigma_\bullet} = (\Gamma_{\mathfrak{W}|(\ker(\Gamma_{\mathfrak{W}}))^\perp})^{-1}$*, and the adjoint of* $\mathfrak{C}_{\Sigma_\bullet}$ *is* $\mathfrak{C}^*_{\Sigma_\bullet} = \Gamma_{\mathfrak{W}}\Gamma^*_{\mathfrak{W}}$*, regarded as an operator with values in* $\mathcal{X}_\bullet$*.*

*Proof of (i).* It is easy to see that $\Gamma_{\mathfrak{W}|(\ker(\Gamma_{\mathfrak{W}}))^\perp}$, regarded as an operator with values in $\mathcal{X}_\bullet$, is a unitary map from $(\ker(\Gamma_{\mathfrak{W}}))^\perp$ onto $\mathcal{X}_\bullet$, and hence $\mathcal{X}_\bullet$ is a Hilbert space. The embedding is dense since $\operatorname{im}(\Gamma_{\mathfrak{W}})$ is dense in $\overline{\operatorname{im}(\Gamma_{\mathfrak{W}})}$. Since $\operatorname{im}(\Gamma_{\mathfrak{W}}) = \operatorname{im}(|\Gamma^*_{\mathfrak{W}}|)$, we can also interpret $\mathcal{X}_\bullet$ as the range space of $|\Gamma^*_{\mathfrak{W}}|$, and $|\Gamma^*_{\mathfrak{W}}|$, interpreted as an operator with values in $\mathcal{X}_\bullet$, is a unitary map of $\mathcal{X}_\circ$ onto $\mathcal{X}_\bullet$.

To compute the adjoint of the embedding $\mathcal{X}_\bullet \hookrightarrow \mathcal{X}_\circ$ we let $x_\bullet \in \mathcal{X}_\bullet$ and $y_\circ \in \mathcal{X}_\circ$, and compute

$$(|\Gamma^*_{\mathfrak{W}}|x_\bullet, y_\circ)_{\mathcal{X}_\circ} = (x_\bullet, |\Gamma^*_{\mathfrak{W}}|y_\circ)_{\mathcal{X}_\circ} = (|\Gamma^*_{\mathfrak{W}}|x_\bullet, |\Gamma^*_{\mathfrak{W}}|^2 y_\circ)_{\mathcal{X}_\bullet}.$$

Since $\operatorname{im}(|\Gamma^*_{\mathfrak{W}}|)_{|\mathcal{X}_\bullet}$ is dense in $\mathcal{X}_\bullet$, we find that for all $x_\bullet \in \mathcal{X}_\bullet$ and $y_\circ \in \mathcal{X}_\circ$,

$$(x_\bullet, y_\circ)_{\mathcal{X}_\circ} = (x_\bullet, |\Gamma^*_{\mathfrak{W}}|^2 y_\circ)_{\mathcal{X}_\bullet}.$$

This proves that the adjoint of the embedding $\mathcal{X}_\bullet \hookrightarrow \mathcal{X}_\circ$ is equal to $|\Gamma^*_{\mathfrak{W}}|^2_{|\mathcal{X}_\circ}$. The embedding is contractive since $|\Gamma^*_{\mathfrak{W}}|^2$ is contractive.

*Proof of (ii)–(iii).* This follows from (i) and Theorem 4.10. □

**Theorem 4.19.** *Introduce the same notations as in Lemma 4.18.*

(i) *Denote* $\mathcal{X}_\odot := \operatorname{im}\left(|\Gamma^*_{\mathfrak{W}}|^{1/2}\right) = \operatorname{im}\left((\Gamma_{\mathfrak{W}}\Gamma^*_{\mathfrak{W}})^{1/2}\right)$*, and equip* $\mathcal{X}_\odot$ *with the range norm*

$$\||\Gamma^*_{\mathfrak{W}}|^{1/2}y\|_{\mathcal{X}_\circ} = \|y\|_{\mathcal{H}_-}, \qquad y \in (\ker(\Gamma_{\mathfrak{W}}))^\perp,$$

*Then* $\mathcal{X}_\odot$ *is a Hilbert space,* $\mathcal{X}_\bullet$ *is contractively and densely embedded in* $\mathcal{X}_\odot$*, and* $\mathcal{X}_\odot$ *is contractively and densely embedded in* $\mathcal{X}_\circ$*. The restriction of* $|\Gamma^*_{\mathfrak{W}}|^{1/2}$ *to* $\mathcal{X}_\circ$ *is a unitary map of* $\mathcal{X}_\circ$ *onto* $\mathcal{X}_\odot$*, and the restriction of* $|\Gamma^*_{\mathfrak{W}}|^{1/2}$ *to* $\mathcal{X}_\odot$ *is a unitary map of* $\mathcal{X}_\odot$ *onto* $\mathcal{X}_\bullet$*. The adjoint of the embedding map* $\mathcal{X}_\bullet \hookrightarrow \mathcal{X}_\odot$ *is the restriction to* $\mathcal{X}_\odot$ *of* $|\Gamma^*_{\mathfrak{W}}|$*, and the adjoint of the embedding map* $\mathcal{X}_\odot \hookrightarrow \mathcal{X}_\circ$ *is the restriction to* $\mathcal{X}_\circ$ *of* $|\Gamma^*_{\mathfrak{W}}|$.

(ii) *Define*

$$V_\odot = \begin{bmatrix} |\Gamma^*_{\mathfrak{W}}|^{1/2} & 0 & 0 \\ 0 & |\Gamma^*_{\mathfrak{W}}|^{1/2} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_\circ. \tag{4.4}$$

*Then* $\Sigma_\odot = (V_\odot; \mathcal{X}_\odot; \mathcal{W})$ *is a minimal balanced realization of* $\mathfrak{W}$*.*

(iii) *The past/present map* $\mathfrak{B}_{\Sigma_\odot}$ *of* $\Sigma_\odot$ *is equal to* $\Gamma_{\mathfrak{W}}$*, regarded as an operator with values in* $\mathcal{X}_\odot$*, and the present/future map* $\mathfrak{C}_{\Sigma_\odot}$ *of* $\Sigma_\odot$ *is the embedding operator* $\mathcal{X}_\odot \hookrightarrow \mathcal{H}_+$*. The adjoint of* $\mathfrak{B}_{\Sigma_\odot}$ *is* $\mathfrak{B}^*_{\Sigma_\odot} = (\Gamma_{\mathfrak{W}|(\ker(\Gamma_{\mathfrak{W}}))^\perp})^{-1}|\Gamma^*_{\mathfrak{W}}|_{|\mathcal{X}_\odot}$ *and the adjoint of* $\mathfrak{C}^*_{\Sigma_\odot}$ *is equal to* $|\Gamma^*_{\mathfrak{W}}|$*, regarded as an operator with values in* $\mathcal{X}_\odot$*.*

*Proof of (i).* The proof of part (i) is analogous to the proof of part (i) of Lemma 4.18.

*Proof of (ii).* Let $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ be a fundamental decomposition of the signal space $\mathcal{W}$. If we denote the node spaces of $\Sigma_\circ$, $\Sigma_\odot$, and $\Sigma_\bullet$ by $\mathfrak{K}_\circ$, $\mathfrak{K}_\odot$, and $\mathfrak{K}_\bullet$, respectively, then we get the three fundamental decompositions

$$\mathfrak{K}_\circ = \mathfrak{K}_{\circ+} \boxplus -\mathfrak{K}_{\circ-}, \quad \mathfrak{K}_\odot = \mathfrak{K}_{\odot+} \boxplus -\mathfrak{K}_{\odot-}, \quad \mathfrak{K}_\bullet = \mathfrak{K}_{\bullet+} \boxplus -\mathfrak{K}_{\bullet-},$$

where

$$\mathfrak{K}_{\circ+} = \left\{ \left[\begin{smallmatrix} -x \\ x \\ w_+ \end{smallmatrix}\right] \middle| x \in \mathcal{X}_\circ,\ w_+ \in \mathcal{U} \right\}, \quad \mathfrak{K}_{\circ-} = \left\{ \left[\begin{smallmatrix} x \\ x \\ w_- \end{smallmatrix}\right] \middle| x \in \mathcal{X}_\circ,\ w_- \in \mathcal{Y} \right\},$$

and $\mathfrak{K}_{\odot\pm}$ and $\mathfrak{K}_{\bullet\pm}$ are defined analogously. We know that $V_\circ$ is maximal nonnegative in $\mathfrak{K}_\circ$, and that $V_\bullet$ is maximal nonnetagive in $\mathfrak{K}_\bullet$, since $\Sigma_\circ$ and $\Sigma_\bullet$ are passive. By Proposition 2.1(i), this implies that $V_\circ$ and $V_\bullet$ have graph representations over $\mathfrak{K}_{\circ+}$ and $\mathfrak{K}_{\bullet+}$ with contractive angle operators $A_{\circ+}$ and $A_{\bullet+}$, respectively. Since $V_\bullet \subset V_\circ$, we find that $A_{\bullet+}$ is the restriction of $A_{\circ+}$ to $\mathfrak{K}_{\bullet+}$, and it follows from the definitions of $V_\bullet$ and $V_\odot$ that $V_\odot$ is the graph of the operator $A_{\odot+}$ that one gets by interpolating between $A_{\circ+}$ and $A_{\bullet+}$ in the sense of [AS05a, Lemma 3.2]. By that lemma, $A_{\odot+}$ is a contraction, and consequently, by Proposition 2.1, $V_\odot$ is maximal nonnegative in $\mathfrak{K}_\odot$. Since $V_\odot \subset V_\circ$, it is clear that $V_\odot$ inherits property (1.1) from $V_\circ$. Consequently, $V_\odot$ generates a passive s/s system.

The inclusions $V_\bullet \subset V_\odot \subset V_\circ$ implies that every classical trajectory of $\Sigma_\bullet$ is also a classical trajectory of $\Sigma_\odot$, and that every classical trajectory of $\Sigma_\odot$ is also a classical trajectory of $\Sigma_\circ$. These two inclusions of classical trajectories imply the corresponding inclusions for generalised trajectories. Since $\Sigma_\bullet$ and $\Sigma_\circ$ have the same behaviour $\mathfrak{W}$, also the behaviour of $\Sigma_\odot$ must coincide with $\mathfrak{W}$. Thus, these three systems are externally equivalent. Since $\Sigma_\bullet$ is controllable and $\mathcal{X}_\bullet$ is dense in $\mathcal{X}_\odot$ the system $\Sigma_\odot$ is controllable, and since $\Sigma_\circ$ is observable also $\Sigma_\odot$ is observable.

That $\Sigma_\odot$ is balanced follows from (iii), which will be proved next.

*Proof of (iii).* By the same argument which we used above to prove (ii) we find that $\mathfrak{B}_{\Sigma_\odot}$ is equal to $\mathfrak{B}_{\Sigma_\bullet}$ composed with the embedding operator $\mathcal{X}_\bullet \hookrightarrow \mathcal{X}_\odot$, and that $\mathfrak{C}_{\Sigma_\odot} = \mathfrak{C}_{\Sigma_\circ|\mathcal{X}_\odot}$. This combined with (i) and Lemmas 4.13 and 4.18 leads to the characterisations of $\mathfrak{B}_{\Sigma_\odot}$ and $\mathfrak{C}_{\Sigma_\odot}$ given in the statement of the theorem. We further conclude that $\mathfrak{B}^*_{\Sigma_\odot}$ is equal to the adjoint of the embedding operator $\mathcal{X}_\bullet \hookrightarrow \mathcal{X}_\odot$ composed with $\mathfrak{B}^*_{\Sigma_\bullet}$ and that $\mathfrak{C}^*_{\Sigma_\odot}$ is equal to $\mathfrak{C}^*_{\Sigma_\circ}$ composed with the adjoint of the embedding operator $\mathcal{X}_\odot \hookrightarrow \mathcal{X}_\circ$. This combined with (i) and Lemmas 4.13 and 4.18 leads to the characterisations of $\mathfrak{B}^*_{\Sigma_\odot}$ and $\mathfrak{C}^*_{\Sigma_\odot}$ given in the statement of the theorem. $\square$

# 5 Passive Real State/Signal Systems and Behaviours

We are now ready to turn to the main subject of this paper, namely four different types of symmetries that a passive s/s system may possess. In this chapter we deal with *real symmetry*, and in the next three chapters we shall discuss *reciprocal symmetry*, *signature invariance*, and *transpose invariance*.

We begin by discussion conjugate-linear unitary similarity between two passive s/s systems.

**Lemma 5.1.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, let $\mathcal{X}_1$ and $\mathcal{W}_1$ be a Hilbert and a Kreĭn space, respectively, and let $\mathcal{V}_\mathcal{X}$ and $\mathcal{V}_\mathcal{W}$ be two conjugate-linear unitary operators in $\overline{\mathcal{B}}(\mathcal{X}; \mathcal{X}_1)$ and $\overline{\mathcal{B}}(\mathcal{W}; \mathcal{W}_1)$, respectively. Define $V_1$ by*

$$V_1 = \begin{bmatrix} \mathcal{V}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{V}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{V}_\mathcal{W} \end{bmatrix} V. \tag{5.1}$$

*Then the following statements are true.*

(i) *$\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1)$ is a passive s/s system.*

(ii) *$\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory of $\Sigma$ on some interval $I$ if and only if $\left[\begin{smallmatrix} \mathcal{V}_\mathcal{X} x \\ \mathcal{V}_\mathcal{W} w \end{smallmatrix}\right]$ is a classical trajectory of $\Sigma_1$ on $I$*

(iii) *$\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is generalised trajectory of $\Sigma$ on some interval $I$ if and only if $\left[\begin{smallmatrix} \mathcal{V}_\mathcal{X} x \\ \mathcal{V}_\mathcal{W} w \end{smallmatrix}\right]$ is a generalised trajectory of $\Sigma_1$ on $I$.*

(iv) *If we denote the past, two-sided, and future behaviours of $\Sigma$ by $\mathfrak{W}_-$, $\mathfrak{W}$, and $\mathfrak{W}_+$, respectively, then the corresponding behaviours of $\Sigma_1$ are equal to $\mathcal{V}_\mathcal{W}\mathfrak{W}_-$, $\mathcal{V}_\mathcal{W}\mathfrak{W}$, and $\mathcal{V}_\mathcal{W}\mathfrak{W}_+$, respectively.*

*Proof.* That $V_1$ is maximal nonnegative follows from the maximal nonnegativity of $V$ together with the fact that the conjugate-linear operator $\left[\begin{smallmatrix} \mathcal{V}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{V}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{V}_\mathcal{W} \end{smallmatrix}\right]$ is a unitary map from the node space of $\Sigma$ onto the node space of $\Sigma_1$. That (ii) holds follows immediately from (5.1), and (iii) follows from (ii). Finally, (iv) follows from (iii). □

We shall be especially interested in the case where the two systems $\Sigma$ and $\Sigma_1$ in Lemma 5.1 coincide and the operators $\mathcal{V}_\mathcal{X}$ and $\mathcal{V}_\mathcal{W}$ are conjugations (i.e., conjugate-linear unitary involutions).

**Definition 5.2.** A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called $(\mathcal{C}_\mathcal{X}; \mathcal{C}_\mathcal{W})$-real if (1.15) holds, where $\mathcal{C}_\mathcal{X}$ and $\mathcal{C}_\mathcal{W}$ are conjugations in $\mathcal{X}$ and $\mathcal{W}$, respectively.

Instead of using the characterisation in Definition 5.2p for reality of a system we can also use the following alternative characterisations.

**Lemma 5.3.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, let $\mathcal{C}_{\mathcal{X}}$ and $\mathcal{C}_{\mathcal{W}}$ be conjugations in $\mathcal{X}$ and $\mathcal{W}$, respectively, and let $I \subset \mathbb{R}$ be a nontrivial interval with finite left end-point. Then the following conditions are equivalent:*

(i) *$\Sigma$ is $(\mathcal{C}_{\mathcal{X}}; \mathcal{C}_{\mathcal{W}})$-real;*

(ii) *$\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory of $\Sigma$ on $I$ if and only if $\left[\begin{smallmatrix} \mathcal{C}_{\mathcal{X}} x \\ \mathcal{C}_{\mathcal{W}} w \end{smallmatrix}\right]$ is a classical trajectory of $\Sigma$ on $I$.*

(iii) *$\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is generalised trajectory of $\Sigma$ on $I$ if and only if $\left[\begin{smallmatrix} \mathcal{C}_{\mathcal{X}} x \\ \mathcal{C}_{\mathcal{W}} w \end{smallmatrix}\right]$ is a generalised trajectory of $\Sigma$ on $I$.*

*Proof.* That (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is seen as in the proof of Lemma 5.1.

Conversely, since a generalised trajectory is classical if and only if it has the necessary smoothness (see Lemma 3.4), (iii) $\Rightarrow$ (ii). Finally, (ii) $\Rightarrow$ (i) since the generating subspace is the set of all initial values of $\left[\begin{smallmatrix} \dot{x} \\ x \\ w \end{smallmatrix}\right]$ at the left end-point of $I$ for the set of all classical trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ at the interval $I$. $\square$

**Lemma 5.4.** *If $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a simple passive s/s $(\mathcal{C}_{\mathcal{X}}; \mathcal{C}_{\mathcal{W}})$-real system, then $\mathcal{C}_{\mathcal{X}}$ is uniquely determined by $\Sigma$ and $\mathcal{C}_{\mathcal{W}}$.*

*Proof.* This follows from Lemma 3.50. $\square$

**Lemma 5.5.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive $(\mathcal{C}_{\mathcal{X}}; \mathcal{C}_{\mathcal{W}})$-real system, and let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be unitarily similar to $\Sigma$ with similarity operator $\mathcal{V}$. Then $\Sigma_1$ is $(\mathcal{C}_{\mathcal{X}_1}; \mathcal{C}_{\mathcal{W}})$-real with $\mathcal{C}_{\mathcal{X}_1} = \mathcal{V}\mathcal{C}_{\mathcal{X}}\mathcal{V}^{-1}$.*

*Proof.* This follows from the fact that

$$\begin{aligned}\begin{bmatrix} \mathcal{C}_{\mathcal{X}_1} & 0 & 0 \\ 0 & \mathcal{C}_{\mathcal{X}_1} & 0 \\ 0 & 0 & \mathcal{C}_{\mathcal{W}} \end{bmatrix} V_1 &= \begin{bmatrix} \mathcal{V}\mathcal{C}_{\mathcal{X}}\mathcal{V}^{-1} & 0 & 0 \\ 0 & \mathcal{V}\mathcal{C}_{\mathcal{X}}\mathcal{V}^{-1} & 0 \\ 0 & 0 & \mathcal{C}_{\mathcal{W}} \end{bmatrix} V_1 \\ &= \begin{bmatrix} \mathcal{V}\mathcal{C}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{V}\mathcal{C}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{C}_{\mathcal{W}} \end{bmatrix} V = \begin{bmatrix} \mathcal{V} & 0 & 0 \\ 0 & \mathcal{V} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V = V_1. \quad \square\end{aligned}$$

**Lemma 5.6.** *A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is $(\mathcal{C}_{\mathcal{X}}; \mathcal{C}_{\mathcal{W}})$-real if and only if the adjoint system $\Sigma_* = (V_*; \mathcal{X}, -\mathcal{W})$ is $(\mathcal{C}_{\mathcal{X}}; \mathcal{C}_{-\mathcal{W}})$-real, where $\mathcal{C}_{-\mathcal{W}} = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\mathcal{C}_{\mathcal{W}}\mathcal{I}_{(-\mathcal{W},\mathcal{W})}$.*

*Proof.* This follows from Lemma 2.11 and Definitions 3.20 and 5.2. $\square$

**Lemma 5.7.** *If the passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real, then the reachable subspace $\mathfrak{R}_\Sigma$, the unobservable subspace $\mathfrak{U}_\Sigma$ and their orthogonal complements are invariant under $\mathcal{C}_\mathcal{X}$, i.e.,*

$$\mathfrak{R}_\Sigma = \mathcal{C}_\mathcal{X}\mathfrak{R}_\Sigma, \quad \mathfrak{U}_\Sigma = \mathcal{C}_\mathcal{X}\mathfrak{U}_\Sigma, \quad \mathfrak{R}_\Sigma^\perp = \mathcal{C}_\mathcal{X}\mathfrak{R}_\Sigma^\perp, \quad \mathfrak{R}_\Sigma^\perp = \mathcal{C}_\mathcal{X}\mathfrak{R}_\Sigma^\perp. \tag{5.2}$$

*Thus, the restriction of $\mathcal{C}_\mathcal{X}$ to each of these subspaces is a conjugation in the corresponding subspace.*

*Proof.* By Lemma 5.3, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable past trajectory of $\Sigma$ if and only if $\left[\begin{smallmatrix} \mathcal{C}_\mathcal{X} x \\ \mathcal{C}_\mathcal{W} w \end{smallmatrix}\right]$ is an externally generated stable past trajectory of $\Sigma$. This implies that the image of $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]}) = \{Q_- w \mid w \in \mathcal{Y}\}$ under $\mathfrak{B}_\Sigma$ is invariant under $\mathcal{C}_\mathcal{X}$. The reachable subspace $\mathfrak{R}$ is the closure of this image in $\mathcal{X}$, and consequently $\mathfrak{R}$ is invariant under $\mathcal{C}_\mathcal{X}$.

That $\mathfrak{U}$ is invariant under $\mathcal{C}_\mathcal{X}$ follows immediately from Lemma 5.3. Finally, the invariance of $\mathfrak{R}^\perp$ and $\mathfrak{U}^\perp$ follows from Lemma 2.11. □

**Lemma 5.8.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real s/s system.*

(i) *The restriction $\Sigma_\mathfrak{R} = (V_\mathfrak{R}; \mathfrak{R}, \mathcal{W})$ of $\Sigma$ onto its reachable subspace $\mathfrak{R}$ is $(\mathcal{C}_\mathfrak{R}, \mathcal{C}_\mathcal{W})$-real, where $\mathcal{C}_\mathfrak{R} = \mathcal{C}_\mathcal{X}|_\mathfrak{R}$.*

(ii) *The orthogonal projection $\Sigma_\mathfrak{U}^\perp = (V_{\mathfrak{U}^\perp}; \mathfrak{U}^\perp, \mathcal{W})$ of $\Sigma$ onto the orthogonal complement to its unobservable subspace $\mathfrak{U}$ is $(\mathcal{C}_{\mathfrak{U}^\perp}, \mathcal{C}_\mathcal{W})$-real, where $\mathcal{C}_{\mathfrak{U}^\perp} = \mathcal{C}_{\mathcal{X}|\mathfrak{U}^\perp}$.*

*Proof.* This follows from Lemma 5.7 and formulas (3.11) and (3.12). □

**Definition 5.9.** Let $\mathcal{C}_\mathcal{W}$ be a conjugation in the Kreĭn space $\mathcal{W}$.

(i) A passive two-sided behaviour $\mathfrak{W}$ on $\mathcal{W}$ is called $\mathcal{C}_\mathcal{W}$-real if

$$\mathfrak{W} = \mathcal{C}_\mathcal{W}\mathfrak{W} \tag{5.3}$$

(here the conjugation $\mathcal{C}_\mathcal{W}$ on $K^2(\mathcal{W})$ induced by the conjugation $\mathcal{C}_\mathcal{W} \in \mathcal{B}(\mathcal{W})$ is defined as in Remark 1.1).

(ii) A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called *externally $\mathcal{C}_\mathcal{W}$-real* if its two-sided behaviour is $\mathcal{C}_\mathcal{W}$-real.

It follows from (3.8)–(3.9) that the equality (5.3) is equivalent to each of the following equalities:

$$\mathfrak{W}_+ = \mathcal{C}_\mathcal{W}\mathfrak{W}_+, \qquad \mathfrak{W}_- = \mathcal{C}_\mathcal{W}\mathfrak{W}_-, \tag{5.4}$$

where $\mathfrak{W}_+$ and $\mathfrak{W}_-$ are the passive future and past behaviours on $\mathcal{W}$ defined in terms of $\mathfrak{W}$ by (3.8). Moreover, (5.3) and (5.4) are equivalent to the corresponding relations

$$\mathfrak{W}^{[\perp]} = \mathcal{C}_{\mathcal{W}}\mathfrak{W}^{[\perp]}, \qquad \mathfrak{W}_{\pm}^{[\perp]} = \mathcal{C}_{\mathcal{W}}\mathfrak{W}_{\pm}^{[\perp]} \tag{5.5}$$

for the orthogonal complements.

**Lemma 5.10.** *If $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a passive $(\mathcal{C}_{\mathcal{X}}, \mathcal{C}_{\mathcal{W}})$-real s/s system, then it is externally $\mathcal{C}_{\mathcal{W}}$-real.*

*Proof.* Let $\mathfrak{W}_+$ be the passive future behaviour of $\Sigma$, and let $w_+ \in \mathfrak{W}_+$. Then there exists a unique stable externally generated future trajectory $\left[\begin{smallmatrix} x_+ \\ w_+ \end{smallmatrix}\right]$ of $\Sigma$ (with signal part $w_+$). By Lemma 5.3, this implies that $\left[\begin{smallmatrix} \mathcal{C}_{\mathcal{X}} x_+ \\ \mathcal{C}_{\mathcal{W}} w_+ \end{smallmatrix}\right]$ is an externally generated stable future trajectory of $\Sigma$. Consequently, $\mathcal{C}_{\mathcal{W}} w_+ \in \mathfrak{W}_+$. This proves that $\mathcal{C}_{\mathcal{W}}\mathfrak{W}_+ \subset \mathfrak{W}_+$. By applying $\mathcal{C}_{\mathcal{W}}$ to both sides of this inclusion and taking into account that $\mathcal{C}_{\mathcal{W}}^2 = 1_{\mathcal{W}}$, we find that $\mathcal{C}_{\mathcal{W}}\mathfrak{W}_+ \subset \mathfrak{W}_+$. Thus $\mathcal{C}_{\mathcal{W}}\mathfrak{W}_+ = \mathfrak{W}_+$, and by the comment after Definition 5.9, $\Sigma$ is externally $\mathcal{C}_{\mathcal{W}}$-real. □

**Lemma 5.11.** *A passive two-sided behaviour $\mathfrak{W}$ on the Kreĭn space $\mathcal{W}$ is $\mathcal{C}_{\mathcal{W}}$-real if and only if the adjoint behaviour $\mathfrak{W}_*$ is $\mathcal{C}_{-\mathcal{W}}$-real, where $\mathcal{C}_{-\mathcal{W}} = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\mathcal{C}_{\mathcal{W}}\mathcal{I}_{(-\mathcal{W},\mathcal{W})}$. In particular, a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is externally $\mathcal{C}_{\mathcal{W}}$-real if and only if the adjoint system $\Sigma_*$ is externally $\mathcal{C}_{-\mathcal{W}}$-real.*

*Proof.* This follows from Lemma 2.11 and 3.25 and Definitions 3.24 and 5.9. □

**Lemma 5.12.** *Let $\mathfrak{W}$ be a $\mathcal{C}_{\mathcal{W}}$-real passive two-sided behaviour on $\mathcal{W}$, with the corresponding past and future behaviours $\mathfrak{W}_-$ and $\mathfrak{W}_+$.*

(i) *$w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ if and only if $\mathcal{C}_{\mathcal{W}} w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$. In this case, $\|Q_- \mathcal{C}_{\mathcal{W}} w_-\|_{\mathcal{H}_-} = \|Q_- w_-\|_{\mathcal{H}_-}$.*

(ii) *$w_+ \in \mathcal{K}(\mathfrak{W}_+)$ if and only if $\mathcal{C}_{\mathcal{W}} w_+ \in \mathcal{K}(\mathfrak{W}_+)$. In this case $\|Q_+ \mathcal{C}_{\mathcal{W}} w_+\|_{\mathcal{H}_+} = \|Q_+ w_+\|_{\mathcal{H}_+}$.*

(iii) *$w \in \mathcal{L}(\mathfrak{W})$ if and only if $\mathcal{C}_{\mathcal{W}} w \in \mathcal{L}(\mathfrak{W})$. In this case $\|Q\mathcal{C}_{\mathcal{W}} w\|_{\mathcal{D}(\mathfrak{W})} = \|Qw\|_{\mathcal{D}(\mathfrak{W})}$.*

*Proof of (i).* We have for each $w_- \in K^2_-(\mathcal{W})$,

$$\begin{aligned}\|Q_-\mathcal{C}_\mathcal{W} w_-\|^2_{\mathcal{H}_-} &= \sup\left\{[\mathcal{C}_\mathcal{W} w_- + z, \mathcal{C}_\mathcal{W} w_- + z]_{K^2_-(\mathcal{W})}\,\middle|\, z \in \mathfrak{W}_-^{[\perp]}\right\} \\ &= \sup\left\{[\mathcal{C}_\mathcal{W} w + \mathcal{C}_\mathcal{W} z, \mathcal{C}_\mathcal{W} w + \mathcal{C}_\mathcal{W} z]_{K^2_-(\mathcal{W})}\,\middle|\, \mathcal{C}_\mathcal{W} z \in \mathfrak{W}_-^{[\perp]}\right\} \\ &= \sup\left\{[w+z, w+z]_{K^2_-(\mathcal{W})}\,\middle|\, z \in \mathfrak{W}_-^{[\perp]}\right\} = \|Q_- w_-\|^2_{\mathcal{H}_-}.\end{aligned}$$

Thus, $\mathcal{C}_\mathcal{W} w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ if and only if $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$, and $\|Q_-\mathcal{C}_\mathcal{W} w_-\|_{\mathcal{H}_-} = \|Q_- w_-\|_{\mathcal{H}_-}$.
*Proof of (ii).* This proof is analogous to the one above.
*Proof of (iii).* Let $w \in \mathcal{L}(\mathfrak{W})$, and choose $x_m \in \mathcal{D}^0(\mathfrak{W})$ such that $x_m \to Qw$ in $\mathcal{D}(\mathfrak{W})$ as $m \to \infty$. Let $R$ be a bounded right-inverse of the quotient map $Q$, and define $w_m := w + R(x_m - Qw)$. Then $Qw_m = x_m \to Qw$ in $\mathcal{D}(\mathfrak{W})$, $w_m \in \mathcal{L}^0(\mathfrak{W})$, and $w_m \to w$ in $K^2(\mathcal{W})$ as $m \to \infty$. Each $w_m$ can be written in the form $w_m = z_m + z_m^\dagger$, where $z_m \in \mathfrak{W}$ and $z_m^\dagger \in \mathfrak{W}^{[\perp]}$, and since both $\mathfrak{W}$ and $\mathfrak{W}^{[\perp]}$ are invariant under $\mathcal{C}_\mathcal{W}$, we conclude that $\mathcal{C}_\mathcal{W} w_m \in \mathcal{L}^0(\mathfrak{W})$, and $\mathcal{C}_\mathcal{W} w_m \to \mathcal{C}_\mathcal{W} w$ in $K^2(\mathcal{W})$ as $m \to \infty$. Moreover,

$$\begin{aligned}\|x_m\|^2_{\mathcal{D}(\mathfrak{W})} = \|Qw_m\|^2_{\mathcal{D}(\mathfrak{W})} &= [\pi_- z_m, \pi_- z_m]_{K^2_-(\mathcal{W})} - [\pi_+ z_m^\dagger, \pi_+ z_m^\dagger]_{K^2_+(\mathcal{W})} \\ &= [\pi_-\mathcal{C}_\mathcal{W} z_m, \pi_-\mathcal{C}_\mathcal{W} z_m]_{K^2_-(\mathcal{W})} - [\pi_+\mathcal{C}_\mathcal{W} z_m^\dagger, \pi_+\mathcal{C}_\mathcal{W} z_m^\dagger]_{K^2_+(\mathcal{W})} \\ &= \|Q\mathcal{C}_\mathcal{W} w_m\|^2_{\mathcal{D}(\mathfrak{W})}.\end{aligned}$$

Applying the same identity to $x_m - x_n$ we find that $Q\mathcal{C}w_m$ is a Cauchy sequence in $\mathcal{D}(\mathfrak{W})$, and hence it converges to some limit, that we denote by $\mathcal{C}x$. Since the restriction of $Q$ to $\mathcal{L}(\mathfrak{W})$ is closed as an operator with values in $\mathcal{D}(\mathfrak{W})$, and since $\mathcal{C}_\mathcal{W} w_m \to \mathcal{C}_\mathcal{W} w$ in $K^2(\mathcal{W})$ as $m \to \infty$, we find that $\mathcal{C}x = Q\mathcal{C}_\mathcal{W} w$. This proves that $\mathcal{C}_\mathcal{W} w \in \mathcal{L}(\mathfrak{W})$. Letting $m \to \infty$ in the equality $\|x_m\|_{\mathcal{D}(\mathfrak{W})} = \|Q\mathcal{C}_\mathcal{W} w_m\|_{\mathcal{D}(\mathfrak{W})}$ we find that $\|x\|_{\mathcal{D}(\mathfrak{W})} = \|Q\mathcal{C}_\mathcal{W} w\|_{\mathcal{D}(\mathfrak{W})}$. □

**Lemma 5.13.** *Let $\mathfrak{W}$ be a $\mathcal{C}_\mathcal{W}$-real passive two-sided behaviour on $\mathcal{W}$, with the corresponding past and future behaviours $\mathfrak{W}_-$ and $\mathfrak{W}_+$.*

(i) *There is a unique conjugation $\mathcal{C}_{\mathcal{H}_-}$ in $\mathcal{H}_-$ such that $\mathcal{C}_{\mathcal{H}_-} Q_- w_- = Q_-\mathcal{C}_\mathcal{W} w_-$ for all $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$.*

(ii) *There is a unique conjugation $\mathcal{C}_{\mathcal{H}_+}$ in $\mathcal{H}_+$ such that $\mathcal{C}_{\mathcal{H}_+} Q_+ w_+ = Q_+\mathcal{C}_\mathcal{W} w_+$ for all $w_+ \in \mathcal{K}(\mathfrak{W}_+)$.*

(iii) *There is a unique conjugation $\mathcal{C}_{\mathcal{D}(\mathfrak{W})}$ in $\mathcal{D}(\mathfrak{W})$ such that $\mathcal{C}_{\mathcal{D}(\mathfrak{W})} Qw = Q\mathcal{C}_\mathcal{W} w$ for all $w \in \mathcal{L}(\mathfrak{W})$.*

*Proof.* By Lemma 5.12(i), $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ if and only if $\mathcal{C}_\mathcal{W} w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$, in which case $\|Q_- \mathcal{C}_\mathcal{W} w_-\|_{\mathcal{H}_-} = \|Q_- w_-\|_{\mathcal{H}_-}$. This enables us to define a unitary operator $\mathcal{C}_{\mathcal{H}_-}$ in $\mathcal{H}_-$ by the formula $\mathcal{C}_{\mathcal{H}_-} Q_- w_-$ by $\mathcal{C}_{\mathcal{H}_-} Q_- w_- = Q_- \mathcal{C}_\mathcal{W} w_-$, $w_- \in \mathcal{K}(\mathfrak{W}_-)$. This operator is conjugate-linear since $Q_-$ is linear and $\mathcal{C}_{\mathcal{H}_-}$ is conjugate-linear. Thus, $\mathcal{C}_{\mathcal{H}_-}$ is a conjugation in $\mathcal{H}_-$.

The operators $\mathcal{C}_{\mathcal{H}_+}$ and $\mathcal{C}_{\mathcal{D}(\mathfrak{W})}$ are defined analogously, and the proofs that also these two operators are conjugations are the same as the proof given above, with part (i) of Lemma 5.12 replaced by parts (ii) and (iii). □

**Theorem 5.14.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a simple conservative externally $\mathcal{C}_\mathcal{W}$-real s/s system. Then there exists a unique conjugation $\mathcal{C}_\mathcal{X}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real.*

*Proof.* Let $x_0 \in \mathcal{X}$, and choose some stable two-sided trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$ such that $x(0) = x_0$ (this is possible since $\Sigma$ is both forward and backward well-posed; see [AKS11b, Remark 4.2]). Then $w \in \mathcal{L}(\mathfrak{W})$. By Lemma 5.12(iii), also $\mathcal{C}_\mathcal{W} w \in \mathcal{L}(\mathfrak{W})$. To this trajectory corresponds a unique stable two-sided trajectory $\left[\begin{smallmatrix} \tilde{x} \\ \mathcal{C}_\mathcal{W} w \end{smallmatrix}\right]$ of $\Sigma$. Define $\mathcal{C}_\mathcal{X} x_0 := \tilde{x}(0)$. We claim that this is a well-defined operator which is a conjugation.

First of all, we need to check that $\mathcal{C}_\mathcal{X}$ is well-defined. However, this follows from the fact that it preserves norms, and this is true because the conjugation $\mathcal{C}_\mathcal{W}$ does not change the norm in $K^2(\mathcal{W})$, and

$$\|x_0\|_\mathcal{X}^2 = [w, w]_{K^2(\mathcal{W})} = [\mathcal{C}_\mathcal{W} w, \mathcal{C}_\mathcal{W} w]_{K^2(\mathcal{W})} = \|\tilde{x}(0)\|_\mathcal{X}^2.$$

Thus $\mathcal{C}_\mathcal{W}$ is isometric. It is also easy to see that $\mathcal{C}_\mathcal{W}$ is an involution, and that $\mathcal{C}_\mathcal{W}$ is conjugate-linear. Being an involution, $\mathcal{C}_\mathcal{W}$ is surjective, and hence unitary. By Lemma 2.10, $\mathcal{C}_\mathcal{W}$ is a conjugation.

By construction, if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable two-sided trajectory of $\Sigma$, then the stable two-sided trajectory $\left[\begin{smallmatrix} \tilde{x} \\ \mathcal{C}_\mathcal{W} w \end{smallmatrix}\right]$ whose signal part is $\mathcal{C}_\mathcal{W} w$ satisfies $\tilde{x}(0) = \mathcal{C}_\mathcal{X} x(0)$. The set of stable two-sided trajectories of $\Sigma$ is shift-invariant, and by applying the same argument to a shifted trajectory we find that $\left[\begin{smallmatrix} \mathcal{C}_\mathcal{X} x \\ \mathcal{C}_\mathcal{W} w \end{smallmatrix}\right]$ is a stable two-sided trajectory of $\Sigma$ if and only if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable two-sided trajectory of $\Sigma$. In particular, the same statement applies to classical stable two-sided trajectories also. Evaluating such trajectories at zero, we find that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$ if and only if $\left[\begin{smallmatrix} \mathcal{C}_\mathcal{X} z \\ \mathcal{C}_\mathcal{X} x \\ \mathcal{C}_\mathcal{W} w \end{smallmatrix}\right] \in V$. This shows that $\Sigma$ is $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real.

The uniqueness of $\mathcal{C}_\mathcal{X}$ follows from Lemma 5.4. □

**Theorem 5.15.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a observable co-energy preserving externally $\mathcal{C}_\mathcal{W}$-real s/s system. Then there exists a unique conjugation $\mathcal{C}_\mathcal{X}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real.*

*Proof.* This proof is analogous to the proof of Theorem 5.14 (with two-sided trajectories replaced by future trajectories), and it is left to the reader. (Recall that the present/future map of an observable and co-energy preserving system is unitary.) □

**Theorem 5.16.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive controllable energy preserving simple conservative externally $\mathcal{C}_{\mathcal{W}}$-real s/s system. Then there exists a unique conjugation $\mathcal{C}_{\mathcal{X}}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{C}_{\mathcal{X}}, \mathcal{C}_{\mathcal{W}})$-real.*

*Proof.* Let $x_0 \in \mathcal{X}$, and choose some stable past trajectory $\left[\begin{smallmatrix} x_- \\ w_- \end{smallmatrix}\right]$ with $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ such that $x_-(0) = x_0$; this is possible since the past/present map $\mathfrak{B}_\Sigma$ is a unitary map of $\mathcal{H}_-$ onto $\mathcal{X}$. By Lemma 5.12(i), also $\mathcal{C}_{\mathcal{W}} w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$. To this trajectory corresponds a unique stable past trajectory $\left[\begin{smallmatrix} \tilde{x}_- \\ \mathcal{C}_{\mathcal{W}} w_- \end{smallmatrix}\right]$ of $\Sigma$. Define $\mathcal{C}_{\mathcal{X}} x_0 := \tilde{x}(0)$. As in the proof of Theorem 5.14 we see that $\mathcal{C}_{\mathcal{X}}$ is a conjugation.

Let $w \in \mathfrak{W}$. and let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be the unique externally generated stable two-sided trajectory of $\Sigma$ with signal part $w$. Then by the preceding argument, $\mathcal{C}_{\mathcal{X}} x(0) = \mathfrak{B}_\Sigma \mathcal{C}_{\mathcal{W}} \pi_- w$. By shifting the trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to the left or right we find that the stable two-sided trajectory whose signal part is $\mathcal{C}_{\mathcal{W}} w$ is equal to $\left[\begin{smallmatrix} \mathcal{C}_{\mathcal{X}} x \\ \mathcal{C}_{\mathcal{W}} w \end{smallmatrix}\right]$. The set of initial states $x(0)$ of the type $x(0) = \mathfrak{B}_\Sigma \pi_- w$ for some $w \in \mathfrak{W}$ is dense in $\mathcal{X}$, and consequently, it is true that if $\left[\begin{smallmatrix} x_+ \\ w_+ \end{smallmatrix}\right]$ is an arbitrary stable future trajectory of $\Sigma$, then also $\left[\begin{smallmatrix} \mathcal{C}_{\mathcal{X}} x_+ \\ \mathcal{C}_{\mathcal{W}} w_+ \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma$. By Lemma 5.3, this implies that $\Sigma$ is $(\mathcal{C}_{\mathcal{X}}; \mathcal{C}_{\mathcal{W}})$-real. □

**Corollary 5.17.** *Let $\mathfrak{W}$ be a $\mathcal{C}_{\mathcal{W}}$-real passive two-sided behaviour on $\mathcal{W}$, with the corresponding past and future behaviours $\mathfrak{W}_-$ and $\mathfrak{W}_+$.*

(i) *The canonical controllable energy preserving realization $\Sigma_{\mathrm{cep}}^{\mathfrak{W}_-}$ of $\mathfrak{W}$ is $(\mathcal{C}_{\mathcal{H}_-}, \mathcal{C}_{\mathcal{W}})$-real.*

(ii) *The canonical observable co-energy preserving realization $\Sigma_{\mathrm{oce}}^{\mathfrak{W}_+}$ of $\mathfrak{W}$ is $(\mathcal{C}_{\mathcal{H}_+}, \mathcal{C}_{\mathcal{W}})$-real.*

(iii) *The canonical simple conservative realization $\Sigma_{\mathrm{sc}}^{\mathfrak{W}}$ of $\mathfrak{W}$ is $(\mathcal{C}_{\mathcal{D}(\mathfrak{W})}, \mathcal{C}_{\mathcal{W}})$-real.*

*Proof.* That these three canonical models are real for some conjugations in their state spaces follows from Theorems 5.16, 5.15, and 5.14 applied to these models. That the conjugations are precisely those listed above can be seen by comparing the proofs of the cited theorems with Lemma 5.13. □

**Corollary 5.18.** *The unique state space conjugation $\mathcal{C}_\mathcal{X}$ in Theorem 5.14 is given by*

$$\mathcal{C}_\mathcal{X} = (\mathfrak{B}_\Sigma^{\mathrm{bil}})^{-1}\mathcal{C}_{\mathcal{D}(\mathfrak{W})}\mathfrak{B}_\Sigma^{\mathrm{bil}} = \mathfrak{C}_\Sigma^{\mathrm{bil}}\mathcal{C}_{\mathcal{D}(\mathfrak{W})}(\mathfrak{C}_\Sigma^{\mathrm{bil}})^{-1}; \tag{5.6}$$

*here $\mathfrak{C}_\Sigma^{\mathrm{bil}}$ and $\mathfrak{B}_\Sigma^{\mathrm{bil}} = (\mathfrak{C}_\Sigma^{\mathrm{bil}})^*$ are the two-sided present/future and past/present maps of the simple conservative system $\Sigma$.*

*Proof.* This follows from Theorem 3.47, Lemma 5.5, and Corollary 5.17. □

**Theorem 5.19.** *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a passive $(\mathcal{C}_\mathcal{X},\mathcal{C}_\mathcal{W})$-real system with past/present map $\mathfrak{B}_\Sigma$, present/future map $\mathfrak{C}_\Sigma$, two-sided behaviour $\mathfrak{W}$, and past/future map $\Gamma_\mathfrak{W}$. Let $\mathcal{C}_{\mathcal{H}_-}$ and $\mathcal{C}_{\mathcal{H}_+}$ be the conjugations in parts (i) and (ii) of Lemma 5.13. Then $\mathfrak{B}_\Sigma$ is $(\mathcal{C}_{\mathcal{H}_-},\mathcal{C}_\mathcal{X})$-real, $\mathfrak{C}_\Sigma$ is $(\mathcal{C}_\mathcal{X},\mathcal{C}_{\mathcal{H}_+})$-real, and $\Gamma_\mathfrak{W}$ is $(\mathcal{C}_{\mathcal{H}_-},\mathcal{C}_{\mathcal{H}_+})$-real.*

*Proof.* If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an externally generated stable past trajectory of $\Sigma$, then by Lemma 5.3 also $\left[\begin{smallmatrix} \mathcal{C}_\mathcal{X} x \\ \mathcal{C}_\mathcal{W} w \end{smallmatrix}\right]$ is an externally generated stable past trajectory of $\Sigma$. This implies that $\mathfrak{B}Q_-\mathcal{C}_\mathcal{W}w = \mathcal{C}_\mathcal{X}\mathfrak{B}Q_-w$ for all $w \in \mathfrak{W}_-$. Here we can replace $Q_-\mathcal{C}_\mathcal{W}w$ by $\mathcal{C}_{\mathcal{H}_-}Q_-w$ to get $\mathfrak{B}\mathcal{C}_{\mathcal{H}_-}Q_-w = \mathcal{C}_\mathcal{X}\mathfrak{B}Q_-w$ for all $w \in \mathfrak{W}_-$. Since $Q_-\mathfrak{W}_-$ is dense in $\mathcal{H}_-$ we find that $\mathfrak{B}\mathcal{C}_{\mathcal{H}_-} = \mathcal{C}_\mathcal{X}\mathfrak{B}$, i.e., $\mathfrak{B}_\Sigma$ is $(\mathcal{C}_{\mathcal{H}_-},\mathcal{C}_\mathcal{X})$-real.

Likewise, if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma$, then by Lemma 5.3 also $\left[\begin{smallmatrix} \mathcal{C}_\mathcal{X} x \\ \mathcal{C}_\mathcal{W} w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma$, i.e., $Q_+\mathcal{C}_\mathcal{W}w = \mathfrak{C}_\Sigma\mathcal{C}_\mathcal{X}x(0)$. Here we can replace $Q_+\mathcal{C}_\mathcal{W}$ by $\mathcal{C}_{\mathcal{H}_+}Q_+w$. This implies that $\mathfrak{C}_\Sigma\mathcal{C}_\mathcal{X} = \mathcal{C}_{\mathcal{H}_+}\mathfrak{C}_\Sigma$, and so $\mathfrak{C}_\Sigma$ is $(\mathcal{C}_\mathcal{X},\mathcal{C}_{\mathcal{H}_+})$-real.

Finally, $\Gamma_\mathfrak{W} = \mathfrak{C}_\Sigma\mathfrak{B}_\Sigma$ is $(\mathcal{C}_{\mathcal{H}_-},\mathcal{C}_{\mathcal{H}_+})$-real since $\mathfrak{B}_\Sigma$ is $(\mathcal{C}_{\mathcal{H}_-},\mathcal{C}_\mathcal{X})$-real and $\mathfrak{C}_\Sigma$ is $(\mathcal{C}_\mathcal{X},\mathcal{C}_{\mathcal{H}_-})$-real. □

**Corollary 5.20.** *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a passive $(\mathcal{C}_\mathcal{X},\mathcal{C}_\mathcal{W})$-real s/s system with past/present map $\mathfrak{B}_\Sigma$ and present/future map $\mathfrak{C}_\Sigma$.*

(i) *If $\Sigma$ is observable, then $\mathcal{C}_\mathcal{X} = \mathfrak{C}_\Sigma^{-1}\mathcal{C}_{\mathcal{H}_+}\mathfrak{C}_\Sigma$.*

(ii) *If $\Sigma$ is controllable, then $\mathcal{C}$ is the closure of the operator*

$$\mathfrak{B}_\Sigma\mathcal{C}_{\mathcal{H}_-}(\mathfrak{B}_{\Sigma|(\ker(\mathfrak{B}_\Sigma))^\perp})^{-1},$$

*which is defined on* $\operatorname{im}(\mathfrak{B}_\Sigma)$.

(iii) *If $\Sigma$ is simple, then $\mathcal{C}_\mathcal{X}$ is the closure of the operator which is defined on* $\operatorname{im}(\mathfrak{B}_\Sigma) + (\ker(\mathfrak{C}_\Sigma))^\perp$ *by*

$$\mathcal{C}_\mathcal{X}x_0 = \begin{cases} \mathfrak{C}_\Sigma^{-1}\mathcal{C}_{\mathcal{H}_+}\mathfrak{C}_\Sigma x_0, & x_0 \in (\ker(\mathfrak{C}_\Sigma))^\perp, \\ \mathfrak{B}_\Sigma\mathcal{C}_{\mathcal{H}_-}(\mathfrak{B}_{\Sigma|(\ker(\mathfrak{B}_\Sigma))^\perp})^{-1}x_0, & x_0 \in \operatorname{im}(\mathfrak{B}_\Sigma)). \end{cases}$$

*Proof.* By Theorem 5.19, $\mathcal{C}_\mathcal{X}\mathfrak{B}_\Sigma = \mathfrak{B}_\Sigma\mathcal{C}_\mathcal{Y}$ and $\mathfrak{C}_\Sigma\mathcal{C}_\mathcal{X} = \mathcal{C}_\mathcal{Y}\mathfrak{C}_\Sigma$. From this claim (iii) follows immediately. Claims (i) and (ii) are special cases of (iii). □

**Theorem 5.21.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a minimal optimal externally $\mathcal{C}_\mathcal{W}$-real s/s system. Then there exists a unique conjugation $\mathcal{C}_\mathcal{X}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real.*

*Proof.* The uniqueness claim follows from Lemma 5.4, so it suffices to prove that $\Sigma$ is $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real for some conjugation $\mathcal{C}_\mathcal{X}$. By Theorem 4.6 and Lemma 5.5, to do this it suffices to prove the corresponding statement for the canonical minimal optimal model $\Sigma^{\mathfrak{W}}_{\mathrm{mo}}$, and by Lemma 5.8, it then suffices to prove the same statement for the canonical observable co-energy preserving model $\Sigma^{\mathfrak{W}_+}_{\mathrm{oce}}$. But according to Theorem 5.15, $\Sigma^{\mathfrak{W}_+}_{\mathrm{oce}}$ is $(\mathcal{C}_{\mathcal{H}_+}, \mathcal{C}_\mathcal{W})$-real. □

**Theorem 5.22.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a minimal $*$-optimal externally $\mathcal{C}_\mathcal{W}$-real s/s system. Then there exists a unique conjugation $\mathcal{C}_\mathcal{X}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real.*

*Proof.* The proof of Theorem 5.22 is analogous to the proof of Theorem 5.21. □

**Theorem 5.23.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a minimal passive balanced externally $\mathcal{C}_\mathcal{W}$-real s/s system. Then there exists a unique conjugation $\mathcal{C}_\mathcal{X}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real.*

*Proof.* The uniqueness of $\mathcal{C}_\mathcal{X}$ again follows from Lemma 5.4, so it suffices to prove the existence of $\mathcal{C}_\mathcal{X}$, and by Lemma 5.5, it suffices to prove that the minimal optimal system $\Sigma_\odot = (V_\odot; \mathcal{X}_\odot, \mathcal{W})$ constructed in Theorem 4.19 is $(\mathcal{C}_{\mathcal{X}_\odot}, \mathcal{C}_\mathcal{W})$-real for some conjugation $\mathcal{C}_{\mathcal{X}_\odot}$ in $\mathcal{X}_\odot$. As we shall see below, $\mathcal{C}_{\mathcal{X}_\odot}$ is the restriction to $\mathcal{X}_\odot$ of $\mathcal{C}_{\mathcal{H}_+}$.

By Lemma 5.19, $\Gamma_\mathfrak{W}\mathcal{C}_{\mathcal{H}_-} = \mathcal{C}_{\mathcal{H}_+}\Gamma_\mathfrak{W}$. As can easily be seen, this implies that $\Gamma_\mathfrak{W}\Gamma_\mathfrak{W}^*$ commutes with $\mathcal{C}_{\mathcal{H}_+}$. Since $|\Gamma_\mathfrak{W}^*|^{1/2} = (\Gamma_\mathfrak{W}\Gamma_\mathfrak{W}^*)^{1/4}$ can be obtained as a uniform limit of powers of $\Gamma_\mathfrak{W}\Gamma_\mathfrak{W}^*$, this implies that $\mathcal{C}_{\mathcal{H}_+}$ commutes with $|\Gamma_\mathfrak{W}^*|^{1/2}$, and hence also with the inverse of the restriction of $|\Gamma_\mathfrak{W}^*|^{1/2}$ to $\mathcal{X}_\circ = \overline{\mathrm{im}\,(\Gamma_\mathfrak{W})} = (\ker(|\Gamma_\mathfrak{W}^*|))^\perp = (\ker\big(|\Gamma_\mathfrak{W}^*|^{1/2}\big))^\perp$. It follows from the definition of $\mathcal{X}_\odot$ that $\mathcal{X}_\odot$ is invariant under $\mathcal{C}_{\mathcal{H}_+}$. Moreover, with the notations of Lemma 4.18 and Theorem 4.19, for all $x \in \mathcal{X}_\odot$ we have

$$\begin{aligned}\|\mathcal{C}_{\mathcal{H}_+}x\|^2_{\mathcal{X}_\odot} &= \|(|\Gamma_\mathfrak{W}^*|_{|\mathcal{X}_\circ})^{-1/2}\mathcal{C}_{\mathcal{H}_+}x\|^2_{\mathcal{X}_\circ} = \|\mathcal{C}_{\mathcal{H}_+}(|\Gamma_\mathfrak{W}^*|_{|\mathcal{X}_\circ})^{-1/2}x\|^2_{\mathcal{X}_\circ}\\ &= \|(|\Gamma_\mathfrak{W}^*|_{|\mathcal{X}_\circ})^{-1/2}x\|^2_{\mathcal{X}_\circ} = \|x\|^2_{\mathcal{X}_\odot}.\end{aligned}$$

Thus, the restriction $\mathcal{C}_{\mathcal{X}_\odot}$ of $\mathcal{C}_{\mathcal{H}_+}$ to $\mathcal{X}_\odot$ is an isometric operator in $\mathcal{X}_\odot$, and hence a conjugation in $\mathcal{X}_\odot$. That $\Sigma_\odot$ is $(\mathcal{C}_{\mathcal{X}_\odot}, \mathcal{C}_\mathcal{W})$-real follows from (4.4) and the fact that $\Sigma_\circ$ is $(\mathcal{C}_{\mathcal{X}_\circ}, \mathcal{C}_\mathcal{W})$-real, where $\mathcal{C}_{\mathcal{X}_\circ}$ is the restriction of $\mathcal{C}_{\mathcal{H}_+}$ to $\mathcal{X}_\circ$. □

# 6 Passive Reciprocal State/Signal Systems and Behaviours

Earlier in this article we have seen two types of transformations of passive systems, namely the transformation which takes a system $\Sigma$ to its dual $\Sigma_*$ introduced in Lemma 2.6, and the conjugate-linear unitary transformation in Lemma 5.1. Here we shall study a third type of transformations which contains the duality transformation in Lemma 2.6 as a special case.

**Lemma 6.1.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with adjoint $\Sigma_* = (V_*; \mathcal{X}, -\mathcal{W})$, let $\mathcal{X}_1$ and $\mathcal{W}_1$ be a Hilbert and a Kreĭn space, respectively, let $\mathcal{V}_\mathcal{X}$ be a linear unitary operator in $\mathcal{B}(\mathcal{X}; \mathcal{X}_1)$, and let $\mathcal{R}_\mathcal{W}$ be a linear skew-unitary operator in $\mathcal{B}(\mathcal{W}; \mathcal{W}_1)$.*

(i) *Define $V_1$ by*

$$V_1 = \begin{bmatrix} -\mathcal{V}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{V}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{R}_\mathcal{W} \end{bmatrix} V^{[\perp]}. \tag{6.1}$$

*Then $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1)$ is a passive s/s system.*

(ii) *$\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory of $\Sigma_*$ on some interval $I$ if and only if $\left[\begin{smallmatrix} \mathcal{V}_\mathcal{X} x \\ \mathcal{R}_\mathcal{W} \mathcal{I}_{(-\mathcal{W},\mathcal{W})} w \end{smallmatrix}\right]$ is a classical trajectory of $\Sigma_1$ on $I$.*

(iii) *$\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a generalised trajectory of $\Sigma_*$ on some interval $I$ if and only if $\left[\begin{smallmatrix} \mathcal{V}_\mathcal{X} x \\ \mathcal{R}_\mathcal{W} \mathcal{I}_{(-\mathcal{W},\mathcal{W})} w \end{smallmatrix}\right]$ is a generalised trajectory of $\Sigma_1$ on $I$.*

(iv) *If we denote the past, two-sided, and full behaviours of $\Sigma$ by $\mathfrak{W}_-$, $\mathfrak{W}$, and $\mathfrak{W}_+$, respectively, then the corresponding behaviours of $\Sigma_1$ are equal to $\mathcal{R}_\mathcal{W} \mathbf{\mathfrak{R}} \mathfrak{W}_+^{[\perp]}$, $\mathcal{R}_\mathcal{W} \mathbf{\mathfrak{R}} \mathfrak{W}^{[\perp]}$, and $\mathcal{R}_\mathcal{W} \mathbf{\mathfrak{R}} \mathfrak{W}_-^{[\perp]}$, respectively.*

*Proof.* This follows from Lemmas 2.6 and 3.23 and the fact that both $\mathcal{V}_\mathcal{X}$ and $\mathcal{I}_\mathcal{W} \mathcal{I}_{(-\mathcal{W},\mathcal{W})}$ are unitary operators in $\mathcal{B}(\mathcal{X}; \mathcal{X}_1)$ and $\mathcal{B}(-\mathcal{W}, \mathcal{W}_1)$, respectively. □

We shall be especially interested in the case where the two systems $\Sigma$ and $\Sigma_1$ in Lemma 6.1 coincide and both $\mathcal{V}_\mathcal{X}$ and $\mathcal{R}_\mathcal{W}$ are involutions, i.e., $\mathcal{V}_\mathcal{X}$ is a signature operator and $\mathcal{R}_\mathcal{W}$ is a skew-signature operator.

**Definition 6.2.** A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called $(\mathcal{J}_\mathcal{X}; \mathcal{I}_\mathcal{W})$-reciprocal if (1.16) (or equivalently, (1.20)) holds, where $\mathcal{J}_\mathcal{X}$ is a signature operator in $\mathcal{X}$ and $\mathcal{I}_\mathcal{W}$ is a skew-signature operator in $\mathcal{W}$.

Instead of using the characterisation give above for reciprocity of a system we can also use the following alternative characterisations.

**Lemma 6.3.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, let $\mathcal{J}_\mathcal{X}$ be a signature operator in $\mathcal{X}$, let $\mathcal{I}_\mathcal{W}$ be and skew-signature operator in $\mathcal{W}$, and let $I \subset \mathbb{R}$ be a nontrivial interval with finite left end-point. Then the following conditions are equivalent:*

(i) $\Sigma$ *is* $(\mathcal{J}_\mathcal{X}; \mathcal{I}_\mathcal{W})$*-reciprocal;*

(ii) $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *is a classical trajectory of* $\Sigma_*$ *on* $I$ *if and only if* $\left[\begin{smallmatrix} \mathcal{J}_\mathcal{X} x \\ \mathcal{I}_\mathcal{W} \mathcal{I}_{(-\mathcal{W},\mathcal{W})} w \end{smallmatrix}\right]$ *is a classical trajectory of* $\Sigma$ *on* $I$

(iii) $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *is a generalised trajectory of* $\Sigma_*$ *on* $I$ *if and only if* $\left[\begin{smallmatrix} \mathcal{J}_\mathcal{X} x \\ \mathcal{I}_\mathcal{W} \mathcal{I}_{(-\mathcal{W},\mathcal{W})} w \end{smallmatrix}\right]$ *is a generalised trajectory of* $\Sigma$ *on* $I$*.*

*Proof.* The proof is analogous to the proof of Lemma 5.3. □

**Lemma 6.4.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive $(\mathcal{J}_\mathcal{X}; \mathcal{I}_\mathcal{W})$-reciprocal system, and let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be unitarily similar to $\Sigma$ with similarity operator $\mathcal{V}$. Then $\Sigma_1$ is $(\mathcal{J}_{\mathcal{X}_1}; \mathcal{I}_\mathcal{W})$-reciprocal with $\mathcal{J}_{\mathcal{X}_1} = \mathcal{V}\mathcal{J}_\mathcal{X}\mathcal{V}^{-1}$.*

*Proof.* The proof is analogous to the proof of Lemma 5.5. □

**Lemma 6.5.** *A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is $(\mathcal{J}_\mathcal{X}; \mathcal{I}_\mathcal{W})$-reciprocal if and only if the adjoint system $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W})$ is $(\mathcal{J}_\mathcal{X}; \mathcal{I}_{-\mathcal{W}})$-reciprocal, where $\mathcal{I}_{-\mathcal{W}} = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\mathcal{I}_\mathcal{W}\mathcal{I}_{(-\mathcal{W},\mathcal{W})}$.*

*Proof.* The proof is analogous to the proof of Lemma 5.6. □

**Lemma 6.6.** *If the passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is $(\mathcal{J}_\mathcal{X}, \mathcal{I}_\mathcal{W})$-reciprocal, then the reachable subspace $\mathfrak{R}_\Sigma$ and the unobservable subspace $\mathfrak{U}_\Sigma$ of $\Sigma$ satisfy*

$$\mathcal{J}_\mathcal{X}\mathfrak{R}_\Sigma = \mathfrak{U}_\Sigma^\perp, \quad \mathcal{J}_\mathcal{X}\mathfrak{U}_\Sigma = \mathfrak{R}_\Sigma^\perp, \quad \mathcal{J}_\mathcal{X}\mathfrak{R}_\Sigma^\perp = \mathfrak{U}_\Sigma, \quad \mathcal{J}_\mathcal{X}\mathfrak{U}_\Sigma^\perp = \mathfrak{R}_\Sigma. \tag{6.2}$$

*In particular, $\Sigma$ is minimal if and only if $\Sigma$ is controllable, or equivalently, if and only if $\Sigma$ is observable.*

*Proof.* The proof is analogous to the proof of Lemma 5.7. □

**Lemma 6.7.** *Let $\Sigma$ be a simple passive s/s system which satisfies* (1.16) *for some unitary operator $\mathcal{J}_\mathcal{X}$ and some skew-signature operator $\mathcal{I}_\mathcal{W}$. Then $\mathcal{J}_\mathcal{X}$ is a signature operator, and hence $\Sigma$ is $(\mathcal{J}_\mathcal{X}; \mathcal{I}_\mathcal{W})$-reciprocal. Moreover, $\mathcal{J}_\mathcal{X}$ is determined uniquely by $\Sigma$ and $\mathcal{I}_\mathcal{W}$.*

*Proof.* Denote $\mathcal{D} = \left[\begin{smallmatrix} -\mathcal{J}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{I}_{\mathcal{W}} \end{smallmatrix}\right]$. It follows from (1.16) that

$$V^{[\perp]} = \mathcal{D}^{-1}V.$$

On the other hand, it is easy to check that $\mathcal{D}$ is skew-unitary, and hence by by (1.16) and Lemma 2.19 with $V$ replaced by $V^{[\perp]}$,

$$V^{[\perp]} = (\mathcal{D}V^{[\perp]})^{[\perp]} = \mathcal{D}V.$$

Thus, $\mathcal{D}V = \mathcal{D}^{-1}V$. Here

$$\mathcal{D}^{-1}V = \begin{bmatrix} -\mathcal{J}_{\mathcal{X}}^{-1} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}}^{-1} & 0 \\ 0 & 0 & \mathcal{I}_{\mathcal{W}}^{-1} \end{bmatrix} V = \begin{bmatrix} -\mathcal{J}_{\mathcal{X}}^{*} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}}^{*} & 0 \\ 0 & 0 & \mathcal{I}_{\mathcal{W}} \end{bmatrix} V.$$

Multiplying this identity by $\mathcal{D}^{-1}$ to the left we get

$$V = \begin{bmatrix} \mathcal{J}_{\mathcal{X}}^{-1}\mathcal{J}_{\mathcal{X}}^{*} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}}^{-1}\mathcal{J}_{\mathcal{X}}^{*} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V.$$

By Lemma 3.50, $\mathcal{J}_{\mathcal{X}}^{-1}\mathcal{J}_{\mathcal{X}}^{*} = 1_{\mathcal{X}}$, i.e., $\mathcal{J}_{\mathcal{X}}$ is a signature operator.

By comparing (1.16) to (1.19) we find that (1.20) holds. By Lemma 3.51, $\mathcal{J}_{\mathcal{X}}$ is determined uniquely by $V$, $V_*$, and $\mathcal{I}_{\mathcal{W}}\mathcal{I}_{(-\mathcal{W},\mathcal{W})}$, and hence by $V$ and $\mathcal{I}_{\mathcal{W}}$. □

**Definition 6.8.** Let $\mathcal{I}_{\mathcal{W}}$ be a skew-signature operator in the Kreĭn space $\mathcal{W}$.

(i) A passive two-sided behaviour $\mathfrak{W}$ on $\mathcal{W}$ is called $\mathcal{I}_{\mathcal{W}}$-reciprocal if

$$\mathfrak{W} = \mathcal{I}_{\mathcal{W}}\boldsymbol{\mathfrak{R}}\mathfrak{W}^{[\perp]} \tag{6.3}$$

(here the skew-signature operator $\mathcal{I}_{\mathcal{W}}$ on $K^2(\mathcal{W})$ induced by the skew-signature operator $\mathcal{I}_{\mathcal{W}} \in \mathcal{B}(\mathcal{W})$ is defined as in Remark 1.1).

(ii) A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called *externally $\mathcal{I}_{\mathcal{W}}$-reciprocal* if its two-sided behaviour is $\mathcal{I}_{\mathcal{W}}$-reciprocal.

It follows from (3.8)–(3.9) that the equality (6.3) is equivalent to each of the following equalities:

$$\mathfrak{W}_{+} = \mathcal{I}_{\mathcal{W}}\boldsymbol{\mathfrak{R}}\mathfrak{W}_{-}^{[\perp]}, \qquad \mathfrak{W}_{-} = \mathcal{I}_{\mathcal{W}}\boldsymbol{\mathfrak{R}}\mathfrak{W}_{+}^{\perp}, \tag{6.4}$$

where $\mathfrak{W}_{+}$ and $\mathfrak{W}_{-}$ are the passive future and past behaviours on $\mathcal{W}$ defined in terms of $\mathfrak{W}$ by (3.8).

**Lemma 6.9.** *If $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a passive $(\mathcal{J}_{\mathcal{X}}, \mathcal{I}_{\mathcal{W}})$-reciprocal s/s system, then it is externally $\mathcal{I}_{\mathcal{W}}$-reciprocal.*

*Proof.* The proof if analogous to the proof of Lemma 5.10. □

**Lemma 6.10.** *A passive two-sided behaviour $\mathfrak{W}$ on the Kreĭn space $\mathcal{W}$ is $\mathcal{I}_{\mathcal{W}}$-reciprocal if and only if the adjoint behaviour $\mathfrak{W}_*$ is $\mathcal{I}_{-\mathcal{W}}$-reciprocal, where $\mathcal{I}_{-\mathcal{W}} = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\mathcal{I}_{\mathcal{W}}\mathcal{I}_{(-\mathcal{W},\mathcal{W})}$. In particular, a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is externally $\mathcal{I}_{\mathcal{W}}$-reciprocal if and only if the adjoint system $\Sigma_*$ is externally $\mathcal{I}_{-\mathcal{W}}$-reciprocal.*

*Proof.* This follows from Lemma 2.19 and 3.25 and Definitions 3.24 and 6.8. □

**Lemma 6.11.** *Let $\mathfrak{W}$ be a $\mathcal{I}_{\mathcal{W}}$-reciprocal passive two-sided behaviour on $\mathcal{W}$, with the corresponding past and future behaviours $\mathfrak{W}_-$ and $\mathfrak{W}_+$.*

(i) *$w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ if and only if $\mathcal{I}_{\mathcal{W}}\mathbf{Я}w_- \in \mathcal{K}(\mathfrak{W}_+)$. In this case, $\|Q_+\mathcal{I}_{\mathcal{W}}\mathbf{Я}w_-\|_{\mathcal{H}_+} = \|Q_-w_-\|_{\mathcal{H}_-}$.*

(ii) *$w_+ \in \mathcal{K}(\mathfrak{W}_+)$ if and only if $\mathcal{C}_{\mathcal{W}}\mathbf{Я}w_+ \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$. In this case $\|Q_-\mathcal{I}_{\mathcal{W}}\mathbf{Я}w_+\|_{\mathcal{H}_-} = \|Q_-w_+\|_{\mathcal{H}_-}$.*

(iii) *$w \in \mathcal{L}(\mathfrak{W})$ if and only if $\mathcal{I}_{\mathcal{W}}\mathbf{Я}w \in \mathcal{L}(\mathfrak{W})$. In this case $\|Q\mathcal{I}_{\mathcal{W}}\mathbf{Я}w\|_{\mathcal{D}(\mathfrak{W})} = \|Qw\|_{\mathcal{D}(\mathfrak{W})}$.*

*Proof.* The proof is analogous to the proof of Lemma 6.11. □

**Lemma 6.12.** *Let $\mathfrak{W}$ be a $\mathcal{I}_{\mathcal{W}}$-reciprocal passive two-sided behaviour on $\mathcal{W}$, with the corresponding past and future behaviours $\mathfrak{W}_-$ and $\mathfrak{W}_+$.*

(i) *There is a unique unitary operator $\mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}$ in $\mathcal{B}(\mathcal{H}_-;\mathcal{H}_+)$ such that $\mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}Q_-w_- = Q_+\mathcal{I}_{\mathcal{W}}\mathbf{Я}w_-$ for all $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$.*

(ii) *There is a unique unitary operator $\mathcal{V}_{(\mathcal{H}_+,\mathcal{H}_-)}$ in $\mathcal{B}(\mathcal{H}_+;\mathcal{H}_-)$ such that $\mathcal{V}_{(\mathcal{H}_+,\mathcal{H}_-)}Q_+w_+ = Q_-\mathcal{I}_{\mathcal{W}}\mathbf{Я}w_+$ for all $w_+ \in \mathcal{K}(\mathfrak{W}_+)$. This operator is the adjoint of the operator $\mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}$ in (i).*

(iii) *There is a unique signature operator $\mathcal{J}_{\mathcal{D}(\mathfrak{W})}$ in $\mathcal{D}(\mathfrak{W})$ such that $\mathcal{J}_{\mathcal{D}(\mathfrak{W})}Qw = Q\mathcal{I}_{\mathcal{W}}\mathbf{Я}w$ for all $w \in \mathcal{L}(\mathfrak{W})$.*

*Proof.* The proof is analogous to the proof of Lemma 5.13. □

**Theorem 6.13.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a simple conservative externally $\mathcal{I}_\mathcal{W}$-reciprocal s/s system. Then there exists a unique signature operator $\mathcal{J}_\mathcal{X}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{J}_\mathcal{X}, \mathcal{I}_\mathcal{W})$-reciprocal.*

*Proof.* The uniqueness of $\mathcal{J}_\mathcal{X}$ follows from Lemma 6.7, so it suffices to prove the existence of $\mathcal{J}_\mathcal{X}$.

The system $\Sigma = (V; \mathcal{X}; \mathcal{W})$ is a simple conservative realization of its behaviour $\mathfrak{W}$, and hence $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W})$ is a simple conservative realization of the dual behaviour $\mathfrak{W}_* = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\mathbf{\mathfrak{R}}\mathfrak{W}^{[\perp]}$. Recall that $V_*$ is given by (1.19). Consequently, the s/s system whose generating subspace is equal to $\begin{bmatrix} -1_\mathcal{X} & 0 & 0 \\ 0 & 1_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{I}_\mathcal{W} \end{bmatrix} V^{[\perp]}$ is a simple conservative realization of $\mathcal{I}_\mathcal{W}\mathbf{\mathfrak{R}}\mathfrak{W}_*$, which is assumed to be equal to $\mathfrak{W}$. Since two simple conservative realizations of the same passive behaviour are unitarily similar, there exists a unitary operator $\mathcal{J}_\mathcal{X}$ in $\mathcal{X}$ such that (1.16) holds. By Lemma 6.7, $\mathcal{J}_\mathcal{X}$ is a signature operator which is uniquely determined by $\Sigma$ and $\mathcal{I}_\mathcal{W}$. □

**Corollary 6.14.** *Let $\mathfrak{W}$ be a $\mathcal{I}_\mathcal{W}$-reciprocal passive two-sided behaviour on $\mathcal{W}$. Then the canonical simple conservative realization $\Sigma_{\text{sc}}^{\mathfrak{W}}$ of $\mathfrak{W}$ is $(\mathcal{J}_{\mathcal{D}(\mathfrak{W})}, \mathcal{I}_\mathcal{W})$-reciprocal, where $\mathcal{J}_{\mathcal{D}(\mathfrak{W})}$ is the operator in Lemma 6.12.*

*Proof.* The proof is analogous to the proof of Corollary 5.17. □

**Corollary 6.15.** *The unique signature operator $\mathcal{J}_\mathcal{X}$ in Theorem 6.13 is given by*

$$\mathcal{J}_\mathcal{X} = (\mathfrak{B}_\Sigma^{\text{bil}})^{-1}\mathcal{J}_{\mathcal{D}(\mathfrak{W})}\mathfrak{B}_\Sigma^{\text{bil}} = \mathfrak{C}_\Sigma^{\text{bil}}\mathcal{D}_{\mathcal{D}(\mathfrak{W})}(\mathfrak{C}_\Sigma^{\text{bil}})^{-1}; \tag{6.5}$$

*here $\mathfrak{C}_\Sigma^{\text{bil}}$ and $\mathfrak{B}_\Sigma^{\text{bil}} = (\mathfrak{C}_\Sigma^{\text{bil}})^*$ are the two-sided present/future and past/present maps of the simple conservative system $\Sigma$ and $\mathcal{J}_{\mathcal{D}(\mathfrak{W})}$ is the signature operator in Lemma 6.12.*

*Proof.* This follows from Theorem 3.47, Lemma 6.4, and Corollary 6.14. □

**Theorem 6.16.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive $(\mathcal{J}_\mathcal{X}, \mathcal{I}_\mathcal{W})$-reciprocal system with past/present map $\mathfrak{B}_\Sigma$, present/future map $\mathfrak{C}_\Sigma$, and two-sided behaviour $\mathfrak{W}$. Let $\mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}$ and $\mathcal{V}_{(\mathcal{H}_+,\mathcal{H}_-)} = \mathcal{V}^*_{(\mathcal{H}_-,\mathcal{H}_+)}$ be the unitary operators in Lemma 6.12. Then*

$$\mathfrak{B}_\Sigma = \mathcal{J}_\mathcal{X}\mathfrak{C}_\Sigma^*\mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}, \quad \mathfrak{C}_\Sigma = \mathcal{V}_{(\mathcal{H}_+,\mathcal{H}_-)}\mathfrak{B}_\Sigma^*\mathcal{J}_\mathcal{X}, \quad \Gamma_\mathfrak{W} = \mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}\Gamma_\mathfrak{W}^*\mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}.$$

*Proof.* The proof of the formula $\mathfrak{B}_\Sigma = \mathcal{J}_\mathcal{X}\mathfrak{C}_\Sigma^*\mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}$ is analogous to the proof of Theorem 5.19. Taking the adjoint of this formula we get the second formula $\mathfrak{C}_\Sigma = \mathcal{V}_{(\mathcal{H}_+,\mathcal{H}_-)}\mathfrak{B}_\Sigma^*\mathcal{J}_\mathcal{X}$. Finally, these two formulas together with the fact that $\Gamma_\mathfrak{W} = \mathfrak{C}_\Sigma\mathfrak{B}_\Sigma$ gives the third formula $\Gamma_\mathfrak{W} = \mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}\Gamma_\mathfrak{W}^*\mathcal{V}_{(\mathcal{H}_-,\mathcal{H}_+)}$. □

**Theorem 6.17.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a minimal passive balanced externally $\mathcal{I}_{\mathcal{W}}$-reciprocal s/s system. Then there exists a unique signature operator $\mathcal{J}_{\mathcal{X}}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{J}_{\mathcal{X}}, \mathcal{I}_{\mathcal{W}})$-reciprocal.*

*Proof.* The proof is essentially the same as the proof of Theorem 6.13. $\square$

# 7 Passive Signature Invariant and Decomposable State/Signal Systems and Behaviours

In this section we study yet another class of symmetries of passive s/s systems and passive behaviours, where the symmetry is with respect to two signature operators $\mathcal{J}_\mathcal{X}$ and $\mathcal{J}_\mathcal{W}$ in the Hilbert state space $\mathcal{X}$ and the Kreĭn signal space $\mathcal{W}$, respectively. It turns out that this class of symmetries is related to the question when a passive s/s system can be decomposed into two independent subsystems.

**Definition 7.1.** A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called $(\mathcal{J}_\mathcal{X}; \mathcal{J}_\mathcal{W})$-signature invariant if (1.17) holds, where $\mathcal{J}_\mathcal{X}$ and $\mathcal{J}_\mathcal{W}$ are signature operators in $\mathcal{X}$ and $\mathcal{W}$, respectively.

**Definition 7.2.** Let $\mathcal{J}_\mathcal{W}$ be a signature operator in the Kreĭn space $\mathcal{W}$.

(i) A passive two-sided behaviour $\mathfrak{W}$ on $\mathcal{W}$ is called $\mathcal{J}_\mathcal{W}$-signature invariant if

$$\mathfrak{W} = \mathcal{J}_\mathcal{W}\mathfrak{W} \tag{7.1}$$

(here the signature operator $\mathcal{J}_\mathcal{W}$ on $K^2(\mathcal{W})$ induced by the signature operator $\mathcal{J}_\mathcal{W} \in \mathcal{B}(\mathcal{W})$ is defined as in Remark 1.1).

(ii) A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called *externally $\mathcal{J}_\mathcal{W}$-signature invariant* if its two-sided behaviour is $\mathcal{J}_\mathcal{W}$-signature invariant.

**Remark 7.3.** It is possible to develop a symmetry theory which is completely analogous to the one in Section 5 by replacing all conjugate-linear operators appearing in that section by linear operators, but keeping the other properties of the operators intact. This has the effect of converting all the conjugations used in Section 5 to signature operators, and it converts the notions of $(\mathcal{J}_\mathcal{X}, \mathcal{J}_\mathcal{W})$-reality and $\mathcal{J}_\mathcal{W}$-reality introduced in Definitions 5.2 and 5.9 into the notions of $(\mathcal{J}_\mathcal{X}, \mathcal{J}_\mathcal{W})$-signature invariance and $\mathcal{J}_\mathcal{W}$-signature invariance introduced in Definitions 7.1 and 7.2. In particular, all the lemmas, theorems, and corollaries in Section 5 remain with these replacements. All the proofs remain the same.

In particular, the following results are true:

**Lemma 7.4.** *If $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a simple passive s/s $(\mathcal{J}_\mathcal{X}; \mathcal{J}_\mathcal{W})$-signature invariant system, then $\mathcal{J}_\mathcal{X}$ is uniquely determined by $\Sigma$ and $\mathcal{J}_\mathcal{W}$.*

*Proof.* This is the linear analogue of Lemma 5.4. □

**Lemma 7.5.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive $(\mathcal{J}_\mathcal{X}; \mathcal{J}_\mathcal{W})$-signature invariant system, and let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be unitarily similar to $\Sigma$ with similarity operator $\mathcal{V}$. Then $\Sigma_1$ is $(\mathcal{J}_{\mathcal{X}_1}; \mathcal{J}_\mathcal{W})$-signature invariant with $\mathcal{J}_{\mathcal{X}_1} = \mathcal{V}\mathcal{J}_\mathcal{X}\mathcal{V}^{-1}$.*

*Proof.* This is the linear analogue of Lemma 5.5. □

**Lemma 7.6.** *If $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a passive $(\mathcal{J}_\mathcal{X}, \mathcal{J}_\mathcal{W})$-signature invariant s/s system, then it is externally $\mathcal{J}_\mathcal{W}$-signature invariant.*

*Proof.* This is the linear analogue of Lemma 5.10. □

**Theorem 7.7.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive externally $\mathcal{J}_\mathcal{W}$-signature invariant s/s system which belongs to one of the classes a)–f) listed in Section 1. Then there exists a unique signature operator $\mathcal{J}_\mathcal{X}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{J}_\mathcal{X}, \mathcal{J}_\mathcal{W})$-signature invariant.*

*Proof.* This is the linear analogue of Theorems 5.14, 5.15, 5.16, 5.21, 5.22, and 5.23, and it can be proved in the same way as the analogous results were proved in Section 5.

For completness, let us also outline a slightly different proof which can be used in the cases where the system is observable and co-energy preserving, or controllable and energy preserving, or simple and conservative. The uniqueness still follows from Lemma 7.4. Thanks to Lemma 7.5, for the proof of existence of the operator $\mathcal{J}_\mathcal{X}$ it suffices to prove existence in the case where $\Sigma$ is one of the canonical models presented in Sections 3. In the case of the observable co-energy preserving model $\Sigma^{\mathfrak{W}+}_{\rm oce}$, the controllable energy preserving model $\Sigma^{\mathfrak{W}-}_{\rm cep}$, and the simple conservative model $\Sigma^{\mathfrak{W}}_{\rm sc}$ one can again start by proving the analogue of Lemma 5.13 (with the same proof as in Section 5), and after that one gives a direct proof of the analogue of Corollary 5.17 by appealing to the explicit descriptions (3.25), (3.26), and (3.27) that we have for the generating subspaces of these three canonical models. □

As we mentioned at the beginning of this section, signature invariance is related to the decomposability of a passive s/s system or behavior.

**Definition 7.8.**

(i) A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is *decomposable* if there exist orthogonal decompositions $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{W} = \mathcal{W}_1 \boxplus \mathcal{W}_2$, out of which at least one is nontrivial, such that, with respect to these decompositions, $V$ has the representation

$$V = \left\{ \begin{bmatrix} z_1 + z_2 \\ x_1 + x_2 \\ w_1 + w_2 \end{bmatrix} \middle| \begin{bmatrix} z_i \\ x_i \\ w_i \end{bmatrix} \in V_i := V \cap \begin{bmatrix} \mathcal{X}_i \\ \mathcal{X}_i \\ \mathcal{W}_i \end{bmatrix}, \ i = 1, 2 \right\}. \qquad (7.2)$$

The system $\Sigma$ is *non-decomposable* if it such a decomposition does *not* exist.

(ii) A passive two-sided behavior on a Kreĭn space $\mathcal{W} \neq \{0\}$ is *decomposable* if there exists some nontrivial orthogonal decomposition $\mathcal{W} = \mathcal{W}_1 \boxplus \mathcal{W}_2$ (nontrivial means that neither $\mathcal{W}_1 = \{0\}$ nor $\mathcal{W}_2 = \{0\}$, or equivalently, neither $\mathcal{W}_1 = \mathcal{W}$ nor $\mathcal{W}_2 = \mathcal{W}$) such that $\mathfrak{W}$ has the representation

$$\mathfrak{W} = \{w_1 + w_2 |\, w_i \in \mathfrak{W}_i := \mathfrak{W} \cap K^2(\mathcal{W}_i),\ i = 1, 2\}\,. \tag{7.3}$$

A passive future or past behavior is decomposable if the corresponding two-sided behavior is decomposable in the above sense. A passive behavior (two-sided, future, or past) is *non-decomposable* if it is *not decomposable* in the above sense.

**Lemma 7.9.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and let $\mathfrak{W}$ be a passive two-sided behavior in $\mathcal{W}$*

(i) *If $\Sigma$ is decomposable, then $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}_i)$, $i = 1, 2$, where $\mathcal{X}_i$, $\mathcal{W}_i$, and $V_i$ are as in Definition 7.8, are passive s/s systems.*

(ii) *If $\mathfrak{W}$ is decomposable, then $\mathfrak{W}_i$, $i = 1, 2$, where $\mathcal{W}_i$ and $\mathfrak{W}_i$ are as in Definition 7.8, are passive two-sided behaviors in $\mathcal{W}$. The same statemet is also true for passive future and past behaviors.*

The easy proof of this lemma is left to the reader.

Thus, a passive s/s system or a passive two-sided behavior is decomposable if and only if it can be split into two independent passive subsystems or passive sub-behaviors, respectively. The same statement is true for passive future and past behaviors, too.

The following theorem establishes a connection between signature invariance and decomposability of a passive system or behavior. It uses the following agreement:

**Agreement 7.10.** A signature operator $\mathcal{J}_\mathcal{W}$ in the Kreĭn space is *nontrivial* if $\mathcal{J}_\mathcal{W} \neq \pm 1_\mathcal{W}$. A pair of signature operators $(\mathcal{J}_\mathcal{X}, \mathcal{J}_\mathcal{W})$ is *nontrivial* if $(1_\mathcal{X}, 1_\mathcal{W}) \neq (\mathcal{J}_\mathcal{X}, \mathcal{J}_\mathcal{W}) \neq (-1_\mathcal{X}, -1_\mathcal{W})$.

**Theorem 7.11.**

(i) *A passive s/s system is $(\mathcal{J}_\mathcal{X}, \mathcal{J}_\mathcal{W})$-signature invariant for some nontrivial pair $(\mathcal{J}_\mathcal{X}, \mathcal{J}_\mathcal{W})$ of signature operators if and only if $\Sigma$ is decomposable.*

(ii) *A passive two-sided behavior $\mathfrak{W}$ is $\mathcal{J}_{\mathcal{W}}$-signature invariant for some nontrivial operator $\mathcal{J}_{\mathcal{W}}$ if and only if $\mathfrak{W}$ is decomposable. The same statemet is also true for passive future and past behaviors.*

*Proof. Proof of (i).* Suppose first that $\Sigma$ is decomposable, and define $\mathcal{X}_i$, $\mathcal{W}_i$, and $V_i$, $i = 1, 2$, as in Definition 7.8. Define $\mathcal{J}_{\mathcal{X}}$ and $\mathcal{J}_{\mathcal{W}}$ in block matrix form with respect to the decompositions $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{W}_1 \boxplus \mathcal{W}_2$ by

$$\mathcal{J}_{\mathcal{X}} := \begin{bmatrix} 1_{\mathcal{X}_1} & 0 \\ 0 & -1_{\mathcal{X}_2} \end{bmatrix}, \qquad \mathcal{J}_{\mathcal{W}} := \begin{bmatrix} 1_{\mathcal{X}_1} & 0 \\ 0 & -1_{\mathcal{X}_2} \end{bmatrix}. \tag{7.4}$$

Since $\left[\begin{smallmatrix} -1_{\mathcal{X}_2} & 0 & 0 \\ 0 & -1_{\mathcal{X}_2} & 0 \\ 0 & 0 & -1_{\mathcal{W}_2} \end{smallmatrix}\right] V_2 = V_2$, it follows from Definitions 7.8 and 7.1 that $\Sigma$ is $(\mathcal{J}_{\mathcal{X}}, \mathcal{J}_{\mathcal{W}})$-signature invariant. Moreover, the $(\mathcal{J}_{\mathcal{X}}, \mathcal{J}_{\mathcal{W}})$ is nontrivial.

Coversely, suppose that $\Sigma$ is $(\mathcal{J}_{\mathcal{X}}, \mathcal{J}_{\mathcal{W}})$-signature invariant. Moreover, supose furthermore that at least one of the operators $\mathcal{J}_{\mathcal{X}}$ and $\mathcal{J}_{\mathcal{W}}$ is nontrivial (i.e., not equal to $\pm 1_{\mathcal{X}}$ or $\pm 1_{\mathcal{W}}$. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be the eigenspaces of $\mathcal{J}_{\mathcal{X}}$ with respect to the eighenvalues $+1$ and $-1$, respectively, and let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the eigenspaces of $\mathcal{J}_{\mathcal{W}}$ with respect to the eighenvalues $+1$ and $-1$, respectively. Then at least one of the decompositions $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{W} = \mathcal{W}_1 \mathcal{W}_2$ is nontrivial, and with respect to these decompositions, $\mathcal{J}_{\mathcal{X}}$ and $\mathcal{J}_{\mathcal{W}}$ has the block decomposition (7.4). From this decomposition follows that $V_i := V \cap \left[\begin{smallmatrix} \mathcal{X}_i \\ \mathcal{X}_i \\ \mathcal{W}_i \end{smallmatrix}\right] \subset V$, and that (7.2) holds. Thus $\Sigma$ is decomposable in this case.

If instead $(\mathcal{J}_{\mathcal{X}}, \mathcal{J}_{\mathcal{W}}) = (1_{\mathcal{X}}, -1_{\mathcal{W}})$ or $(\mathcal{J}_{\mathcal{X}}, \mathcal{J}_{\mathcal{W}}) = (-1_{\mathcal{X}}, 1_{\mathcal{W}})$ then both the above decompositions are trivial, and we have to proceed differently. In these cases it follows from the signature invariance of $\Sigma$ that (7.4) holds in both cases with

$$V_1 := V \cap \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \{0\} \end{bmatrix} \subset V \text{ and } V_2 := V \cap \begin{bmatrix} \{0\} \\ \{0\} \\ \mathcal{W} \end{bmatrix} \subset V.$$

Hence, also in this case $\Sigma$ is decomposable (into two noninteracting systems, one with a zero state space, and the other with a zero signal space).

*Proof of (ii).* The proof of (ii) is analogous to the proof of (i) (but slightly simpler), and it is left to the reader. □

**Theorem 7.12.** *If $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a passive s/s system which belongs to one of the classes a)–f) listed in Section 1, then $\Sigma$ is non-decomposable if and only if its two-sided behavior is non-decomposable (or equivalently, its future or past behavior is non-decomposable).*

*Proof.* This follows from Theorems 7.7 and 7.11. □

# 8 Passive Transpose Invariant State/Signal Systems and Behaviours

In this section we present one final class of symmetries of passive s/s systems and passive behaviours, called transpose invariance. This notion is can be regarded as a slightly modified version reciprocity, where one has replaced the adjoint $\Sigma_*$ of a system $\Sigma$ by a *transpose* $\Sigma^T$ of $\Sigma$. The difference between $\Sigma^T$ and $\Sigma_*$ is analogous to the difference between a transpose $A^T$ of a matrix $A$ and the Hermitian adjoint $A^*$ of $A$. The mapping from $A$ into $A^T$ is linear, whereas the mapping of $A$ into $A^*$ is conjugate-linear. One way to define the matrix $A^T$ is to identify it with the operator that one gets by multiplying the operator induced by $A^*$ by conjugation operators to the left and the right. The same idea can be used to define the notion of a transpose of a general operator $A \in \mathcal{B}(\mathcal{U};\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn (or Hilbert) spaces: one fixes two conjugation operators $\mathcal{C}_\mathcal{U}$ and $\mathcal{C}_\mathcal{Y}$ in $\mathcal{U}$ and $\mathcal{Y}$, respectively, and calls the operator $A^T = \mathcal{C}_\mathcal{Y} A^* \mathcal{C}_\mathcal{U} \in \mathcal{B}(\mathcal{Y};\mathcal{U})$ a *transpose of* $A$. Clearly $A^T$ depends not only on $A$, but also on the two conjugation operators $C_\mathcal{U}$ and $\mathcal{C}_\mathcal{Y}$. The notion of a *transpose* $\Sigma^T$ *of a passive s/s system* $\Sigma = (V;\mathcal{X},\mathcal{W})$ can be defined in an analogous way by fixing a conjugation $\mathcal{C}_\mathcal{X}$ in $\mathcal{X}$ and a skew-conjugation operator $\mathcal{B}_\mathcal{W}$ in $\mathcal{X}$ and $\mathcal{W}$, respectively, and letting $\Sigma^T = (V^T;\mathcal{X},\mathcal{W})$ be the s/s system whose generating subspace is

$$V^T = \begin{bmatrix} \mathcal{C}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{C}_\mathcal{W} & 0 \\ 0 & 0 & \mathcal{B}_\mathcal{W}\mathcal{I}_{(-\mathcal{W},\mathcal{W})} \end{bmatrix} V_*.$$

where $V_*$ is the generating subspace of the adjoint $\Sigma_*$ of $\Sigma$ and $\mathcal{B}_\mathcal{W}\mathcal{I}_{(-\mathcal{W},\mathcal{W})}$ is a unitary conjugate-linear operator from $-\mathcal{W}$ to $\mathcal{W}$. It is easy to see that $\Sigma^T$ is a passive s/s system. According to (1.18) and (1.21), $\Sigma$ is $(\mathcal{C}_\mathcal{X},\mathcal{B}_\mathcal{W})$-transpose invariant if $V = V^T$. Comparing this to the definition of reciprocal symmetry of $\Sigma$, wee see that the difference between the transpose symmetry and the reciprocal symmetry is that we replace the signature operator $\mathcal{J}_\mathcal{X}$ and the anti-signature operator $\mathcal{I}_\mathcal{W}$ in (1.16) and (1.20) by a conjugation $\mathcal{C}_\mathcal{X}$ and a anti-conjugation $\mathcal{B}_\mathcal{W}$, respectively.

Motivated by this discussion, we arrive at the following definition.

**Definition 8.1.** A passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ is called $(\mathcal{C}_\mathcal{X};\mathcal{B}_\mathcal{W})$-transpose invariant if (1.18) (or equivalently, (1.21)) holds, where $\mathcal{C}_\mathcal{X}$ is a conjugation $\mathcal{C}_\mathcal{X}$ in $\mathcal{X}$ and $\mathcal{B}_\mathcal{W}$ is a skew-conjugation in $\mathcal{W}$.

**Definition 8.2.** Let $\mathcal{B}_\mathcal{W}$ be a skew-conjugation in the Kreĭn space $\mathcal{W}$.

(i) A passive two-sided behaviour $\mathfrak{W}$ on $\mathcal{W}$ is called $\mathcal{B}_{\mathcal{W}}$-transpose invariant if

$$\mathfrak{W} = \mathcal{B}_{\mathcal{W}} \mathbf{A} \mathfrak{W}^{[\perp]} \tag{8.1}$$

(here the skew-conjugation $\mathcal{B}_{\mathcal{W}}$ on $K^2(\mathcal{W})$ induced by the skew-conjugation $\mathcal{B}_{\mathcal{W}} \in \mathcal{B}(\mathcal{W})$ is defined as in Remark 1.1).

(ii) A passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called *externally $\mathcal{B}_{\mathcal{W}}$-transpose invariant* if its two-sided behaviour is $\mathcal{B}_{\mathcal{W}}$-transpose invariant.

**Remark 8.3.** It is possible to develop a symmetry theory which is completely analogous to the one in Section 6 by replacing the signature operator $\mathcal{J}_{\mathcal{X}}$ by a conjugation $\mathcal{C}_{\mathcal{X}}$ and the skew-signature operator $\mathcal{I}_{\mathcal{W}}$ by a skew-conjugation $\mathcal{B}_{\mathcal{W}}$. This has the effect of converting the notions of $(\mathcal{J}_{\mathcal{X}}, \mathcal{I}_{\mathcal{W}})$-reciprocity and $\mathcal{I}_{\mathcal{W}}$-reciprocity introduced in Definitions 6.2 and 6.8 into the notions of $(\mathcal{C}_{\mathcal{X}}, \mathcal{B}_{\mathcal{W}})$-transpose invariance and $\mathcal{B}_{\mathcal{W}}$-transpose invariance introduced in Definitions 8.1 and 8.2. In particular, all the lemmas, theorems, and corollaries in Section 6 remain valid with these replacements. All the proofs remain essentially the same.

In particular, the following results are true:

**Lemma 8.4.** *If $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a simple passive s/s $(\mathcal{C}_{\mathcal{X}}; \mathcal{B}_{\mathcal{W}})$-transpose invariant system, then $\mathcal{C}_{\mathcal{X}}$ is uniquely determined by $\Sigma$ and $\mathcal{B}_{\mathcal{W}}$.*

*Proof.* This is the conjugate-linear analogue of Lemma 6.7. □

**Lemma 8.5.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive $(\mathcal{C}_{\mathcal{X}}; \mathcal{B}_{\mathcal{W}})$-transpose invariant system, and let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be unitarily similar to $\Sigma$ with similarity operator $\mathcal{V}$. Then $\Sigma_1$ is $(\mathcal{C}_{\mathcal{X}_1}; \mathcal{B}_{\mathcal{W}})$-transpose invariant with $\mathcal{C}_{\mathcal{X}_1} = \mathcal{V} \mathcal{C}_{\mathcal{X}} \mathcal{V}^{-1}$.*

*Proof.* This is the conjugate-linear analogue of Lemma 5.5. □

**Lemma 8.6.** *If $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a passive $(\mathcal{C}_{\mathcal{X}}, \mathcal{B}_{\mathcal{W}})$-transpose invariant s/s system, then it is externally $\mathcal{B}_{\mathcal{W}}$-transpose invariant.*

*Proof.* This is the conjugate-linear analogue of Lemma 6.9. □

**Theorem 8.7.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive externally $\mathcal{B}_{\mathcal{W}}$-transpose invariant s/s system which is either simple and conservative or minimal and passive balanced. Then there exists a unique conjugation $\mathcal{C}_{\mathcal{X}}$ in $\mathcal{X}$ such that $\Sigma$ is $(\mathcal{C}_{\mathcal{X}}, \mathcal{B}_{\mathcal{W}})$-transpose invariant.*

*Proof.* This is the conjugate-linear analogue of Theorems 6.13 and 6.17. □

# 9 Doubly Symmetric Passive State/Signal Systems and Behaviours

In this section we study passive s/s systems and behaviours that are invariant with respect to two different symmetries of the types that we have considered in Sections 5–8, and in addition, the operators associated with the two symmetries commute with each other. We show that in this case the given system or behavior is invariant also with respect to a third symmetry, namely the product of the operators that define the two original symmetries.

In order to be able to discuss all the different types of symmetries in a coherent way we start by making the following agreement.

**Agreement 9.1.** Let $\mathcal{W}$ be a Kreĭn space.

(i) In this section, by a *symmetry in* $\mathcal{W}$ we mean an operator $\mathcal{G}$ which is a signature operator, or a conjugation, or a skew-signature operator, or a skew-conjugation.

(ii) Let $\mathcal{G}$ be a symmetry in $\mathcal{W}$. We call a two-sided passive behavior $\mathfrak{W}$ on $\mathcal{W}$ $\mathcal{G}$*-symmetric* if

  (a) $\mathfrak{W}$ is $\mathcal{G}$-real in the case where $\mathcal{G}$ is a conjugation;

  (b) $\mathfrak{W}$ is $\mathcal{G}$-reciprocal in the case where $\mathcal{G}$ is a skew-signature operator;

  (c) $\mathfrak{W}$ is $\mathcal{G}$-signature invariant in the case where $\mathcal{G}$ is a signature operator;

  (d) $\mathfrak{W}$ is $\mathcal{G}$-transpose invariant in the case where $\mathcal{G}$ is a skew-conjugation.

**Lemma 9.2.** *If* $\mathcal{G}$ *is a symmetry of one of the types listed in Agreement 9.1, then* $-\mathcal{G}$ *is a symmtery of the same type.*

This is obvious.

**Agreement 9.3.** Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two symmetries of the type listed in Agreement 9.1. We we say that $\mathcal{G}_1$ and $\mathcal{G}_2$ are essentially different if $\mathcal{G}_1 \neq \pm\mathcal{G}_2$.

As the following lemma shows, a commuting product of two symmetries is again a symmetry.

**Lemma 9.4.** *Let* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *be two symmetries of the type listed in Agreement 9.1, and suppose that* $\mathcal{G}_1\mathcal{G}_2 = \mathcal{G}_2\mathcal{G}_1$*. Then also* $\mathcal{G}_3 := \mathcal{G}_1\mathcal{G}_2 = \mathcal{G}_2\mathcal{G}_1$ *is a symmetry of the type listed in Agreement 9.1. All the symmetries* $\mathcal{G}_i$*,* $i = 1, 2, 3$*, commmute with each other, and the product of two of these symmetries is equal to the third. The exact type of the three symmetries* $\mathcal{G}_i$*,* $i = 1, 2, 3$ *can be determined from the following rules:*

(i) *All the above symmetries are of the same type if and only if they are all signature operators;*

(ii) *Two of the above symmetries are of the same type if and only if the third symmetry is a signature operator;*

(iii) *The above symmetries are all of different type if and only if one of them is a conjugation, another is a skew-signature operator, and the third is a skew-conjugation.*

The easy proof is left to the reader.

**Theorem 9.5.** *Let $\mathfrak{W}$ be a passive behavior in $\mathcal{W}$, and suppose that $\mathfrak{W}$ is both $\mathcal{G}_1$-symmetric and $\mathcal{G}_2$-symmetric, where each of these symmetries belongs to one of the classes listed in Agreement 9.1. In addition, suppose that $\mathcal{G}_1\mathcal{G}_2 = \mathcal{G}_2\mathcal{G}_1$. Then $\mathfrak{W}$ is also $\mathcal{G}_3$-symmetric, where $\mathcal{G}_3 = \mathcal{G}_1\mathcal{G}_2 = \mathcal{G}_2\mathcal{G}_1$. The type of the third symmetry can be determined from Lemma 9.4.*

*Proof.* The proofs of the different subcases are analogous to each other, so let us only prove the case which is maybe most interesting, namely the one where $\mathcal{G}_1$ is a conjugation and $\mathcal{G}_2$ is a skew-signature operator (or the other way around), which means that $\mathfrak{W}$ is both $\mathcal{G}_1$-real and $\mathcal{G}_2$-reciprocal. In this case Theorem 9.5 claims that $\mathfrak{W}$ is also $\mathcal{G}_3$-transpose symmetric, where $\mathcal{G}_3 = \mathcal{G}_1\mathcal{G}_2 = \mathcal{G}_2\mathcal{G}_1$. This can be shown as follows. For simplicity, let us denote $\mathcal{G}_1$ by $\mathcal{C}_\mathcal{W}$, $\mathcal{G}_2$ by $\mathcal{I}_\mathcal{W}$, and $\mathcal{G}_3$ by $\mathcal{B}_\mathcal{W}$. By Lemma 9.4, $\mathcal{B}_\mathcal{W} = \mathcal{C}_\mathcal{W}\mathcal{I}_\mathcal{W}$ is a skew-conjugation. Moreover, by Definitions 5.9 and 6.8,

$$\mathcal{B}_\mathcal{W}\mathbf{\mathfrak{R}}\mathfrak{W}^{[\perp]} = \mathcal{C}_\mathcal{W}\mathcal{I}_\mathcal{W}\mathbf{\mathfrak{R}}\mathfrak{W}^{[\perp]} = \mathcal{C}_\mathcal{W}\mathfrak{W} = \mathfrak{W}.$$

According to Definition 8.2, this means shows that $\mathfrak{W}$ is $\mathcal{B}_\mathcal{W}$-transpose invariant. □

Above we have been looking at passive two-sided *behaviors* which are doubly symmetric in the sense that they are invariant with respect to two commuting symmetries. An analogous result is true for passive s/s systems with two commuting symmetries.

**Agreement 9.6.** Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, let $\mathcal{G}_\mathcal{X}$ be a signature operator or a conjugation in $\mathcal{X}$, and let $\mathcal{G}_\mathcal{W}$ be a symmetry in $\mathcal{W}$. We call $\Sigma$ $(\mathcal{G}_\mathcal{X}, \mathcal{G}_\mathcal{W})$*-symmetric* if

(i) $\Sigma$ is $(\mathcal{G}_\mathcal{X}, \mathcal{G}_\mathcal{W})$-real in the case where $\mathcal{G}_\mathcal{X}$ and $\mathcal{G}_\mathcal{W}$ are conjugations;

(ii) $\Sigma$ is $(\mathcal{G}_\mathcal{X}, \mathcal{G}_\mathcal{W})$-reciprocal in the case where $\mathcal{G}_\mathcal{X}$ is a signature operator and $\mathcal{G}_\mathcal{W}$ is a skew-signature operator;

(iii) $\Sigma$ is $(\mathcal{G}_\mathcal{X}, \mathcal{G}_\mathcal{W})$-signature invariant in the case where $\mathcal{G}_\mathcal{X}$ and $\mathcal{G}_\mathcal{W}$ are signature operators;

(iv) $\Sigma$ is $(\mathcal{G}_\mathcal{X}, \mathcal{G}_\mathcal{W})$-transpose invariant in the case where $\mathcal{G}_\mathcal{X}$ is a conjugation operator and $\mathcal{G}_\mathcal{W}$ is a skew-conjugation.

**Theorem 9.7.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and suppose that $\Sigma$ is both $(\mathcal{G}_\mathcal{X}^{(1)}, \mathcal{G}_\mathcal{W}^{(1)})$-symmetric and $(\mathcal{G}_\mathcal{X}^{(2)}, \mathcal{G}_\mathcal{W}^{(2)})$-symmetric, where each of these symmetries belongs to one of the classes listed in Agreement 9.6.*

(i) *If $\mathcal{G}_\mathcal{W}^{(1)}\mathcal{G}_\mathcal{W}^{(2)} = \mathcal{G}_\mathcal{W}^{(2)}\mathcal{G}_\mathcal{W}^{(1)}$ and $\mathcal{G}_\mathcal{X}^{(1)}\mathcal{G}_\mathcal{X}^{(2)} = \mathcal{G}_\mathcal{X}^{(2)}\mathcal{G}_\mathcal{X}^{(1)}$, then $\Sigma$ is $(\mathcal{G}_\mathcal{X}^{(3)}, \mathcal{G}_\mathcal{W}^{(3)})$-symmetric, where $\mathcal{G}_\mathcal{X}^{(3)} = \mathcal{G}_\mathcal{X}^{(1)}\mathcal{G}_\mathcal{X}^{(2)} = \mathcal{G}_\mathcal{X}^{(2)}\mathcal{G}_\mathcal{X}^{(1)}$ and $\mathcal{G}_\mathcal{W}^{(3)} = \mathcal{G}_\mathcal{W}^{(1)}\mathcal{G}_\mathcal{W}^{(2)} = \mathcal{G}_\mathcal{W}^{(2)}\mathcal{G}_\mathcal{W}^{(1)}$. All the symmetries $\mathcal{G}_\mathcal{W}^{(i)}$, $i = 1, 2, 3$, commmute with each other, and the product of two of these symmetries is equal to the third, and the same result holds for the three symmetries $\mathcal{G}_\mathcal{X}^{(i)}$, $i = 1, 2, 3$, too. The exact type of the three pairs of symmetries $(\mathcal{G}_\mathcal{X}^{(i)}, \mathcal{G}_\mathcal{W}^{(i)})$ can be determined from the following rules:*

   *(a) All the above symmetries are of the same type if and only if they are all signature invariances;*

   *(b) Two of the above symmetries are of the same type if and only if the third symmetry is a signature invariance;*

   *(c) The above symmetries are all of different type if and only if one of them is a reality, another is a reciprocity, and the third is a transpose invariance.*

(ii) *If $\mathcal{G}_\mathcal{W}^{(1)}\mathcal{G}_\mathcal{W}^{(2)} = \mathcal{G}_\mathcal{W}^{(2)}\mathcal{G}_\mathcal{W}^{(1)}$ and $\Sigma$ is simple, then $\mathcal{G}_\mathcal{X}^{(1)}\mathcal{G}_\mathcal{X}^{(2)} = \mathcal{G}_\mathcal{X}^{(2)}\mathcal{G}_\mathcal{X}^{(1)}$ (and hence the conclusion of (i) holds).*

*Proof. Proof of (i).* The proofs of the different subcases are again analogous to each other, so let us only prove, for example, the case where $\Sigma$ is both $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real and $(\mathcal{J}_\mathcal{X}, \mathcal{I}_\mathcal{W})$-reciprocal, and we claim that $\Sigma$ is also $(\mathcal{C}_\mathcal{X}^{(3)}, \mathcal{B}_\mathcal{W})$-transpose invariant, where $\mathcal{C}_\mathcal{X}^{(3)} := \mathcal{C}_\mathcal{X}\mathcal{J}_\mathcal{X}$ and $\mathcal{B}_\mathcal{W} := \mathcal{C}_\mathcal{W}\mathcal{I}_\mathcal{W}$. By Lemma 9.4, $\mathcal{C}_\mathcal{X}^{(3)}$ is a conjugation and $\mathcal{B}_\mathcal{W}$ is a skew-conjugation. Moreover, by Definitions 5.2 and 6.2,

$$\begin{aligned}\begin{bmatrix} -\mathcal{C}_\mathcal{X}^{(3)} & 0 & 0 \\ 0 & \mathcal{C}_\mathcal{X}^{(3)} & 0 \\ 0 & 0 & \mathcal{B}_\mathcal{W} \end{bmatrix} V^{[\perp]} &= \begin{bmatrix} \mathcal{C}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{C}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{C}_\mathcal{W} \end{bmatrix} \begin{bmatrix} -\mathcal{J}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{J}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{I}_\mathcal{W} \end{bmatrix} V^{[\perp]} \\ &= \begin{bmatrix} \mathcal{C}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{C}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{C}_\mathcal{W} \end{bmatrix} V = V. \end{aligned} \tag{9.1}$$

According to Definition 8.1, this means shows that $\Sigma$ is $(\mathcal{C}_{\mathcal{X}}^{(3)}, \mathcal{B}_{\mathcal{W}})$-transpose invariant.

*Proof of (ii).* Let us again only prove the case where $\Sigma$ is simple and both $(\mathcal{C}_{\mathcal{X}}, \mathcal{C}_{\mathcal{W}})$-real and $(\mathcal{J}_{\mathcal{X}}, \mathcal{I}_{\mathcal{W}})$-reciprocal. We can continue the computation in (9.1) to get

$$\begin{aligned}\begin{bmatrix} -\mathcal{C}_{\mathcal{X}}\mathcal{J}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{C}_{\mathcal{X}}\mathcal{J}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{B}_{\mathcal{W}} \end{bmatrix} V^{[\perp]} = V &= \begin{bmatrix} \mathcal{C}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{C}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{C}_{\mathcal{W}} \end{bmatrix} V^{[\perp]} \\ &= \begin{bmatrix} -\mathcal{J}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{I}_{\mathcal{W}} \end{bmatrix} \begin{bmatrix} \mathcal{C}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{C}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{C}_{\mathcal{W}} \end{bmatrix} V^{[\perp]} \\ &= \begin{bmatrix} -\mathcal{J}_{\mathcal{X}}\mathcal{C}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}}\mathcal{C}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{B}_{\mathcal{W}} \end{bmatrix} V^{[\perp]}.\end{aligned}$$

The equality between the first and last terms in this chain can be rewritten as

$$\begin{bmatrix} \mathcal{C}_{\mathcal{X}}\mathcal{J}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{J}_{\mathcal{X}}\mathcal{C}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{B}_{\mathcal{W}} \end{bmatrix} V_* = \begin{bmatrix} \mathcal{J}_{\mathcal{X}}\mathcal{C}_{\mathcal{X}} & 0 & 0 \\ 0 & \mathcal{C}_{\mathcal{X}}\mathcal{J}_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{B}_{\mathcal{W}} \end{bmatrix} V_*.$$

By Lemma 3.51, $\mathcal{C}_{\mathcal{X}}\mathcal{J}_{\mathcal{X}} = \mathcal{J}_{\mathcal{X}}\mathcal{C}_{\mathcal{X}}$. □

As the following theorem shows, it is also true that double external symmetry of a passive s/s systems which is either simple and conservative or minimal and balanced implies double full symmetry.

**Theorem 9.8.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system which is either simple and conservative or minimal and balanced, and suppose that the behavior $\mathfrak{W}$ of of $\Sigma$ is both $\mathcal{G}_{\mathcal{W}}^{(1)}$-symmetric and $\mathcal{G}_{\mathcal{W}}^{(2)}$-symmetric, where $\mathcal{G}_{\mathcal{W}}^{(1)}\mathcal{G}_{\mathcal{W}}^{(2)} = \mathcal{G}_{\mathcal{W}}^{(2)}\mathcal{G}_{\mathcal{W}}^{(1)}$ and each of these symmetries belongs to one of the classes listed in Agreement 9.1. Then there exists unique symmetries $\mathcal{G}_{\mathcal{X}}^{(1)}$ and $\mathcal{G}_{\mathcal{X}}^{(2)}$ in $\mathcal{X}$ such that $\Sigma$ is both $(\mathcal{G}_{\mathcal{X}}^{(1)}, \mathcal{G}_{\mathcal{W}}^{(1)})$-symmetric and $(\mathcal{G}_{\mathcal{X}}^{(2)}, \mathcal{G}_{\mathcal{W}}^{(2)})$-symmetric. Moreover, $\mathcal{G}_{\mathcal{X}}^{(1)}\mathcal{G}_{\mathcal{X}}^{(2)} = \mathcal{G}_{\mathcal{X}}^{(2)}\mathcal{G}_{\mathcal{X}}^{(1)}$, and $\Sigma$ is also $(\mathcal{G}_{\mathcal{X}}^{(3)}, \mathcal{G}_{\mathcal{W}}^{(3)})$-symmetric, where $\mathcal{G}_{\mathcal{X}}^{(3)} = \mathcal{G}_{\mathcal{X}}^{(1)}\mathcal{G}_{\mathcal{X}}^{(2)} = \mathcal{G}_{\mathcal{X}}^{(2)}\mathcal{G}_{\mathcal{X}}^{(1)}$ and $\mathcal{G}_{\mathcal{W}}^{(3)} = \mathcal{G}_{\mathcal{W}}^{(1)}\mathcal{G}_{\mathcal{W}}^{(2)} = \mathcal{G}_{\mathcal{W}}^{(2)}\mathcal{G}_{\mathcal{W}}^{(1)}$. All the symmetries $\mathcal{G}_{\mathcal{W}}^{(i)}$, $i = 1, 2, 3$, commmute with each other, and the product of two of these symmetries is equal to the third, and the same result holds for the three symmetries $\mathcal{G}_{\mathcal{X}}^{(i)}$, $i = 1, 2, 3$, too. The exact type of the three pairs of symmetries $(\mathcal{G}_{\mathcal{X}}^{(i)}, \mathcal{G}_{\mathcal{W}}^{(i)})$ can be determined from Theorem 9.7.*

*Proof.* This follows from Theorems 5.14, 5.23, 6.13, 6.17, 7.7, 8.7, and 9.7 combined with Lemma 9.4. □

**Theorem 9.9.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system which belongs to one of the classes a)–f) listed in Section 1. If the behavior $\mathfrak{W}$ of $\Sigma$ is both $\mathcal{G}_{\mathcal{W}}^{(1)}$-symmetric and $\mathcal{G}_{\mathcal{W}}^{(2)}$-symmetric, where $\mathcal{G}_{\mathcal{W}}^{(1)}\mathcal{G}_{\mathcal{W}}^{(2)} = \mathcal{G}_{\mathcal{W}}^{(2)}\mathcal{G}_{\mathcal{W}}^{(1)}$ and each of $\mathcal{G}_{\mathcal{W}}^{(1)}$ and $\mathcal{G}_{\mathcal{W}}^{(2)}$ is either a signature operator or a conjugation, then there exists unique symmetries $\mathcal{G}_{\mathcal{X}}^{(1)}$ and $\mathcal{G}_{\mathcal{X}}^{(2)}$ in $\mathcal{X}$, which are either signature operators or conjugations such that $\Sigma$ is both $(\mathcal{G}_{\mathcal{X}}^{(1)}, \mathcal{G}_{\mathcal{W}}^{(1)})$-symmetric and $(\mathcal{G}_{\mathcal{X}}^{(2)}, \mathcal{G}_{\mathcal{W}}^{(2)})$-symmetric. Moreover, $\mathcal{G}_{\mathcal{X}}^{(1)}\mathcal{G}_{\mathcal{X}}^{(2)} = \mathcal{G}_{\mathcal{X}}^{(2)}\mathcal{G}_{\mathcal{X}}^{(1)}$, and $\Sigma$ is also $(\mathcal{G}_{\mathcal{X}}^{(3)}, \mathcal{G}_{\mathcal{W}}^{(3)})$-symmetric, where $\mathcal{G}_{\mathcal{X}}^{(3)} = \mathcal{G}_{\mathcal{X}}^{(1)}\mathcal{G}_{\mathcal{X}}^{(2)} = \mathcal{G}_{\mathcal{X}}^{(2)}\mathcal{G}_{\mathcal{X}}^{(1)}$ and $\mathcal{G}_{\mathcal{W}}^{(3)} = \mathcal{G}_{\mathcal{W}}^{(1)}\mathcal{G}_{\mathcal{W}}^{(2)} = \mathcal{G}_{\mathcal{W}}^{(2)}\mathcal{G}_{\mathcal{W}}^{(1)}$ are signature operators or conjugations. All the symmetries $\mathcal{G}_{\mathcal{W}}^{(i)}$, $i = 1, 2, 3$, commmute with each other, and the product of two of these symmetries is equal to the third, and the same result holds for the three symmetries $\mathcal{G}_{\mathcal{X}}^{(i)}$, $i = 1, 2, 3$, too. The exact type of the three pairs of symmetries $(\mathcal{G}_{\mathcal{X}}^{(i)}, \mathcal{G}_{\mathcal{W}}^{(i)})$ can be determined from following rule: Either all of these symmetries are signature invariances, or two of them are realities, and the third is a singature invariance.*

*Proof.* This follows from Theorems 5.14, 5.15, 5.16, 5.21, 5.22, 5.23, 7.7, and 9.7 combined with Lemma 9.4. □

**Agreement 9.10.** Two symmetries $\mathcal{G}_1$ and $\mathcal{G}_2$ of the types listed in Agreement 9.1 are *essentially different* if $\mathcal{G}_1 \neq \pm\mathcal{G}_2$. Two pairs of symmetries $(\mathcal{G}_{\mathcal{X}}^{(i)}, \mathcal{G}_{\mathcal{W}}^{(i)})$, $i = 1, 2$ are *essentially different* if $(\mathcal{G}_{\mathcal{X}}^{(1)}, \mathcal{G}_{\mathcal{W}}^{(1)}) \neq (\mathcal{G}_{\mathcal{X}}^{(2)}, \mathcal{G}_{\mathcal{W}}^{(2)})$ and $(\mathcal{G}_{\mathcal{X}}^{(2)}, \mathcal{G}_{\mathcal{W}}^{(2)}) \neq (-\mathcal{G}_{\mathcal{X}}^{(1)}, -\mathcal{G}_{\mathcal{W}}^{(1)})$.

**Theorem 9.11.**

(i) *A non-decomposable passive two-sided behavior $\mathfrak{W}$ in $\mathcal{W}$ cannot have two essentially different commuting symmetries $\mathcal{G}_{\mathcal{W}}^{(i)}$, $i = 1, 2$, which both belong to the* same class *of symmteries considered in Agreement 9.1.*

(ii) *A non-decomposable passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ cannot have two essentially different commuting symmetries $(\mathcal{G}_{\mathcal{W}}^{(i)}, \mathcal{G}_{\mathcal{W}}^{(i)})$, $i = 1, 2$, which both belong to the* same class *of symmteries considered in Agreement 9.6.*

*Proof.* This follows from Theorems 7.11, 9.5, and 9.7.

□

# 10 The Characteristic Bundles of Passive State/Signal Systems and Behaviours

In this section we return to the notions of the characteristic node and signal boundels of a passive s/s system that was mentioned in the introduction, and which serve as frequency domain characteristics of such systems.

## 10.1 The characteristic node bundle

**Definition 10.1.** The *characteristic node bundle* $\widehat{\mathfrak{E}}$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is the family (1.9) of subspaces $\{\widehat{\mathfrak{E}}(\lambda)\}_{\lambda\in\mathbb{C}}$, of the Kreĭn node space $\mathfrak{K}$. The subspace $\widehat{\mathfrak{E}}(\lambda)$ is called the *fiber* of $\widehat{\mathfrak{E}}$ at $\lambda \in \mathbb{C}$.

**Lemma 10.2.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with characteristic node bundle $\widehat{\mathfrak{E}}$, and let $\lambda \in \mathbb{C}$. Then*

$$\begin{aligned}
\widehat{\mathfrak{E}}(\lambda) &= \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V, & V &= \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \widehat{\mathfrak{E}}(\lambda), \\
\widehat{\mathfrak{E}}(\lambda)^{[\perp]} &= \begin{bmatrix} 1_{\mathcal{X}} & \overline{\lambda} & 0 \\ 0 & -1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V^{[\perp]}, & V^{[\perp]} &= \begin{bmatrix} 1_{\mathcal{X}} & \overline{\lambda} & 0 \\ 0 & -1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \widehat{\mathfrak{E}}(\lambda)^{[\perp]}.
\end{aligned} \tag{10.1}$$

*Proof.* This follows from (1.9), Lemma 2.3, and the fact that

$$\begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix}^* = \begin{bmatrix} 1_{\mathcal{X}} & \overline{\lambda} & 0 \\ 0 & -1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \text{ and } \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix}^{-1} = \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \tag{10.2}$$

where the adjoint has been computed with respect to the inner product (1.6) in the node space $\mathfrak{K}$. □

**Remark 10.3.** Formulas (10.1) show that any one of the fibers $\widehat{\mathfrak{E}}(\lambda)$ together with the value of $\lambda$ determines the generating subspace $V$ and all the whole characteristic bundle $\widehat{\mathfrak{E}}$ uniquely. Of course, the generating subspace $V$ itself also determines $\widehat{\mathfrak{E}}$ uniquely.

**Theorem 10.4.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with characteristic node bundle $\widehat{\mathfrak{E}}$, and let $\Sigma_* = (V_*; \mathcal{X}, -\mathcal{W})$ be the adjoint system with characteristic node bundle $\widehat{\mathfrak{E}}_*$. Then*

$$\widehat{\mathfrak{E}}_*(\lambda) = \begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 \\ 0 & -1_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{I}_{(\mathcal{W},-\mathcal{W})} \end{bmatrix} \widehat{\mathfrak{E}}(\overline{\lambda})^{[\perp]}, \qquad \lambda \in \mathbb{C}. \tag{10.3}$$

*Proof.* By (1.19) and (10.1), appled both to the original system $\Sigma$ and the adoint $\Sigma_*$, for all $\lambda \in \mathbb{C}$,

$$\begin{aligned}\widehat{\mathfrak{E}}_*(\lambda) = \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{-\mathcal{W}} \end{bmatrix} V_* &= \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{-\mathcal{W}} \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{I}_{(\mathcal{W},-\mathcal{W})} \end{bmatrix} V^{[\perp]} \\ &= \begin{bmatrix} 1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{I}_{(\mathcal{W},-\mathcal{W})} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & \lambda & 0 \\ 0 & -1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \widehat{\mathfrak{E}}(\overline{\lambda})^{[\perp]} \\ &= \begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 \\ 0 & -1_{\mathcal{X}} & 0 \\ 0 & 0 & \mathcal{I}_{(\mathcal{W},-\mathcal{W})} \end{bmatrix} \widehat{\mathfrak{E}}(\overline{\lambda})^{[\perp]}. \quad \square\end{aligned}$$

## 10.2 The connection between stable future trajectories and the characteristic node bundle

In this subsection we establish a connection between the Laplace transforms of stable future trajectories of $\Sigma$ and the characteristic node bundle of $\Sigma$. We begin with some preliminary lemmas.

**Lemma 10.5.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and let $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ be a fundamental decomposition of $\mathcal{W}$. Then, for each $x_0 \in \mathcal{X}$ and each $u \in L^2(\mathbb{R}^+; \mathcal{U})$, there exists a unique stable future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$ satisfying $x(0) = x_0$ and $P_{\mathcal{U}} w = u$.*

*Proof.* See [AKS11b, Lemma 3.4(i)]. $\square$

**Lemma 10.6.** *If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of the passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, then there exists a sequence of classical generated stable future trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ of $\Sigma$ such that $x_n \to x$ uniformly on $\mathbb{R}^+$ and $w_n \to w$ in $L^2(\mathbb{R}^+; \mathcal{W})$ as $n \to \infty$. If $x(0) = 0$, then we can require, in addition, that $x_n(0) = 0$ and $w_n(0) = 0$.*

*Proof.* See [AKS11b, Lemmas 3.6 and 3.9(i)]. $\square$

For each $w \in K^2_+(\mathcal{W})$ and $x \in L^\infty(\mathbb{R}^+; \mathcal{X})$ we define the *Laplace transform* of $w$ and $x$ by

$$\widehat{w}(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda t} w(t)\, \mathrm{d}t, \quad \widehat{x}(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda t} x(t)\, \mathrm{d}t, \quad \lambda \in \mathbb{C}^+. \tag{10.4}$$

The image of the Kreĭn space $K^2_+(\mathcal{W})$ under the Laplace transform is another Kreĭn space that we denote by $\widehat{K}^2_+(\mathcal{W})$. Thus,

$$\widehat{K}^2_+(\mathcal{W}) := \left\{ \widehat{w} \,\middle|\, w \in L^2(\mathbb{R}^+; \mathcal{W}) \right\}. \tag{10.5}$$

As a topological vector space the space $\widehat{K}^2_+(\mathcal{W})$ coincides with the Hardy space $H^2(\mathbb{C}^+;\mathcal{W})$ of holomorphic $\mathcal{W}$-valued functions on $\mathbb{C}^+$ with finite $H^2$-norm, defined by

$$\|\widehat{w}\|^2_{H^2(\mathbb{C}^+;\mathcal{W})} = \sup_{\mu>0} \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\widehat{w}(\mu+i\omega)\|^2_{\mathcal{W}}\,\mathrm{d}\omega,$$

where $\|\cdot\|_{\mathcal{W}}$ is some admissible norm in $\mathcal{W}$. The inner product in $\widehat{K}^2_+(\mathcal{W})$ is given by

$$[\widehat{w}_1,\widehat{w}_2]_{\widehat{K}^2_+(\mathcal{W})} := \frac{1}{2\pi}\int_{-\infty}^{\infty} [\widehat{w}_1(i\omega),\widehat{w}_2(i\omega)]_{\mathcal{W}}\,\mathrm{d}\omega, \tag{10.6}$$

where $\widehat{w}_1$ and $\widehat{w}_2$ have been defined a.e. on the imaginary axis to be equal to their nontangential limits from the right. By the Payley–Wiener theorem, the Laplace transform is a unitary map of $K^2_+(\mathcal{W})$ onto $\widehat{K}^2_+(\mathcal{W})$, and if $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ is a fundamental decomposition of $\mathcal{W}$, then

$$\widehat{K}^2_+(\mathcal{W}) = H^2(\mathbb{C}^+;\mathcal{U}) \boxplus -H^2(\mathbb{C}^+;\mathcal{Y}) \tag{10.7}$$

is a fundamental decomposition of $\widehat{K}^2_+(\mathcal{W})$.

**Theorem 10.7.** *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a passive s/s system with characteristic node bundle $\widehat{\mathfrak{E}}$.*

(i) *If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future trajectory of $\Sigma$ with initial state $x_0$, then the Laplace transform $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$ of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ satisfies*

$$\begin{bmatrix} x_0 \\ \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda), \qquad \lambda \in \mathbb{C}^+. \tag{10.8}$$

*Here $\widehat{w} \in K^2_+(\mathcal{W})$.*

(ii) *Conversely, if (10.8) holds for some triple $\begin{bmatrix} \widehat{x} \\ x_0 \\ \widehat{w} \end{bmatrix}$, where $x_0 \in \mathcal{X}$ is fixed, $\widehat{w} \in K^2_+(\mathcal{W})$, and $\widehat{x}$ is an $\mathcal{X}$-valued function in $\mathbb{C}^+$, then $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$ is the Laplace transform of a stable future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$ with initial state $x_0$.*

(iii) *To each $\lambda \in \mathbb{C}^+$ and each $\begin{bmatrix} x_0 \\ x_\lambda \\ w_\lambda \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda)$ there exist at least one stable future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$ with initial value $x_0$ such that $\left[\begin{smallmatrix} x_\lambda \\ w_\lambda \end{smallmatrix}\right] = \begin{bmatrix} \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{bmatrix}$.*

*Thus, for all $\lambda \in \mathbb{C}^+$,*

$$\widehat{\mathfrak{E}}(\lambda) = \left\{ \begin{bmatrix} x_0 \\ x_\lambda \\ w_\lambda \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \,\middle|\, \begin{matrix} \left[\begin{smallmatrix} x_\lambda \\ w_\lambda \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{smallmatrix}\right] \textit{ for some stable future} \\ \textit{trajectory } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \textit{ of } \Sigma \textit{ with initial state } x_0 \end{matrix} \right\}. \tag{10.9}$$

*Proof. Proof of (i).* If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable classical future trajectory of $\Sigma$ with initial state $x_0$, then by multiplying (1.2) by $e^{-\lambda t}$ and integrating over $\mathbb{R}^+$ we find that the Laplace transform $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$ of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ satisfies (10.8). That the same statement is true also for stable generalised trajectories follows from Lemma 10.6 and the closedness of $\widehat{\mathfrak{E}}(\lambda)$ (which follows from the closedness of $V$).

*Proof of (ii).* Let $\left[\begin{smallmatrix} x_0 \\ \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$ satisfy the assumption of (ii). Let $w \in L^2(\mathbb{R}^+; \mathcal{W})$ be the inverse Laplace transform of $\widehat{w}$, and let $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ be a fundamental decomposition of $\mathcal{W}$, and let $u = P_\mathcal{U} w$. By Lemma 10.5, there exists a stable future trajectory $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right]$ of $\Sigma$ with $P_\mathcal{U} w_1 = u$ and $x_1(0) = x_0$. By part (i), (10.8) holds with $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix} \widehat{x}_1 \\ \widehat{w}_1 \end{smallmatrix}\right]$, so it also holds with $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix} \widehat{x}_2 \\ \widehat{w}_2 \end{smallmatrix}\right]$ and $x_0$ replaced by zero, where $\left[\begin{smallmatrix} \widehat{x}_2 \\ \widehat{w}_2 \end{smallmatrix}\right] := \left[\begin{smallmatrix} \widehat{x}_1 \\ \widehat{w}_1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$. Moreover, $P_\mathcal{U} \widehat{w}_2(\lambda) = P_\mathcal{U} \widehat{w}_2(\lambda) - P_\mathcal{U} \widehat{w}(\lambda) = 0$. By (10.1) and the nonnegativity of $V$ in $\mathfrak{K}$ we get

$$0 \le -2\Re\lambda \|x_2(\lambda)\|^2_\mathcal{X} - \|P_\mathcal{Y} w_2(\lambda)\|^2_\mathcal{Y},$$

and hence $\begin{bmatrix} x_2(\lambda) \\ w_2(\lambda) \end{bmatrix} = 0$, i.e., $\begin{bmatrix} \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{x}_1(\lambda) \\ \widehat{w}_1(\lambda) \end{bmatrix}$ for all $\lambda \in \mathbb{C}^+$. Thus, $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$ is the Laplace transform of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$.

*Proof of (iii).* Let $\left[\begin{smallmatrix} x_0 \\ x_\lambda \\ w_\lambda \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda)$, let $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ be a fundamental decomposition of $\mathcal{W}$, and let $u_\lambda = P_\mathcal{U} w_\lambda$. Choose some arbitrary function $u \in L^2(\mathbb{R}^+;\mathcal{U})$ such that $\widehat{u}(\lambda) = u_\lambda$ (for example, let $u = u_\lambda u_0$, where $u_0$ is a scalar function satisfying $\hat{u}_0(\lambda) = 1$). By Lemma 10.5, there exists a stable future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$ with initial state $x_0$ such that $P_\mathcal{U} w = u$. By part (i), $\left[\begin{smallmatrix} x_0 \\ \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda)$. In addition, $P_\mathcal{U} \widehat{w}(\lambda) = \widehat{u}(\lambda) = u_\lambda = P_\mathcal{U} w_\lambda$. By the same argument as we used in the proof of part (ii), a vector $\left[\begin{smallmatrix} x_0 \\ x_\lambda \\ w_\lambda \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda)$ is determined uniquely by $x_0$ and $P_\mathcal{U} w_\lambda$. Consequently, $\left[\begin{smallmatrix} x_\lambda \\ w_\lambda \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{smallmatrix}\right]$. □

## 10.3 The characteristic bundle of a passive future behavior

**Definition 10.8.** By a *passive frequency domain behaviour* in the Kreĭn space $\mathcal{W}$ we mean a maximal nonnegative subspace $\widehat{\mathfrak{W}}_+$ of $\widehat{K}^2_+(\mathcal{W})$ which is shift-invariant in the sense that the function $\lambda \mapsto e^{-t\lambda} \widehat{w}(\lambda)$ belongs to $\widehat{\mathfrak{W}}_+$ whenever $\widehat{w} \in \widehat{\mathfrak{W}}_+$ and $t \in \mathbb{R}^+$.

**Lemma 10.9.** *If $\mathfrak{W}_+$ is a passive future behaviour in $\mathcal{W}$, then the image $\widehat{\mathfrak{W}}_+ := \{\widehat{w} \,|\, w \in \mathfrak{W}_+\}$ of $\mathfrak{W}_+$ under the Laplace transform is a passive frequency domain behaviour in $\mathcal{W}$, and conversely, the inverse image under the Laplace transform of a passive frequency domain behavior in $\mathcal{W}$ is a passive future behavior in $\mathcal{W}$.*

*Proof.* This follows from Definitions 3.11 and 10.8 and the fact that the Laplace transform of the function $\tau_+^t w$ is the function $\lambda \mapsto e^{t\lambda}\widehat{w}(\lambda)$. □

**Lemma 10.10.** *Let $\widehat{\mathfrak{W}}_+$ be a passive frequency domain behavior in the Kreĭn space $\mathcal{W}$. Then, to each fundamental decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ of $\mathcal{W}$ there corresponds a unique $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued Schur function $\widehat{\mathfrak{D}}_+$ on $\mathbb{C}^+$ (i.e., an analytic function whose values are contractive operators), such that $\widehat{\mathfrak{W}}_+$ has the representation*

$$\widehat{\mathfrak{W}}_+ = \left\{ \left[\begin{smallmatrix} \widehat{u} \\ \widehat{y} \end{smallmatrix}\right] \in \left[\begin{smallmatrix} H^2_+(\mathcal{U}) \\ -H^2_+(\mathcal{Y}) \end{smallmatrix}\right] \,\middle|\, \widehat{y}(\lambda) = \widehat{\mathfrak{D}}_+(\lambda)\widehat{u}(\lambda),\ \lambda \in \mathbb{C}^+ \right\}. \tag{10.10}$$

*Proof.* Let $\mathfrak{W}_+$ be the inverse image of $\widehat{\mathfrak{W}}_+$ under the Laplace transform, and let us dente $\mathcal{U} := \mathcal{U}$ and $\mathcal{Y} := \mathcal{Y}$. By, e.g., [AKS11b, Formula (3.15)], there is an linear contraction $\mathfrak{D}_+ \colon L^2(\mathbb{R}^+;\mathcal{U}) \to L^2(\mathbb{R}^+;\mathcal{Y})$ such that

$$\mathfrak{W}_+ = \left\{ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} L^2_+(\mathcal{U}) \\ -L^2_+(\mathcal{Y}) \end{smallmatrix}\right] \,\middle|\, y = \mathfrak{D}_+ u \right\}. \tag{10.11}$$

The operator $\mathfrak{D}_+$ intertwines the two right-shifts in $L^2(\mathbb{R}^-;\mathcal{U})$ and $L^2(\mathbb{R}^-;\mathcal{Y})$, and therefore the image $\widehat{\mathfrak{W}}_+$ of $\mathfrak{W}_+$ has the representation (10.10), where $\widehat{\mathfrak{D}}_+$ is a $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued Schur function; see, e.g., [Sta05, Corollary 4.6.10 and Theorem 10.3.5]. □

**Definition 10.11.** Let $\mathcal{W}$ be a Kreĭn space.

(i) By the *characteristic bundle of a passive frequency domain behavior* $\widehat{\mathfrak{W}}_+$ we mean the family $\widehat{\mathfrak{F}} := \{\widehat{\mathfrak{F}}(\lambda)\}_{\lambda\in\mathbb{C}^+}$ of subspaces of $\mathcal{W}$ defined by

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ \widehat{w}(\lambda) \,\middle|\, \widehat{w} \in \widehat{\mathfrak{W}}_+ \right\}, \qquad \lambda \in \mathbb{C}^+. \tag{10.12}$$

The subspace $\widehat{\mathfrak{F}}(\lambda)$ is called the *fiber* of $\widehat{\mathfrak{F}}$ at $\lambda$.

(ii) By the *characteristic bundle of a passive future behaviour* $\mathfrak{W}_+$ we mean the characteristic bundle of the image $\widehat{\mathfrak{W}}_+$ of $\mathfrak{W}_+$ under the Laplace transform.

(iii) By the *characteristic bundle of a passive two-sided behaviour* $\mathfrak{W}$ we mean the characteristic bundle of the future behavior $\mathfrak{W}_+$ induced by $\mathfrak{W}$.

(iv) By the *characteristic signal bundle of a passive s/s system* $\Sigma = (V;\mathcal{X},\mathcal{W})$ we mean the characteristic bundle of the future behavior $\mathfrak{W}_+$ of $\Sigma$.

**Lemma 10.12.** *Let $\widehat{\mathfrak{W}}_+$ be a passive frequency domain behavior in the Kreĭn space $\mathcal{W}$. Then, to each fundamental decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ of $\mathcal{W}$ there corresponds a unique $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued Schur function $\widehat{\mathfrak{D}}_+$ on $\mathbb{C}^+$ (which is the same function as in Lemma 10.10) such that the fibers of the characteristic bundle of $\widehat{\mathfrak{W}}_+$ have the representation*

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ \left[ \begin{smallmatrix} u \\ \widehat{\mathfrak{D}}_+(\lambda)u \end{smallmatrix} \right] \;\middle|\; u \in \mathcal{U} \right\}. \tag{10.13}$$

*Proof.* This follows from Lemma 10.10, Definition 10.11, and the fact that for each $\lambda \in \mathbb{C}^+$ and each $u_\lambda \in \mathcal{U}$ there is a function $\widehat{u} \in H^2(\mathbb{C}^+;\mathcal{U})$ such that $\widehat{u}(\lambda) = u_\lambda$. □

**Remark 10.13.** If $\varphi$ is a bounded analytic $\mathcal{B}(\mathcal{U},\mathcal{Y})$-valued function in $\mathbb{C}^+$, then the bounded linear operator $\Phi\colon H^2(\mathbb{C}^+;\mathcal{U}) \to H^2(\mathbb{C}^+;\mathcal{Y})$ defined by $(\Phi\widehat{u})(\lambda) = \varphi(\lambda)\widehat{u}(\lambda)$, $\lambda \in \mathbb{C}^+$, is usually called the *Laurent operator* induced by $\varphi$, and $\varphi$ is called the *symbol* of $\Phi$. It is also called the symbol of the shift-invariant operator $\check{\Phi} := \mathcal{L}^{-1}\Phi\mathcal{L}\colon L^2(\mathbb{R}^+;\mathcal{U}) \to L^2(\mathbb{R}^+;\mathcal{Y})$, where $\mathcal{L}$ stands for the Laplace tranfsorm (this operator was denoted by $\mathfrak{D}_+$ in the proof of Lemma 10.10). Below we shall call the function $\widehat{\mathfrak{D}}$ in Lemmas 10.10 and 10.12 the *scattering matrix of the passive future behaviour* $\mathfrak{W}_+ := \mathcal{L}^{-1}\widehat{\mathfrak{W}}_+$ *induced by the fundamental decomposition* $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ *of* $\mathcal{W}$. We also use the same name with $\mathfrak{W}_+$ replaced by $\mathfrak{W}$, where $\mathfrak{W}$ is the two-sided behavior induced by $\mathfrak{W}_+$, or replaced by $\Sigma$ in the case where $\Sigma$ is a passive s/s system with future behavior $\mathfrak{W}_+$. By Lemmas 10.10 and 10.12, once the passive future behavior $\mathfrak{W}_+$ has been fixed, the scattering matrix $\widehat{\mathfrak{D}}_+$ is determined uniquely by the fundamental decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ (but it will, of course, depend on this decompsition). Conversely, the decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ and the scattering matrix $\widehat{\mathfrak{D}}_+$ also determine $\mathfrak{W}_+$ uniquely. Thus, a passive s/s system and a passive two-sided or future behavior has a *unique characteristic (signal) bundle*, but it has *infintely many scattering matrices* corresponding to different fundamental decompositions of the signal space (except in the degenerate cases where $\mathcal{W}$ is a Hilbert space or an anti-Hilbert space). (Other types of direct sum decompositions of the signal space give rise to other types of *transfer functions*, which have different names depending on the type of decomposition. We shall return to this elsewhere.)

**Theorem 10.14.** *Let $\mathfrak{F} = \{\widehat{\mathfrak{F}}(\lambda)\}_{\lambda \in \mathbb{C}^+}$ be the characteristic bundle of the passive frequency domain behavior $\widehat{\mathfrak{W}}_+$.*

(i) *The fibers $\widehat{\mathfrak{F}}(\lambda)$ of $\widehat{\mathfrak{F}}$, $\lambda \in \mathbb{C}^+$, are maximal nonnegative subspaces of $\mathcal{W}$.*

(ii) *A function* $\widehat{w} \in \widehat{K}^2_+(\mathcal{W})$ *belongs to* $\widehat{\mathfrak{W}}_+$ *if and only if*

$$\widehat{w}(\lambda) \in \widehat{\mathfrak{F}}(\lambda), \qquad \lambda \in \mathbb{C}^+. \tag{10.14}$$

*Proof. Proof of (i).* This follows from Proposition 2.1 and Lemma 10.12.

*Proof of (ii).* One direction of the above claim follows directly from Definition 10.11, so it suffices to prove the opposite direction. Thus, let $\widehat{w} \in \widehat{K}^2_+(\mathcal{W})$, and suppose that (10.14) holds. Let $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ be a fundamental decomposition of $\mathcal{W}$, and let $\widehat{u} = P_{\mathcal{U}}\widehat{w}$. Since $\widehat{K}^2_+(\mathcal{W}) = H^2(\mathbb{C}^+;\mathcal{U}) \boxplus -H^2(\mathbb{C}^+;\mathcal{Y})$ is a fundamental decomposition of $K^2_+(\mathcal{W})$, and since $\widehat{\mathfrak{W}}_+$ is a maximal nonnegative subspace of $\widehat{K}^2_+(\mathcal{W})$, there is a unique function $\widehat{w}_1 \in \widehat{\mathfrak{W}}_+$ such that $P_{\mathcal{U}}\widehat{w}_1 = P_{\mathcal{U}}\widehat{w}$. Thus, for all $\lambda \in \mathbb{C}^+$, both $\widehat{w}(\lambda) \in \mathfrak{F}(\lambda)$ and $\widehat{w}_1(\lambda) \in \mathfrak{F}(\lambda)$ and $P_{\mathcal{U}}\widehat{w}_1(\lambda) = P_{\mathcal{U}}\widehat{w}(\lambda)$. Since $\mathfrak{F}(\lambda)$ is a maximal nonnegative subspace of $\mathcal{W}$, a vector in $\mathfrak{F}(\lambda)$ is determined uniquely by its orthogonal projection onto $\mathcal{U}$, and consequently $\widehat{w}(\lambda) = \widehat{w}_1(\lambda)$, $\lambda \in \mathbb{C}^+$. Thus shows that $\widehat{w} = \widehat{w}_1$, and since $\widehat{w}_1 \in \widehat{\mathfrak{W}}_+$, also $\widehat{w} \in \widehat{\mathfrak{W}}_+$, as claimed. □

**Corollary 10.15.** *Two passive s/s systems* $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ *and* $\Sigma_2 = (V_2, \mathcal{X}_2, \mathcal{W})$ *are externally equivalent if and only if their characteristic signal bundles coincide.*

*Proof.* This follows from Definition 10.11 and Theorem 10.14. □

**Theorem 10.16.** *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a passive s/s system with characteristic node bundle* $\widehat{\mathfrak{E}} = \{\widehat{\mathfrak{E}}(\lambda)\}_{\lambda\in\mathbb{C}}$ *and characteristic signal bundle* $\widehat{\mathfrak{F}} = \{\widehat{\mathfrak{F}}(\lambda)\}_{\lambda\in\mathbb{C}^+}$. *Then* (1.10) *holds and*

$$\widehat{\mathfrak{F}}(\lambda)^{[\perp]} = \left\{ w^\dagger \in \mathcal{W} \;\middle|\; \begin{bmatrix} \overline{\lambda}x^\dagger \\ -x^\dagger \\ w^\dagger \end{bmatrix} \in V^{[\perp]} \text{ for some } x^\dagger \in \mathcal{X} \right\}, \qquad \lambda \in \mathbb{C}^+. \tag{10.15}$$

*These two identities can alternatively be written in the forms*

$$\widehat{\mathfrak{F}}(\lambda) = P_{\mathcal{W}}\left(\widehat{\mathfrak{E}}(\lambda) \cap \left[\begin{smallmatrix} 0 \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]\right), \qquad \lambda \in \mathbb{C}^+, \tag{10.16}$$

$$\widehat{\mathfrak{F}}(\lambda)^{[\perp]} = P_{\mathcal{W}}\left(\widehat{\mathfrak{E}}(\lambda)^{[\perp]} \cap \left[\begin{smallmatrix} 0 \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]\right), \qquad \lambda \in \mathbb{C}^+. \tag{10.17}$$

*Proof.* Clearly (1.10) and (10.15) are equivalent to (10.16) and (10.15), respectively.

Let $w_\lambda \in \mathcal{W}$, and suppose that there exists some $x_\lambda \in \mathcal{X}$ such that $\left[\begin{smallmatrix} \lambda x_\lambda \\ x_\lambda \\ w_\lambda \end{smallmatrix}\right] \in V$, or equivalently, $\left[\begin{smallmatrix} 0 \\ x_\lambda \\ w_\lambda \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda)$. By Theorem 10.7(iii), there exists some stable future externally generated trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of $\Sigma$ such that $\left[\begin{smallmatrix} x_\lambda \\ w_\lambda \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{smallmatrix}\right]$. Since $w \in \mathfrak{W}_+$, this means that $w_\lambda \in \widehat{\mathfrak{F}}(\lambda)$. Thus, the right-hand side of (10.16) is contained in $\widehat{\mathfrak{F}}$.

Conversely, suppose that $w_\lambda \in \widehat{\mathfrak{F}}_\lambda$. Then $w_\lambda = \widehat{w}(\lambda)$ for some $w \in \mathfrak{W}_+$. To this $w$ corresponds a unique function $x$ such that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a stable future externally generated trajectory of $\Sigma$. By Theorem 10.7(i), $\left[\begin{smallmatrix} 0 \\ \widehat{x}(\lambda) \\ \widehat{w}(\lambda) \end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\lambda)$, and consequently, $w_\lambda = \widehat{w}(\lambda)$ belongs to the right-hand side of (10.16). Thus, (1.10) and (10.16) hold.

For each $\lambda \in \mathbb{C}^+$ we denote

$$\mathcal{Z}_+(\lambda) := \left\{ \begin{bmatrix} \overline{\lambda} x \\ -x \end{bmatrix} \middle| x \in \mathcal{X} \right\}, \qquad \mathcal{Z}_-(\lambda) := \left\{ \begin{bmatrix} \lambda x \\ x \end{bmatrix} \middle| x \in \mathcal{X} \right\}.$$

Then $\mathcal{Z}_+(\lambda) \boxplus -\mathcal{Z}_-(\lambda)$ is a fundamental decomposition of the internal part $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right]$ of the node space $\mathfrak{K}$ equipped with inner product induced by the operator $\left[\begin{smallmatrix} 0 & -1_\mathcal{X} \\ -1_\mathcal{X} & 0 \end{smallmatrix}\right]$. If we let $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ be a fundamental decomposition of $\mathcal{W}$, then $\left[\begin{smallmatrix} \mathcal{Z}_+(\lambda) \\ \mathcal{U} \end{smallmatrix}\right] \boxplus - \left[\begin{smallmatrix} \mathcal{Z}_-(\lambda) \\ \mathcal{Y} \end{smallmatrix}\right]$ is a fundamental decomposition of the node space $\mathfrak{K}$. Since $V$ is maximal nonnegative it follows from Proposition 2.1(i) there exists a contraction $A(\lambda) = \left[\begin{smallmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) \end{smallmatrix}\right] \in \mathcal{B}\left(\left[\begin{smallmatrix} \mathcal{Z}_+(\lambda) \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{Z}_-(\lambda) \\ \mathcal{Y} \end{smallmatrix}\right]\right)$ such that (the vectors on the right-hand side have been split in accordance with the natural decomposition $\mathfrak{K} = \mathcal{Z}_+(\lambda) \boxplus -\mathcal{Z}_-(\lambda) \boxplus \mathcal{U} \boxplus -\mathcal{Y}$ of $\mathfrak{K}$)

$$V = \left\{ \begin{bmatrix} z_+ \\ A_{11}(\lambda) z_+ + A_{12}(\lambda) u \\ u \\ A_{21}(\lambda) z_+ + A_{22}(\lambda) u \end{bmatrix} \middle| \begin{bmatrix} z_+ \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{Z}_+(\lambda) \\ \mathcal{U} \end{bmatrix} \right\}, \tag{10.18}$$

$$V^{[\perp]} = \left\{ \begin{bmatrix} A_{11}(\lambda)^* z_- + A_{21}(\lambda)^* y \\ z_- \\ A_{12}(\lambda)^* z_- + A_{22}(\lambda)^* y \\ y \end{bmatrix} \middle| \begin{bmatrix} z_- \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{Z}_-(\lambda) \\ \mathcal{Y} \end{bmatrix} \right\}. \tag{10.19}$$

A vector produced by the right-hand side of (10.18) belongs to $\widehat{\mathfrak{E}}(\lambda) \cap \left[\begin{smallmatrix} 0 \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ if and only $z_+ = 0$, and a vector produced by the right-hand side of (10.19) belongs to $\widehat{\mathfrak{E}}(\lambda)^{[\perp]} \cap \left[\begin{smallmatrix} 0 \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ if and only if $z_- = 0$. Thus,

$$\widehat{\mathfrak{E}}(\lambda) \cap \left[\begin{smallmatrix} 0 \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] = \left\{ \begin{bmatrix} A_{12}(\lambda)u \\ u \\ A_{22}(\lambda)u \end{bmatrix} \middle| u \in \mathcal{U} \right\},$$

$$\widehat{\mathfrak{E}}(\lambda)^{[\perp]} \cap \left[\begin{smallmatrix} \mathcal{X} \\ 0 \\ \mathcal{W} \end{smallmatrix}\right] = \left\{ \begin{bmatrix} A_{21}(\lambda)^* y \\ A_{22}(\lambda)^* y \\ y \end{bmatrix} \middle| y \in \mathcal{Y} \right\},$$

and

$$P_{\mathcal{W}}\left(\widehat{\mathfrak{E}}(\lambda) \cap \left[\begin{smallmatrix} 0 \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]\right) = \left\{ \begin{bmatrix} u \\ A_{22}(\lambda)u \end{bmatrix} \middle| u \in \mathcal{U} \right\},$$

$$P_{\mathcal{W}}\left(\widehat{\mathfrak{E}}(\lambda)^{[\perp]} \cap \left[\begin{smallmatrix} 0 \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]\right) = \left\{ \begin{bmatrix} A_{22}(\lambda)^* y \\ y \end{bmatrix} \middle| y \in \mathcal{Y} \right\}.$$

Here the two right-hands sides are orthogonal complements to each other in $\mathcal{W}$, and thus (10.15) holds. □

**Theorem 10.17.** *Let $\mathfrak{W}_+$ be a passive future behavior with characteristic bundle $\widehat{\mathfrak{F}}$ and corresponding past behavior $\mathfrak{W}_-$, and let $\mathfrak{W}_{*+} = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\boldsymbol{\mathfrak{R}}\mathfrak{W}_-^{[\perp]}$ be the adjoint of $\mathfrak{W}_-$ with characteristic bundle $\widehat{\mathfrak{F}}_*$ (cf. Definition 3.24. Then*

$$\widehat{\mathfrak{F}}_*(\lambda) = \mathcal{I}_{(\mathcal{W},-\mathcal{W})}\widehat{\mathfrak{F}}(\overline{\lambda})^{[\perp]}, \qquad \lambda \in \mathbb{C}^+. \tag{10.20}$$

*Proof.* We recall the representation (10.13) of the fibers of $\widehat{\mathfrak{F}}_{\mathfrak{W}_+}$. Taking the orthogonal complements of both sides we get

$$\widehat{\mathfrak{F}}(\lambda)^{[\perp]} = \left\{ \left[\begin{smallmatrix} \widehat{\mathfrak{D}}_+(\lambda)^* y \\ y \end{smallmatrix}\right] \middle| y \in \mathcal{Y} \right\}, \qquad \lambda \in \mathbb{C}^+. \tag{10.21}$$

The fibers of $\widehat{\mathfrak{F}}_*$ have analogous representations, namely

$$\widehat{\mathfrak{F}}_*(\lambda) = \left\{ \left[\begin{smallmatrix} \widehat{\mathfrak{D}}_{*+}(\lambda) y \\ y \end{smallmatrix}\right] \middle| y \in \mathcal{Y} \right\}, \qquad \lambda \in \mathbb{C}^+, \tag{10.22}$$

where $\widehat{\mathfrak{D}}_{*+}$ is the scattering matrix of the operator $\mathfrak{D}_{*+} := \boldsymbol{\mathfrak{R}}\mathfrak{D}_- * \boldsymbol{\mathfrak{R}}$ and $\mathfrak{D}_-$ is defined as in [AKS11b, Formula (3.17)]. Thus, in order to prove (10.20) it suffices to show that

$$\widehat{\mathfrak{D}}_{*+}(\lambda) = \widehat{\mathfrak{D}}_+(\overline{\lambda})^*, \qquad \lambda \in \mathbb{C}^+. \tag{10.23}$$

Let $w^\dagger = \left[\begin{smallmatrix} u^\dagger \\ y^\dagger \end{smallmatrix}\right] \in \boldsymbol{\mathfrak{R}}\mathfrak{W}_-^{[\perp]}$, i.e., $[w^\dagger, \boldsymbol{\mathfrak{R}}w]_{K_+^2(\mathcal{W})} = 0$ for all $w \in \mathfrak{W}_-$. By, e.g., [Sta05, Corollary 4.6.10], for all $\lambda \in \mathbb{C}^+$ and $u_0 \in \mathcal{U}$ the function $w$

defined by $w(t) = \begin{bmatrix} u_0 \\ \widehat{\mathfrak{D}}(\lambda)u_0 \end{bmatrix} \mathrm{e}^{\lambda t}$, $t \in \mathbb{R}^-$, belongs to $\mathfrak{W}_-$, and consequently $[w^\dagger, \boldsymbol{Я} w]_{K^2_+(\mathcal{W})} = 0$. Explicitly, this means that

$$\begin{aligned} 0 &= \int_0^\infty [w^\dagger(t), w(-t)]_{\mathcal{W}} \\ &= \int_0^\infty (u^\dagger(t), \mathrm{e}^{-\lambda t} u_0)_{\mathcal{U}} - \int_0^\infty (y^\dagger(t), \mathrm{e}^{-\lambda t} \widehat{\mathfrak{D}}(\lambda) u_0)_{\mathcal{Y}} \\ &= (\widehat{u}^\dagger(\overline{\lambda}) - \widehat{\mathfrak{D}}(\lambda)^* \widehat{y}^\dagger(\overline{\lambda}), u_0). \end{aligned}$$

This being true for all $u_o \in \mathcal{U}$ we find that

$$\widehat{u}^\dagger(\overline{\lambda}) = \widehat{\mathfrak{D}}(\lambda)^* \widehat{y}^\dagger(\overline{\lambda}), \qquad \lambda \in \mathbb{C}^+.$$

On the other hand, since $w^\dagger = \begin{bmatrix} u^\dagger \\ y^\dagger \end{bmatrix} \in \boldsymbol{Я}\mathfrak{W}_-^{[\perp]}$, we have $\widehat{w}^\dagger = \begin{bmatrix} \widehat{u}^\dagger \\ \widehat{u}^\dagger \end{bmatrix} \in \mathcal{I}_{(\mathcal{W},-\mathcal{W})} \widehat{\mathfrak{W}}_{*+}$. Consequently, $\left\{ \begin{bmatrix} \widehat{\mathfrak{D}}(\lambda)^* y \\ y \end{bmatrix} \middle| \, y \in \mathcal{Y} \right\} \subset \widehat{\mathfrak{F}}_{*+}(\overline{\lambda})$. Since $\widehat{\mathfrak{D}}(\lambda)^*$ is a contraction, the left-hand side of this inclusion is maximal nonnegative in $-\mathcal{W}$, whereas the right-hand side is nonnegative in $-\mathcal{W}$, so the inclusion is, in fact, an equality. Comparing this to (10.22) we find that (10.23) holds. □

## 11 Frequency Domain Characterizations of Symmetries

In this section we study how symmetries of a passive s/s system can be described in terms of the frequency domain characteristics of the system. In particular, we show that the frequency domain characterizations (1.11)–(1.14) of our four basic symmetries are equivalent to those characterizations that we give in Sections 5–(8) in terms of the two-sided passive behaviors.

**Theorem 11.1.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with characteristic node bundle $\widehat{\mathfrak{E}}$. Let $\mathcal{J}_\mathcal{X}$ and $\mathcal{C}_\mathcal{X}$ be a signature operator and a conjugation in $\mathcal{X}$, respectively, and let $\mathcal{J}_\mathcal{W}$, $\mathcal{C}_\mathcal{W}$, $\mathcal{I}_\mathcal{W}$, and $\mathcal{B}_\mathcal{W}$ be a singature operator, a conjugation, a skew-signature operator, and a skew-conjugation in $\mathcal{W}$, respectively.*

(i) $\Sigma$ *is $(\mathcal{C}_\mathcal{X}, \mathcal{C}_\mathcal{W})$-real if and only if for* some $\lambda \in \mathbb{C}$, *or equivalently, for* all $\lambda \in \mathbb{C}$,

$$\widehat{\mathfrak{E}}(\lambda) = \begin{bmatrix} \mathcal{C}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{C}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{C}_\mathcal{W} \end{bmatrix} \widehat{\mathfrak{E}}(\overline{\lambda}). \tag{11.1}$$

(ii) $\Sigma$ *is $(\mathcal{J}_\mathcal{X}, \mathcal{I}_\mathcal{W})$-reciprocal if and only if for* some $\lambda \in \mathbb{C}$, *or equivalently, for* all $\lambda \in \mathbb{C}$,

$$\widehat{\mathfrak{E}}(\lambda) = \begin{bmatrix} \mathcal{J}_\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{J}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{I}_\mathcal{W} \end{bmatrix} \widehat{\mathfrak{E}}(\overline{\lambda})^{[\perp]} \tag{11.2}$$

(iii) $\Sigma$ *is $(\mathcal{J}_\mathcal{X}, \mathcal{J}_\mathcal{W})$-signature invariant if and only if for* some $\lambda \in \mathbb{C}$, *or equivalently, for* all $\lambda \in \mathbb{C}$,

$$\widehat{\mathfrak{E}}(\lambda) = \begin{bmatrix} \mathcal{J}_\mathcal{X} & 0 & 0 \\ 0 & \mathcal{J}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{J}_\mathcal{W} \end{bmatrix} \widehat{\mathfrak{E}}(\lambda). \tag{11.3}$$

(iv) $\Sigma$ *is $(\mathcal{J}_\mathcal{X}, \mathcal{I}_\mathcal{W})$-transpose invariant if and only if for* some $\lambda \in \mathbb{C}$, *or equivalently, for* all $\lambda \in \mathbb{C}$,

$$\widehat{\mathfrak{E}}(\lambda) = \begin{bmatrix} \mathcal{C}_\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{C}_\mathcal{X} & 0 \\ 0 & 0 & \mathcal{B}_\mathcal{W} \end{bmatrix} \widehat{\mathfrak{E}}(\lambda)^{[\perp]}. \tag{11.4}$$

*Proof.* The proof of this theorem consists of four easy algebraic computations based on (1.15)–(1.18), (10.1), and (10.2). Se also Remark 10.3. □

**Theorem 11.2.** *Let $\widehat{\mathfrak{F}} = \{\widehat{\mathfrak{F}}(\lambda)\}_{\lambda\in\mathbb{C}^+}$ be the characteristic bundle of a passive two-sided behavior $\mathfrak{W}$ on the Kreĭn space $\mathcal{W}$ (or of a passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$), and let $\mathcal{J}_\mathcal{W}$, $\mathcal{C}_\mathcal{W}$, $\mathcal{I}_\mathcal{W}$, and $\mathcal{B}_\mathcal{W}$ be a singature operator, a conjugation, a skew-signature operator, and a skew-conjugation in $\mathcal{W}$, respectively.*

(i) *$\mathfrak{W}$ is $\mathcal{C}_\mathcal{W}$-real (or $\Sigma$ is externally $\mathcal{C}_\mathcal{W}$-real) if and only if $\widehat{\mathfrak{F}}$ satisfies* (1.11).

(ii) *$\mathfrak{W}$ is $\mathcal{I}_\mathcal{W}$-reciprocal (or $\Sigma$ is externally $\mathcal{J}_\mathcal{W}$-reciprocal) if and only if $\widehat{\mathfrak{F}}$ satisfies* (1.12).

(iii) *$\mathfrak{W}$ is $\mathcal{J}_\mathcal{W}$-signature invariant (or $\Sigma$ is externally $\mathcal{J}_\mathcal{W}$-signature invariant) if and only if $\widehat{\mathfrak{F}}$ satisfies* (1.13).

(iv) *$\mathfrak{W}$ is $\mathcal{B}_\mathcal{W}$-transpose invariant (or $\Sigma$ is externally $\mathcal{B}_\mathcal{W}$-transpose invariant) if and only if $\widehat{\mathfrak{F}}$ satisfies* (1.14).

*Proof.* The proof of (i) is analogous to the proof of (iii) and the proof of (ii) is analogous to the proof of (iv), so here we only prove (i) and (ii).

*Proof of (i).* Let $\mathfrak{W}_+$ be the future behavior induced by $\mathfrak{W}$. The reality condition (5.3) is equivalent to the condition

$$\mathfrak{W}_+ = \mathcal{C}_\mathcal{W}\mathfrak{W}_+. \tag{11.5}$$

For each $w \in K^2_+(\mathcal{W})$, the the Laplace transform of $\mathcal{C}_\mathcal{W} w$ at a point $\lambda \in \mathbb{C}^+$ is given by

$$\widehat{\mathcal{C}_\mathcal{W} w}(\lambda) = \int_{\mathbb{R}^+} \mathrm{e}^{-\lambda t}\mathcal{C}_\mathcal{W} w(t)\,\mathrm{d}t = \mathcal{C}_\mathcal{W}\int_{\mathbb{R}^+} \mathrm{e}^{-\overline{\lambda} t} w(t)\,\mathrm{d}t = \mathcal{C}_\mathcal{W}\widehat{w}(\overline{\lambda}).$$

This together with (11.5) and Definitions 6.8 and 10.11 gives that $\mathfrak{W}$ is $\mathcal{C}_\mathcal{W}$-real if and only if (1.11) holds.

*Proof of (ii).* Let $\mathfrak{W}_+$ and $\mathfrak{W}_{*+}$ be the future and behavior induced by $\mathfrak{W}$ and the adjoint behavior $\mathfrak{W}_*$, respectively. The reciprocity condition (6.3) is equivalent to the condition $\mathfrak{W} = \mathcal{I}_{(-\mathcal{W},\mathcal{W})}\mathcal{J}_\mathcal{W}\mathfrak{W}_*$, which by Definition 10.11, this is equivalent to

$$\widehat{\mathfrak{F}}(\lambda) = \mathcal{J}_\mathcal{W}\mathcal{I}_{(-\mathcal{W},\mathcal{W})}\widehat{\mathfrak{F}}_*(\lambda), \qquad \lambda \in \mathbb{C}^+,$$

where $\widehat{\mathfrak{F}}_*$ is the characteristic bundle of $\mathfrak{W}_*$. Combining this with (10.20) and Definition 8.2 we find that $\mathfrak{W}$ is $\mathcal{J}_\mathcal{W}$-reciprocal if and only if (1.12) holds. □

The symmetry results that we have developed for passive s/s systems and behaviors in this article are motivated by analogous symmetry results for i/s/o systems, i/o maps, and transfer functions, and they can be used to recover many of these results. Because of lack of space we are forced to postpone a more detailed discussion of how this is done to a later time. However, to get a flavor of what can be achieved we below discuss how one can derive symmetry results for scattering functions by using a fundamental decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ which is in a certain sense invariant under a symmerty of a passive behavior $\mathfrak{W}$ in this signal space. Analogous results where the fundamental decomposition of $\mathcal{W}$ has been replaced by other types of decompositions (such as Lagrangian decompositions and general orthogonal decompositions) will be given elsewhere.

**Theorem 11.3.** *Let $\mathfrak{W}$ be a passive two-sided behavior on the Kreĭn space $\mathcal{W}$, and let $\widehat{\mathfrak{D}}$ be the scattering matrix corresponding to some fundamental decomposition $\mathcal{W} = \mathcal{U} \boxplus \mathcal{Y}$ of $\mathcal{W}$. Let $\mathcal{J}_\mathcal{W}$, $\mathcal{C}_\mathcal{W}$, $\mathcal{I}_\mathcal{W}$, and $\mathcal{B}_\mathcal{W}$ be a singature operator, a conjugation, a skew-signature operator, and a skew-conjugation in $\mathcal{W}$, respectively. Moreover, suppose that these operators satisfy*

$$\begin{aligned}
&\mathcal{J}_\mathcal{W}\mathcal{U} = \mathcal{U} \text{ (and hence } \mathcal{J}_\mathcal{W}\mathcal{Y} = \mathcal{Y}\text{)},\\
&\mathcal{C}_\mathcal{W}\mathcal{U} = \mathcal{U} \text{ (and hence } \mathcal{C}_\mathcal{W}\mathcal{U} = \mathcal{U}\text{)},\\
&\mathcal{I}_\mathcal{W}\mathcal{U} = \mathcal{Y} \text{ (and hence } \mathcal{I}_\mathcal{W}\mathcal{Y} = \mathcal{U}\text{)},\\
&\mathcal{B}_\mathcal{W}\mathcal{U} = \mathcal{Y} \text{ (and hence } \mathcal{B}_\mathcal{W}\mathcal{Y} = \mathcal{U}\text{)},
\end{aligned} \tag{11.6}$$

*Then the following claims hold:*

(i) $\mathfrak{W}$ *is $\mathcal{C}_\mathcal{W}$-real if and only if*

$$\widehat{\mathfrak{D}}(\lambda) = \mathcal{C}_\mathcal{W}\widehat{\mathfrak{D}}(\overline{\lambda})\mathcal{C}_\mathcal{W}|_\mathcal{U}, \qquad \lambda \in \mathbb{C}^+. \tag{11.7}$$

(ii) $\mathfrak{W}$ *is $\mathcal{I}_\mathcal{W}$-reciprocal if and only if*

$$\widehat{\mathfrak{D}}(\lambda) = \mathcal{I}_\mathcal{W}\widehat{\mathfrak{D}}(\overline{\lambda})^*\mathcal{I}_\mathcal{W}|_\mathcal{U}, \qquad \lambda \in \mathbb{C}^+. \tag{11.8}$$

(iii) $\mathfrak{W}$ *is $\mathcal{J}_\mathcal{W}$-signature invariant if and only if*

$$\widehat{\mathfrak{D}}(\lambda) = \mathcal{J}_\mathcal{W}\widehat{\mathfrak{D}}(\lambda)\mathcal{J}_\mathcal{W}|_\mathcal{U}, \qquad \lambda \in \mathbb{C}^+. \tag{11.9}$$

(iv) $\mathfrak{W}$ *is $\mathcal{B}_\mathcal{W}$-transpose-invariant if and only if*

$$\widehat{\mathfrak{D}}(\lambda) = \mathcal{B}_\mathcal{W}\widehat{\mathfrak{D}}(\lambda)^*\mathcal{B}_\mathcal{W}|_\mathcal{U}, \qquad \lambda \in \mathbb{C}^+. \tag{11.10}$$

*Proof.* Again the proof of (i) is analogous to the proof of (iii) and the proof of (ii) is analogous to the proof of (iv), so here we only prove (i) and (ii). In this proof we denote the charactiristic buldle of $\mathfrak{W}$ by $\mathfrak{F}$ and use the representation (10.13) of its fibers.

*Proof of (i).* $\mathcal{C}_\mathcal{W}$-reality of $\mathcal{W}$ is equivalent to (1.11), which by (10.13) and (11.6) is equivalent to

$$\operatorname{im}\left(\begin{bmatrix} 1_\mathcal{U} \\ \widehat{\mathfrak{D}}_+(\lambda) \end{bmatrix}\right) = \operatorname{im}\left(\begin{bmatrix} \mathcal{C}_\mathcal{W}|_\mathcal{U} \\ \mathcal{C}_\mathcal{W}\widehat{\mathfrak{D}}_+(\overline{\lambda}) \end{bmatrix}\right), \qquad \lambda \in \mathbb{C}.$$

The range of the operator on the right-hand side does not change if we multiply it by $\mathcal{C}_\mathcal{W}|_\mathcal{U}$ to the right, and hence

$$\operatorname{im}\left(\begin{bmatrix} 1_\mathcal{U} \\ \widehat{\mathfrak{D}}_+(\lambda) \end{bmatrix}\right) = \operatorname{im}\left(\begin{bmatrix} 1_\mathcal{U} \\ \mathcal{C}_\mathcal{W}\widehat{\mathfrak{D}}_+(\overline{\lambda})\mathcal{C}_\mathcal{W}|_\mathcal{U} \end{bmatrix}\right), \qquad \lambda \in \mathbb{C}.$$

This is equivalent to (11.7).

*Proof of (ii).* $\mathcal{C}_\mathcal{W}$-reciprocity of $\mathcal{W}$ is equivalent to (1.12), which by (10.13), (10.21), and (11.6) is equivalent to

$$\operatorname{im}\left(\begin{bmatrix} 1_\mathcal{U} \\ \widehat{\mathfrak{D}}_+(\lambda) \end{bmatrix}\right) = \operatorname{im}\left(\begin{bmatrix} \mathcal{I}_\mathcal{W}|_\mathcal{Y} \\ \mathcal{I}_\mathcal{W}\widehat{\mathfrak{D}}_+(\overline{\lambda})^* \end{bmatrix}\right), \qquad \lambda \in \mathbb{C}.$$

The range of the operator on the right-hand side does not change if we multiply it by $\mathcal{I}_\mathcal{W}|_\mathcal{U}$ to the right, and hence

$$\operatorname{im}\left(\begin{bmatrix} 1_\mathcal{U} \\ \widehat{\mathfrak{D}}_+(\lambda) \end{bmatrix}\right) = \operatorname{im}\left(\begin{bmatrix} 1_\mathcal{U} \\ \mathcal{I}_\mathcal{W}\widehat{\mathfrak{D}}_+(\overline{\lambda})^*\mathcal{I}_\mathcal{W}|_\mathcal{U} \end{bmatrix}\right), \qquad \lambda \in \mathbb{C}.$$

This is equivalent to (11.8).

□

**Remark 11.4.** Condition 11.6 is equivalent to the condition that the operator $\mathcal{J}_\mathcal{W}$, $\mathcal{C}_\mathcal{W}$, $\mathcal{I}_\mathcal{W}$, and $\mathcal{B}_\mathcal{W}$ can be decomposed in accordance with the decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ as

$$\mathcal{J}_\mathcal{W} = \begin{bmatrix} \mathcal{J}_\mathcal{U} & 0 \\ 0 & \mathcal{J}_\mathcal{Y} \end{bmatrix}, \ \mathcal{C}_\mathcal{W} = \begin{bmatrix} \mathcal{C}_\mathcal{U} & 0 \\ 0 & \mathcal{C}_\mathcal{Y} \end{bmatrix}, \ \mathcal{I}_\mathcal{W} = \begin{bmatrix} 0 & \mathcal{I}_\mathcal{Y} \\ \mathcal{I}_\mathcal{Y}^* & 0 \end{bmatrix}, \ \mathcal{B}_\mathcal{W} = \begin{bmatrix} 0 & \mathcal{B}_\mathcal{Y} \\ \mathcal{B}_\mathcal{Y}^* & 0 \end{bmatrix}, \tag{11.11}$$

where $\mathcal{J}_\mathcal{U} = \mathcal{J}_\mathcal{W}|_\mathcal{U}$, etc. Here $\mathcal{J}_\mathcal{U}$ and $\mathcal{J}_\mathcal{U}$ are signature operators, $\mathcal{C}_\mathcal{U}$ and $\mathcal{C}_\mathcal{Y}$ are conjugations, $\mathcal{I}_\mathcal{Y}$ is linear and unitary, and $\mathcal{B}_\mathcal{Y}$ is conjugate-linear and unitary. In particular, none of these operators is a skew-signature operator or a skew-conjugation, in spite of the fact that $\mathcal{I}_\mathcal{W}$ is a skew-singature operator and $\mathcal{B}_\mathcal{W}$ is a skew-conjugation in $\mathcal{W}$. This is possible due to the fact that whereas $\mathcal{U}$ is as a Hilbert subspace of $\mathcal{W}$, it is the anti-Hilbert space $-\mathcal{Y}$ and not the Hilbert space $\mathcal{Y}$ itself which appears in the fundamental decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ of $\mathcal{W}$.

In Theorem 11.3 we derive symmetry results for passive i/o maps and scattering matrices from our symmetry results for passive behaviors. On the surface it looks like we should get a one-to-one correspondence between symmetry results for scattering matrices and symmetry results for passive behaviors, but this is not the case, due to the fact that there do exist symmetries in the Kreĭn space $\mathcal{W}$ such that $\mathcal{W}$ does not have any fundamental decomposition satisfying the appropriate invariance condition in (11.6). One such example is the following.

**Example 11.5.** We let $\mathcal{W} = \mathbb{C}^2$, and let $\mathcal{C}_{\mathcal{W}}$ be the standard complex conjugation in $\mathbb{C}^2$. We take the inner product in $\mathcal{W}$ to be

$$\left[\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right]_{\mathcal{W}} = \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)_{\mathbb{C}^2} = i(x_1\overline{y}_2 - y_1\overline{x}_2).$$

Then $\operatorname{ind}_+\mathcal{W} = \operatorname{ind}_-\mathcal{W} = 1$, and, for example,

$$\mathcal{W} = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]\mathbb{C} \boxplus \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]\mathbb{C}$$

is a fundamental decomposition of $\mathcal{W}$. However, the subspaces in this decomposition are not invariant under conjugation (instead conjugation maps one of these subspaces into the other). It is not difficult to se that a one-dimensional subspace of $\mathcal{W}$ is invariant under conjugation if and only if it is of the form $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]\mathbb{C}$, where $\alpha$, $\beta \in \mathbb{R}$ and $|\alpha| + |\beta| \neq 0$, and it is equally easy to see that every such subspace is Lagrangian. The converse is also true: every Lagrangian subspace is invariant under conjugation. Thus, the components of a a direct sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of $\mathcal{W}$ are invariant under conjugation if and only both $\mathcal{U}$ and $\mathcal{Y}$ are Lagrangian subspaces of $\mathcal{W}$. In particular, in this example no fundamental decompositions exist in which the two components would be invariant under conjugation.

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