Optimal State Feedback Input-Output Stabilization of Infinite-Dimensional Discrete Time-Invariant Linear Systems

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Abstract. We study the optimal input-output stabilization of discrete time-invariant linear systems in Hilbert spaces by state feedback. We show that a necessary and sufficient condition for this problem to be solvable is that the transfer function has a right factorization over H-infinity. A necessary and sufficient condition in terms of an (arbitrary) realization is that each state which can be reached in a finite time from the zero initial state has a finite cost. Another equivalent condition is that the control Riccati equation has a solution (in general unbounded and even non densely defined). The optimal state feedback input-output stabilization problem can then be solved explicitly in terms of the smallest solution of this control Riccati equation. We further show that after renorming the state space in terms of the solution of the control Riccati equation, the closed-loop system is not only input-output stable, but also strongly internally stable.

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1. Introduction

This is the first in a series of articles dealing in a novel way with the quadratic cost minimization problem for infinite-dimensional time-invariant linear systems in discrete and continuous time. Much of the motivation comes from the continuous time case, but since that case is technically more difficult, and since the continuous time theory to some extent can be reduced to the discrete time theory, we begin

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with the discrete time case. In this article we investigate the full information infinite-horizon LQ (Linear Quadratic) problem, and our next article will deal with the discrete time infinite-horizon optimal output injection problem.

We consider a linear dynamical system in discrete time defined by

$$x_{n+1} = Ax_n + Bu_n, \quad y_n = Cx_n + Du_n, \quad n \in \mathbb{Z}^+; \qquad x_0 = z, \tag{1.1}$$

where $A\colon \mathcal{X} \to \mathcal{X}$, $B\colon \mathcal{U} \to \mathcal{X}$, $C\colon \mathcal{X} \to \mathcal{Y}$, and $D\colon \mathcal{U} \to \mathcal{Y}$ are bounded linear operators, $\mathcal{X}, \mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, and $\mathbb{Z}^+$ is the set of nonnegative integers. A classical problem is to modify the properties of this system by using state feedback of the type $u_n = Kx_n + v_n$, where $K\colon \mathcal{X} \to \mathcal{U}$ is another bounded linear operator. For example, one may require that the resulting closed loop system

$$\begin{aligned} x_{n+1} &= (A+BK)x_n + Bv_n, && n \in \mathbb{Z}^+, \\ y_n &= (C+DK)x_n + Dv_n, && n \in \mathbb{Z}^+, \\ u_n &= Kx_n + v_n, && n \in \mathbb{Z}^+, \\ x_0 &= z, \end{aligned} \tag{1.2}$$

is stable, or at least input-output stable in the sense that if we take $z = 0$ in (1.2) then the mapping from the input sequence $\{v_n\}_{n\in\mathbb{Z}^+}$ to the two output sequences $\{y_n\}_{n\in\mathbb{Z}^+}$ and $\{u_n\}_{n\in\mathbb{Z}^+}$ is bounded from $\ell^2(\mathbb{Z}^+;\mathcal{U})$ to $\ell^2(\mathbb{Z}^+;\mathcal{Y}\times\mathcal{U})$. In the optimal version of this problem one does not only require this input-output map to be bounded, but to have the smallest possible norm.

A special solution to this optimal control problem is well known in the case where $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are finite-dimensional: If for each $z \in \mathcal{X}$ we choose the sequence $\{u_n\}_{n\in\mathbb{Z}^+}$ in (1.1) so that it minimizes $\sum_{n=0}^{\infty}(\|y_n\|_{\mathcal{Y}}^2 + \|u_n\|_{\mathcal{U}}^2)$, then u_n is of feedback form, i.e., $u_n = Kx_n$ for some bounded feedback operator K, and this operator K minimizes the norm of the map from $\{v_n\}_{n\in\mathbb{Z}^+}$ to $\{y_n\}_{n\in\mathbb{Z}^+}$ and $\{u_n\}_{n\in\mathbb{Z}^+}$ in (1.2). Of course, in order for the existence of an optimal sequence $\{u_n\}_{n\in\mathbb{Z}^+}$ for each $z \in \mathcal{X}$ we must require that the *finite cost condition* holds, i.e., that for every $z \in \mathcal{X}$ there is a control $\{u_n\}_{n\in\mathbb{Z}^+}$ such that the solution of (1.1) satisfies $\sum_{n=0}^{\infty}(\|y_n\|_{\mathcal{Y}}^2 + \|u_n\|_{\mathcal{U}}^2) < \infty$. Then the optimal cost of a given initial state $z \in \mathcal{X}$ can be written in the form $\langle z, Qz\rangle_{\mathcal{X}}$ for some bounded nonnegative self-adjoint operator Q, and the feedback operator K is explicitly given by $K = -S^{-1}(B^*QA + D^*C)$ where $S = I + D^*D + B^*QB$. The optimal cost operator Q is the minimal nonnegative self-adjoint solution of the so called control Riccati equation. The feedback K that we get in this way is optimal even in a stronger sense: if we replace v_n in (1.2) by Ew_n for some bounded linear operator $E\colon \mathcal{W} \to \mathcal{U}$, then it is still true that the same feedback operator minimizes the ℓ^2 operator norm from w to the pair $[y;u]$. The optimal norm of this operator is equal to the norm of $(E^*SE)^{1/2}$. Additionaly, the same feedback operator also minimizes the l^1 to l^2 operator norm from w to the pair $[y;u]$.

In the above formulation of the input-output stabilization problem there is a hidden assumption which is redundant in the finite-dimensional case, but not in the infinite-dimensional case. Let us denote the different transfer functions $u \mapsto y$,

$v \mapsto u$, and $v \mapsto y$ of the systems (1.1)–(1.2) by, respectively,

$$\begin{aligned} G_{u,y}(z) &= zC(I - zA)^{-1}B + D, \\ G_{v,u}(z) &= zK(I - z(A + BK))^{-1}B + I, \\ G_{v,y}(z) &= z(C + DK)z(I - z(A + BK))^{-1}B + D. \end{aligned} \tag{1.3}$$

Then all of these are defined in a neighborhood of the origin and satisfy $G_{u,y}(z) = G_{v,y}(z)G_{v,u}(z)^{-1}$ in this neighborhood. The input-output stability of (1.2) implies that both $G_{v,y}$ and $G_{v,u}$ can be extended to H^∞-functions (i.e., bounded analytic functions) in the open unit disc $\mathbb{D}$. Thus, a *necessary condition for the input-output stabilizability of* (1.1) *is that the transfer function* $G_{u,y}$ *has a right* H^∞ *factorization in the unit disc.* This factorization condition is always satisfied in the finite-dimensional case. Moreover, in the finite-dimensional case every controllable realization satisfies the finite cost condition, so that the above outlined procedure can be applied after one restricts the system to the controllable subspace. In the infinite-dimensional case the situation is considerably more complicated. Obtaining a realization that satisfies the finite cost condition is no longer a matter of simply restricting to the controllable subspace: one has to choose the realization (and especially the norm in the state space) with care. In the continuous-time case this question is strongly related to choosing the proper function spaces on which to consider a given (formal) partial differential equation, a problem that is well-known to be extremely delicate. In the continuous-time sequel to the present article we will consider this connection with partial differential equations in more detail. Choosing the proper state space is usually considered to be something that has to be done before one can solve control problems. One of the main points of the present series of articles is that it should instead be considered as an *integral part* of the control problem.

We note that, using abstract realization theory, one can show that a function that has a right H^∞ factorization in the unit disc has a realization that satisfies the finite cost condition. This abstract realization procedure has the downside that the partial differential equation *itself* is changed, not just the space on which it is studied. The (continuous-time version of the) method outlined below only changes the space on which the partial differential equation is studied, not the equation itself.

The first main novelty in the present article is the introduction of a condition, which we call *the finite future incremental cost condition*, which is weaker than the finite cost condition. This condition simply says that each state which can be reached in a finite time from the zero initial state should have a finite cost. We show (in Theorem 6.3) that every realization of a function that has a right H^∞ factorization satisfies this finite future incremental cost condition and that conversely the transfer function of any system that satisfies the finite future incremental cost condition has a right H^∞ factorization. Theorem 6.3 also gives a third equivalent condition: the control Riccati equation has a solution. This solution may not be bounded, or even densely defined. By allowing the optimal cost operator and the optimal feedback operator to be unbounded we are able (in Theorem 5.1) to extend the procedure described above so that *it can always be applied to the given system, as soon as the necessary condition that $G_{u,y}$ has a right H^∞ factorization holds.* The resulting closed loop system will be input-output stable and have a minimal input-output norm, but it is not necessarily internally stable (it may not even be internally well-posed). By changing the norm in the state space (where the new norm is defined in terms of the solution of the input-output stabilization problem) and keeping the same formal operators we construct a new realization that does satisfy the finite cost condition and whose closed-loop system is strongly internally stable. This change of norm for the open-loop system is considered in Theorems 4.12 and 4.13 and for the closed-loop system in Theorems 5.3 and 5.4. Both cases depend crucially on the results in Appendix B, where we generalize one of the usual procedures of obtaining a minimal realization from a given realization.

Another notable novelty in the present article is our treatment of the control Riccati equation. As mentioned above, we (are forced to) consider unbounded and even non densely defined solutions of this equation. We overcome some of the technical difficulties due to this by rewriting the control Riccati equation in terms of sesquilinear forms. The usual operator formulation is considered in Appendix A. Our research led us to the observation that the control Riccati equation is most properly viewed as the synthesis of two separate equations: one that we coin the *optimal continuation equation* and one that we name the *orthogonality equation*. The first of these is obtained in Section 2 without any condition on the system whatsoever (apart from well-posedness). It is only in the orthogonality equation that the finite future incremental cost condition comes into play (see Section 3).

Below, instead of talking about the "finite cost condition" as we have done above, we shall use the name *finite future cost condition*. The inclusion of the word "future" is motivated by the fact that the optimal cost that we have described above can be interpreted as a "future" cost, i.e., it is the minimal cost associated to a given initial condition. There is a related problem where one instead of the future cost looks at the past cost, which is the cost that is associated to a given terminal condition. The latter problem is closely connected to the optimal filtering problem, and we shall return to this in our next article in this series [13].

The present article extends or complements several existing studies. In particular we mention the work of DaPrato and Delfour on unbounded solutions of Riccati equations [5], [6]; the work of Arov, Kaashoek and Pik on unbounded solutions of the Kalman-Yakubovich-Popov inequality and its relation to Schur functions in discrete-time [1] and the continuous-time analogue of this last result by Arov and Staffans [3].

2. Existence and uniqueness of the optimal control

In this section we discuss the solution of the following problem that was mentioned in the introduction: for a given $z \in \mathcal{X}$, choose the sequence $\{u_n\}_{n\in\mathbb{Z}^+}$ in (1.1) that minimizes $\sum_{n=0}^{\infty}(\|y_n\|^2_{\mathcal{Y}} + \|u_n\|^2_{\mathcal{U}})$. We show in this section that this problem is uniquely solvable if $z \in \mathcal{X}$ is such that there is a control $\{u_n\}_{n\in\mathbb{Z}^+}$ such that the solution of (1.1) satisfies $\sum_{n=0}^{\infty}(\|y_n\|^2_{\mathcal{Y}} + \|u_n\|^2_{\mathcal{U}}) < \infty$. We further show that the optimal cost is given by a closed nonnegative symmetric sesquilinear form and that the optimal control is given by a (in general unbounded and non densely defined) state feedback. The pair consisting of the sesquilinear form that gives the optimal cost and the state feedback operator that produces the optimal control solves what we call the *optimal continuation equation*. This equation can be seen as one of the constituents of the control Riccati equation (the other constituent being the *orthogonality equation* that we will introduce in a later section).

We note that some of our results in this section parallel those that DaPrato and Delfour [5], [6] obtained for continuous-time systems. Our proofs are however radically different and in our opinion much simpler.

The principal ingredient in the solution of the above optimal control problem is the following well-known result, which is often referred to as the orthogonal projection lemma.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K}$ a nonempty closed subspace of $\mathcal{H}$. Define, for $h_0 \in \mathcal{H}$, the affine set*

$$\mathcal{K}(h_0) := \{h \in \mathcal{H} : h = h_0 + k \text{ for some } k \in \mathcal{K}\}.$$

Then there exists a unique $h_{\min} \in \mathcal{K}(h_0)$ such that

$$\|h_{\min}\| = \min_{h\in\mathcal{K}(h_0)} \|h\|.$$

The vector $h_{\min}$ is characterized by the fact that $\mathcal{K}(h_0) \cap \mathcal{K}^{\perp} = \{h_{\min}\}$.

Proof. A proof can be found in many books, e.g. [8, Section 3.2]. □

To put our problem into the framework of the orthogonal projection lemma, we first analyze a certain set associated with the system. For a given system consider the affine set of *stable input-output pairs*

$$\mathcal{V}(z) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \begin{bmatrix} l^2(\mathbb{Z}^+;\mathcal{U}) \\ l^2(\mathbb{Z}^+;\mathcal{Y}) \end{bmatrix} : (1.1) \text{ is satisfied} \right\}. \tag{2.1}$$

Definition 2.2. An element z of the state space is said to have *finite future cost* if there exists an input $u \in l^2(\mathbb{Z}^+;\mathcal{U})$ such that the output y defined by (1.1) is in $l^2(\mathbb{Z}^+;\mathcal{Y})$. The set of finite future cost states is denoted by Ξ_+.

The node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is said to satisfy the *finite future cost condition* if $\Xi_+ = \mathcal{X}$.

Note that the set of finite future cost states is a subspace and that $\mathcal{V}(z)$ is nonempty if and only if $z \in \Xi_+$. The set $\mathcal{V}(z)$ will play the role of $\mathcal{K}(h_0)$ in the orthogonal projection lemma. Since $\mathcal{V}(0)$ then plays the role of $\mathcal{K}$ we have to show that it is closed.

Lemma 2.3. *$\mathcal{V}(0)$ is a nonempty closed subspace of $l^2(\mathbb{Z}^+;\mathcal{U}\times\mathcal{Y})$.*

Proof. We first note that the $\mathcal{V}(0)$ is nonempty since it contains zero. If $[u;y]\in\mathcal{V}(0)$, then

$$y_n=\sum_{k=0}^{n-1}CA^kBu_{n-k-1}+Du_n. \tag{2.2}$$

From this it is easily seen that $\mathcal{V}(0)$ is a linear space. We now prove that $\mathcal{V}(0)$ is closed. Let $[u^m;y^m]\in\mathcal{V}(0)$ and assume that there exist $u\in l^2(\mathbb{Z}^+;\mathcal{U})$ and $y\in l^2(\mathbb{Z}^+;\mathcal{Y})$ such that $u^m\to u$ in $l^2(\mathbb{Z}^+;\mathcal{U})$ and $y^m\to y$ in $l^2(\mathbb{Z}^+;\mathcal{Y})$. Then $u_n^m\to u_n$ in $\mathcal{U}$, from which we obtain

$$y_n^m=\sum_{k=0}^{n-1}CA^kBu_{n-k-1}^m+Du_n^m\to\sum_{k=0}^{n-1}CA^kBu_{n-k-1}+Du_n,$$

since we also have $y_n^m\to y_n$ in $\mathcal{Y}$ we obtain that y is the output corresponding to u. This shows that $\mathcal{V}(0)$ is closed. □

The optimal control problem considered in this section can be reformulated as follows: find the element of minimal norm in $\mathcal{V}(z)$. The next lemma shows that such an element indeed exists and is unique (provided that $z\in\Xi_+$).

Lemma 2.4. *For $z\in\Xi_+$, there exists a unique element $[u_z^{\min};y_z^{\min}]\in\mathcal{V}(z)$ with minimal norm. This element is characterized by the fact that it is the unique element of $\mathcal{V}(z)\cap\mathcal{V}(0)^\perp$.*

Proof. We apply Theorem 2.1 with $\mathcal{H}=l^2(\mathbb{Z}^+,\mathcal{U}\times\mathcal{Y})$ and $\mathcal{K}=\mathcal{V}(0)$. Note that if $(u^1,y^1),(u^2,y^2)\in\mathcal{V}(z)$, then $(u^1-u^2,y^1-y^2)\in\mathcal{V}(0)$. So $\mathcal{V}(z)$ is a translation of the closed subspace $\mathcal{V}(0)$ just like $\mathcal{K}(h_0)$ is a translation of the closed set $\mathcal{K}$. That $\mathcal{V}(0)$ is a nonempty closed subspace is the content of Lemma 2.3. The above shows that all the conditions of Theorem 2.1 are fulfilled. This theorem now gives the desired result. □

The operator that assigns the optimal input and output to a given initial state plays an important role.

Definition 2.5. The operator

$$\mathcal{I}_f:\Xi_+\to l^2(\mathbb{Z}^+;\mathcal{U}\times\mathcal{Y}),\ \mathcal{I}_fz:=\begin{bmatrix}u_z^{\min}\\ y_z^{\min}\end{bmatrix},$$

that assigns to $z\in\Xi_+$ the elements of $\mathcal{V}(z)$ with minimal norm, is called the *future minimizing operator* of the node $\left[\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right]$.

Lemma 2.6. *The future minimizing operator is a closed linear operator.*

Proof. Let $z_1,z_2\in\Xi_+$. We shall prove that $\mathcal{I}_f(z_1+z_2)=\mathcal{I}_fz_1+\mathcal{I}_fz_2$. Since the system is linear we have that the output for initial state z_1+z_2 and input $u_{z_1}^{\min}+u_{z_2}^{\min}$ is $y_{z_1}^{\min}+y_{z_2}^{\min}$. Hence $\mathcal{I}_fz_1+\mathcal{I}_fz_2\in\mathcal{V}(z_1+z_2)$. Since $\mathcal{I}_fz_1$ and $\mathcal{I}_fz_2$ are both in it $\mathcal{V}(0)^\perp$ it follows that $\mathcal{I}_fz_1+\mathcal{I}_fz_2$ is. So $\mathcal{I}_fz_1+\mathcal{I}_fz_2$ is in $\mathcal{V}(z_1+z_2)\cap\mathcal{V}(0)^\perp$. Since by Lemma 2.4 the element of $\mathcal{V}(z_1+z_2)$ with minimal norm is the unique element of this set it follows that $\mathcal{I}_f z_1 + \mathcal{I}_f z_2$ is the element of minimal norm in $\mathcal{V}(z_1 + z_2)$. Hence $\mathcal{I}_f(z_1 + z_2) = \mathcal{I}_f z_1 + \mathcal{I}_f z_2$.

We now show that $\mathcal{I}_f$ is closed. Let $z^k \in \Xi_+ \to z^\infty$ in $\mathcal{X}$, $\mathcal{I}_f z^k = [u_{z^k}^{\min}; y_{z^k}^{\min}] \to [u^\infty; y^\infty]$ in $l^2(\mathbb{Z}^+; \mathcal{U} \times \mathcal{Y})$. We need to show that $z^\infty \in \Xi_+$ and $\mathcal{I}_f z^\infty = [u^\infty; y^\infty]$. The output y for initial condition z and input u is given by

$$y_n = CA^n z + \sum_{i=0}^{n-1} CA^i B u_{n-1-i} + D u_n.$$

Applying this with $z = z^k$ and $u = u_{z^k}^{\min}$ we obtain

$$\left(y_{z^k}^{\min}\right)_n = CA^n z^k + \sum_{i=0}^{n-1} CA^i B \left(u_{z^k}^{\min}\right)_{n-1-i} + D \left(u_{z^k}^{\min}\right)_n .$$

Taking the limit for $k \to \infty$ we obtain

$$y_n^\infty = CA^n z^\infty + \sum_{i=0}^{n-1} CA^i B u_{n-1-i}^\infty + D u_n^\infty.$$

This shows that the output for initial state z^∞ and input u^∞ is y^∞. This shows that $z^\infty \in \Xi_+$ and $[u^\infty; y^\infty] \in \mathcal{V}(z^\infty)$. Since $\mathcal{V}(0)^\perp$ is closed and $\mathcal{I}_f z^k \in \mathcal{V}(0)^\perp$, it follows that $[u^\infty; y^\infty] \in \mathcal{V}(0)^\perp$. So $[u^\infty; y^\infty] \in \mathcal{V}(z^\infty) \cap \mathcal{V}(0)^\perp$, from which it follows that we have $[u^\infty; y^\infty] = \mathcal{I}_f z^\infty$. So $\mathcal{I}_f$ is closed. □

Corollary 2.7. *If a node satisfies the finite future cost condition, then its future minimizing operator is bounded.*

Proof. This follows from the closed graph theorem. □

Since $\mathcal{I}_f$ is closed, we can define a closed nonnegative symmetric sesquilinear form q_f in $\mathcal{X}$ by $q_f(z_1, z_2) := \langle \mathcal{I}_f z_1, \mathcal{I}_f z_2 \rangle$. Note that this sesquilinear form may not be densely defined.

Definition 2.8. The closed nonnegative symmetric sesquilinear form q_f on Ξ_+ defined by

$$q_f(z_1, z_2) = \left\langle \begin{bmatrix} u_{z_1}^{\min} \\ y_{z_1}^{\min} \end{bmatrix}, \begin{bmatrix} u_{z_2}^{\min} \\ y_{z_2}^{\min} \end{bmatrix} \right\rangle_{l^2(\mathbb{Z}^+; \mathcal{U} \times \mathcal{Y})}$$

is called the *future optimal cost sesquilinear form* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. The operator $z \mapsto (u_z^{\min})_0$, with domain Ξ_+, is called the *future optimal feedback operator* and is denoted by K_f.

The following theorem easily follows from the results obtained sofar in this section.

Theorem 2.9. *For every $z \in \Xi_+$ there exists a unique input $u^{\min}$ such that*

$$\sum_{n=0}^{\infty} (\|y_n^{\min}\|_{\mathcal{Y}}^2 + \|u_n^{\min}\|_{\mathcal{U}}^2) = \inf_u \sum_{n=0}^{\infty} (\|y_n\|_{\mathcal{Y}}^2 + \|u_n\|_{\mathcal{U}}^2)$$

under the constraint (1.1). *The minimum is equal to* $q_f(z,z)$.

Proof. This is just a reformulation of Lemma 2.4 and the definition of q_f. □

The following result is known as Bellman's principle of optimality.

Lemma 2.10. *For* $z \in \Xi_+$ *and* $u_0 \in \mathcal{U}$ *we have (with* $x_1 = Az+Bu_0$, $y_0 = Cz+Du_0$*)*

$$q_f(z,z) \leq q_f(x_1,x_1) + \|u_0\|^2 + \|y_0\|^2,$$

where we have equality if and only if $u_0 = K_f z$. *(Here* $q_f(x_1,x_1)$ *should be interpreted as infinity if* $x_1 \notin \Xi_+$*).*

Proof. This follows using a simple contradiction argument and uniqueness of the optimal control. □

Definition 2.11. The pair (q,K) is called a (nonnegative) solution of the *optimal continuation equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if

1. q is a closed nonnegative symmetric sesquilinear form in $\mathcal{X}$.
2. $K : D(q) \to \mathcal{U}$ is a linear operator.
3. For all $z \in D(q)$, we have $(A+BK)z \in D(q)$ and

$$q((A+BK)z,(A+BK)z) + \|(C+DK)z\|^2_{\mathcal{Y}} + \|Kz\|^2_{\mathcal{U}} = q(z,z). \tag{2.3}$$

The solutions of the optimal continuation equation are ordered according to the sesquilinear form q. We recall that for sesquilinear forms t_1 and t_2 (defined in the same space) we have $t_1 \leq t_2$ by definition if $D(t_1) \supset D(t_2)$ and $t_1(h,h) \leq t_2(h,h)$ for all $h \in D(t_2)$.

Lemma 2.12. *The pair* (q_f, K_f) *is the smallest solution of the optimal continuation equation.*

Proof. We first show that (q_f,K_f) is indeed a solution. We have $(A+BK_f)z \in \Xi_+$ since $\mathcal{V}((A+BK_f)z)$ contains the element $[\tilde{u};\tilde{y}]$ defined by $[\tilde{u};\tilde{y}]_n := (\mathcal{I}_f z)_{n+1}$. That (q_f,K_f) satisfies (2.3) follows immediately from the equality case in Bellman's principle of optimality, Lemma 2.10.

To show that (q_f,K_f) is the smallest solution we assume that (q,K) is another solution and apply the feedback input $u_n = Kx_n$ to the system (1.1), where $z \in D(q)$. By induction we obtain from (2.3)

$$q(z,z) = q(x_n,x_n) + \sum_{k=0}^{n-1} \|y_k\|^2 + \|u_k\|^2,$$

which implies

$$\sum_{k=0}^{n-1} \|y_k\|^2 + \|u_k\|^2 \leq q(z,z).$$

By letting $n \to \infty$, it follows that $[u; y] \in l^2(\mathbb{Z}^+, \mathcal{U} \times \mathcal{Y})$, so $z \in \Xi_+$, and

$$q_f(z,z) = \sum_{k=0}^{\infty} (\|y_k^{\min}\|^2_{\mathcal{Y}} + \|u_k^{\min}\|^2_{\mathcal{U}}) \leq \sum_{k=0}^{\infty} (\|y_k\|^2_{\mathcal{Y}} + \|u_k\|^2_{\mathcal{U}}) \leq q(z,z).$$

So indeed $q_f \leq q$. □
We remind the reader that the kernel of a nonnegative symmetric sesquilinear form q consists of all the elements z in the domain of the sesquilinear form for which $q(z,z) = 0$.

Lemma 2.13. *Let q be a closed nonnegative symmetric sesquilinear form. Then the kernel $\mathcal{N}(q)$ of q has the following properties:*

1. *$\mathcal{N}(q)$ is closed.*
2. *$\mathcal{N}(q_1) \supset \mathcal{N}(q_2)$ if $q_1 \leq q_2$.*

If (q, K) is a solution of the optimal continuation equation, then $\mathcal{N}(q)$ is A-invariant.

Proof. 1. This follows from the fact that q is closed.
2. Let $z \in \mathcal{N}(q_2)$. It follows that $z \in D(q_1)$ and $q_1(z,z) \leq q_2(z,z) = 0$, so $q_1(z,z) = 0$. Hence $z \in \mathcal{N}(q_1)$.
The claim that $\mathcal{N}(q)$ is A-invariant is established as follows. Let $z \in \mathcal{N}(q)$. Then the right-hand side of (2.3) is zero. From this we obtain that $Kz = 0$ and $q((A+BK)z, (A+BK)z) = 0$. This implies that $q(Az, Az) = 0$, so $Az \in \mathcal{N}(q)$.
□

3. The control Riccati equation

We saw in Section 2 that the pair consisting of the sesquilinear form that gives the optimal cost and the state feedback operator that produces the optimal control provide a solution to the optimal continuation equation. In the classical situation of a bounded feedback more can be said: the above mentioned pair satisfies the *control Riccati equation*. In this section we show that in the case of an unbounded feedback operator the same is true if and only if the *finite future incremental cost condition* holds. Alternative formulations of the control Riccati equation are given in Appendix A.

To precisely formulate this finite future incremental cost condition we need the following concept.

Definition 3.1. An element w of the state space is said to be *finite time reachable* if there exists a N and an input $u : \mathbb{Z}^+ \to \mathcal{U}$ such that the state x defined by (1.1) with $z = 0$ satisfies $x_N = w$. The set of finite-time reachable states is denoted by Ξ_-.

Remark 3.2. It is easily seen that the set of finite time reachable states is an A-invariant subspace.

The following concept is new and is fundamental to the present article.

Definition 3.3. A node satisfies the *finite future incremental cost condition* if every element of the state space that is finite time reachable has finite future cost.

Note that a node that satisfies the finite future cost condition satisfies the finite future incremental cost condition, but that the converse may not be true.

The following lemma provides a further connection between the optimal control problem and the finite future incremental cost condition.

Lemma 3.4. *The finite future incremental cost condition holds if and only if all the elements in the image of B have finite cost. If this condition holds, then Ξ_+ is A-invariant.*

Proof. That the finite future incremental cost condition implies that all the elements in the image of B have finite cost is trivial.

We next show that if all the elements in the image of B have finite cost, then Ξ_+ is A-invariant. Let $z \in \Xi_+$. Then there exists a $[u; y] \in \mathcal{V}(z)$. It follows that $x_1 := Az + Bu_0 \in \Xi_+$ since $[(u_1, \ldots), (y_1, \ldots)] \in \mathcal{V}(x_1)$. Since $Bu_0 \in \Xi_+$ it then follows that $Az = x_1 - Bu_0 \in \Xi_+$.

Assume that all the elements in the image of B have finite cost. It is easily seen that Ξ_- is the smallest A-invariant subspace that contains the image of B. In the preceding paragraph it was proven that Ξ_+ is also an A-invariant subspace that contains the image of B. Hence $\Xi_- \subset \Xi_+$, i.e., the finite future incremental cost condition condition holds. □

The sesquilinear form that we introduce in the following definition, as we will show later, is closely related to the minimal norm of the mapping from the input sequence $\{v_n\}_{n\in\mathbb{Z}^+}$ to the two output sequences $\{y_n\}_{n\in\mathbb{Z}^+}$ and $\{u_n\}_{n\in\mathbb{Z}^+}$ (with initial condition $z = 0$) in (1.2).

Definition 3.5. If the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the finite future incremental cost condition, then we define its *optimal sensitivity sesquilinear form* s_f on $\mathcal{U}$ by $s_f(u_1, u_2) = q_f(Bu_1, Bu_2) + \langle u_1, u_2\rangle + \langle Du_1, Du_2\rangle$.

Note that s_f is a *bounded* nonnegative symmetric sesquilinear form on $\mathcal{U}$, since it is closed and everywhere defined.

The following *orthogonality equation* together with the optimal continuation equation will provide the control Riccati equation.

Definition 3.6. The pair (q, K) is called a (nonnegative) solution of the *orthogonality equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if

1. q is a closed nonnegative symmetric sesquilinear form in $\mathcal{X}$ whose domain satisfies $AD(q) \subset D(q)$, $B\mathcal{U} \subset D(q)$.
2. $K : D(q) \to \mathcal{U}$ is a linear operator.
3. For all $z \in D(q)$ and $u \in \mathcal{U}$ we have
$$\langle (C + DK)z, Du\rangle_{\mathcal{Y}} + \langle Kz, u\rangle_{\mathcal{U}} + q((A + BK)z, Bu) = 0. \tag{3.1}$$

The following lemma shows that the pair consisting of the sesquilinear form that gives the optimal cost and the state feedback operator that produces the optimal control satisfy the orthogonality equation.

Lemma 3.7. *Assume that the finite future incremental cost condition holds. Then the pair (q_f, K_f) is a solution of the orthogonality equation.*

Proof. By Lemma 2.4 $[u_z^{\min}; y_z^{\min}]$ is orthogonal to every element of $\mathcal{V}(0)$. It is easily seen that $[v; y]$ defined by $[v; y]_0 := [u; Du]$, $[v; y]_n := (\mathcal{I}_f Bu)_{n-1}$ is an element of $\mathcal{V}(0)$. So

$$\left\langle \begin{bmatrix} u_z^{\min} \\ y_z^{\min} \end{bmatrix}_0, \begin{bmatrix} u \\ Du \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} u_z^{\min} \\ y_z^{\min} \end{bmatrix}_{\geq 1}, \begin{bmatrix} u_{Bu}^{\min} \\ y_{Bu}^{\min} \end{bmatrix} \right\rangle = 0.$$

Using that (q_f, K_f) is a solution of the optimal continuation equation (Lemma 2.12) we obtain

$$\left\langle \begin{bmatrix} K_f z \\ (C + DK_f)z \end{bmatrix}, \begin{bmatrix} u \\ Du \end{bmatrix} \right\rangle + \langle \mathcal{I}_f (A + BK_f)z, \mathcal{I}_f Bu \rangle = 0,$$

which gives the desired result. □

The optimal cost continuation equation and the orthogonality equation can be combined into one equation.

Definition 3.8. The triple (q, s, K) is called a (nonnegative) solution of the *closed-loop control Riccati equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if

1. q is a closed nonnegative symmetric sesquilinear form in $\mathcal{X}$ whose domain satisfies $AD(q) \subset D(q)$, $B\mathcal{U} \subset D(q)$.
2. s is a bounded nonnegative symmetric sesquilinear form on $\mathcal{U}$.
3. $K : D(q) \to \mathcal{U}$ is a linear operator.
4. For all $z \in D(q)$ and $v \in \mathcal{U}$ we have

$$q((A+BK)z+Bv, (A+BK)z+Bv)+\|Kz+v\|_{\mathcal{U}}^2+\|(C+DK)z+Dv\|_{\mathcal{Y}}^2 = q(z, z)+s(v, v). \tag{3.2}$$

Lemma 3.9. *The triple (q, s, K) is a solution of the closed-loop control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if and only if the pair (q, K) satisfies both the optimal continuation equation and the orthogonality equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and s is given by*

$$s(u_1, u_2) := \langle u_1, u_2 \rangle + \langle Du_1, Du_2 \rangle + q(Bu_1, Bu_2).$$

Proof. The proof is entirely similar to the proof of Lemma A.2 and hence omitted. □

Definition 3.10. The triple (q, s, K) is called a (nonnegative) solution of the *control Riccati equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if

1. q is a closed nonnegative symmetric sesquilinear form in $\mathcal{X}$ whose domain satisfies $AD(q) \subset D(q)$, $B\mathcal{U} \subset D(q)$.

2. s is a bounded nonnegative symmetric sesquilinear form on $\mathcal{U}$.
3. $K : D(q) \to \mathcal{U}$ is a linear operator.
4. For all $z \in D(q)$, $u \in \mathcal{U}$ we have

$$q(Az + Bu, Az + Bu) + \|Cz + Du\|_{\mathcal{Y}}^2 + \|u\|_{\mathcal{U}}^2 = q(z, z) + s(Kz - u, Kz - u). \quad (3.3)$$

The solution is called *classical* when $D(q) = \mathcal{X}$.

Remark 3.11. Note that for a classical solution of the control Riccati equation the sesquilinear form q is bounded by the closed graph theorem. It can be shown that in this case K is bounded as well.

Remark 3.12. Solutions of the control Riccati equation are orderded by the usual ordering of unbounded sesquilinear forms, i.e. $q_1 \leq q_2$ if $D(q_1) \supset D(q_2)$ and $q_1(z, z) \leq q_2(z, z)$ for all $z \in D(q_2)$. Note that we only consider q and not s and K in this definition of order.

The following lemma shows that the closed-loop control Riccati equation and the control Riccati equation are simply reformulations of each other.

Lemma 3.13. *The triple* (q, s, K) *satisfies the closed-loop control Riccati equation of* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *if and only if it satisfies the control Riccati equation of* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

Proof. We use Lemma 3.9 and several results from Appendix A here.

Assume (q, s, K) satisfies the control Riccati equation. That (q, K) satisfies the optimal continuation equation follows by substituting $u = Kz$ in (3.3). The only thing left to check is the equation (3.1). By Lemma A.2 the triple (q, s, K) satisfies the Lure control Riccati equation. It follows that, for $z \in D(q)$ and $u, u_1, u_2 \in \mathcal{U}$,

$$\begin{aligned} s(u_1, u_2) &= \langle u_1, u_2\rangle_{\mathcal{U}} + \langle Du_1, Du_2\rangle_{\mathcal{Y}} + q(Bu_1, Bu_2), \\ -s(Kz, u) &= \langle Cz, Du\rangle_{\mathcal{Y}} + q(Az, Bu) \end{aligned} \quad (3.4)$$

(to obtain the first equation we polarize the middle equation in (A.1)). Writing out the left-hand side of (3.1) in full gives

$$\langle Cz, Du\rangle_{\mathcal{Y}} + \langle DKz, Du\rangle_{\mathcal{Y}} + \langle Kz, u\rangle_{\mathcal{U}} + q(Az, Bu) + q(BKz, Bu). \quad (3.5)$$

Applying (3.4) with $u_1 = Kz$, $u_2 = u$ shows that the expression (3.5) equals zero as desired.

Now assume that (q, K) satisfies the optimal continuation equation and the orthogonality equation. It follows that the expression (3.5) equals zero. Using the definition of s, it follows that the second equation of (3.4) holds true. Using Lemma A.2 we see that this only leaves to show, that for all $z \in D(q)$,

$$q(Az, Az) + \|Cz\|_{\mathcal{Y}}^2 = q(z, z) + s(Kz). \quad (3.6)$$

Writing out the given (2.3) in full results in

$$\begin{aligned} &q(Az, Az) + q(BKz, BKz) + q(Az, BKz) + q(BKz, Az) + \|Cz\|_{\mathcal{Y}}^2 \\ &\quad + \|DKz\|_{\mathcal{Y}}^2 + \langle Cz, DKz\rangle_{\mathcal{Y}} + \langle DKz, Cz\rangle_{\mathcal{Y}} + \|Kz\|_{\mathcal{U}}^2 = q(z, z). \end{aligned}$$

Using the definition of s to evaluate $s(Kz, Kz)$ and the just proven second equation of (3.4) with $u = Kz$, this gives (3.6). $\square$

The following theorem shows that the sesquilinear form that gives the optimal cost gives rise to the smallest solution of the control Riccati equation.

Theorem 3.14. *Assume that the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the finite future incremental cost condition. Then the triple* (q_f, s_f, K_f) *is a solution of the control Riccati equation of* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$*. Moreover, it is the smallest solution.*

Proof. That the pair (q_f, K_f) satisfies the optimal continuation equation follows from Lemma 2.12. That it is a solution of the orthogonality equation is the content of Lemma 3.7. By Lemmas 3.9 and 3.13 the triple (q_f, s_f, K_f) is a solution of the control Riccati equation.

Since by Lemma 2.12 the pair (q_f, K_f) is the smallest solution of the optimal continuation equation and by Lemmas 3.9 and 3.13 every solution (q, s, K) of the control Riccati equation gives a solution (q, K) of the optimal continuation equation, it follows that (q_f, s_f, K_f) is the smallest solution of the control Riccati equation. $\square$

The next two lemmas gives some consequences of the control Riccati equation that will be useful in the forthcoming sections.

Lemma 3.15. *Let* (q, s, K) *be a solution of the control Riccati equation of the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$*. Then* $\Xi_- \subset D(q) \subset \Xi_+$.

Proof. That $\Xi_- \subset D(q)$ follows from the domain inclusions in the definition of the control Riccati equation. That $D(q) \subset \Xi_+$ follows from the fact that q_f is the smallest solution of the control Riccati equation by Theorem 3.14 and $D(q_f) = \Xi_+$. $\square$

Lemma 3.16. *Assume that* (q, s, K) *is a solution of the control Riccati equation of* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$*. Then, for the sequences* v, u, y, x *related by* (1.2)*, we have for all* $n \in \mathbb{Z}^+$

$$q(x_{n+1}, x_{n+1}) + \sum_{k=0}^{n} \|u_k\|^2_{\mathcal{U}} + \|y_k\|^2_{\mathcal{Y}} = q(z, z) + \sum_{k=0}^{n} s(v_k, v_k). \tag{3.7}$$

Proof. By (1.2) and (3.2) with z replaced by x_k and v replaced by v_k, for all $k \in \mathbb{Z}^+$,

$$q(x_{k+1}, x_{k+1}) + \|u_k\|^2_{\mathcal{U}} + \|y_k\|^2_{\mathcal{Y}} = q(x_k, x_k) + s(v_k, v_k).$$

Adding these identities over $k = 0, 1, \ldots, n$ we get (3.7). $\square$

4. Special realizations

In this section we show, using an unbounded solution of the control Riccati equation, how it is possible to construct a node with the same transfer function as the original node and with a classical solution to its control Riccati equation. We make extensive use of Appendix B on completions of compressions of systems.

We first study observability and the observable part of a node.

Definition 4.1. An element z of the state space is called *unobservable* if for zero input u the output y defined by (1.1) is zero.

Remark 4.2. The set of all unobservable elements is easily seen to be a closed A-invariant subspace of the state space. It is called the unobservable subspace and is denoted by $\mathcal{N}$.

Definition 4.3. A node is called *observable* if its unobservable subspace is the zero vectorspace.

The following lemma relates observability and the optimal control problem.

Lemma 4.4. *We have* $\mathcal{N}(q_f) = \mathcal{N}$.

Proof. Suppose that $z \in \mathcal{N}(q_f)$. Then the optimal cost for initial state z is zero, so the output for zero input and initial state z is zero. So $z \in \mathcal{N}$.

If for zero input and initial state z the output is zero, then obviously $z \in \Xi_+$ and the optimal cost is zero, so $q_f(z,z) = 0$, so $z \in \mathcal{N}(q_f)$. □

In the next definition we extend the classical definition of observable part.

Definition 4.5. Let q be a closed nonnegative symmetric sesquilinear form such that $\mathcal{N}(q)$ is A-invariant and is contained in $\mathcal{N}$. The *q-observable part* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is

$$\begin{bmatrix} A_o^q & B_o^q \\ C_o^q & D_o^q \end{bmatrix} := \begin{bmatrix} P_{\mathcal{N}(q)^\perp} A|_{\mathcal{N}(q)^\perp} & P_{\mathcal{N}(q)^\perp} B \\ C|_{\mathcal{N}(q)^\perp} & D \end{bmatrix},$$

with state space $\mathcal{N}(q)^\perp$ (with the subspace topology).

In the special case where $q = q_f$ this node is simply called the *observable part*.

If q is not just any closed nonnegative symmetric sesquilinear form, but a solution of the optimal continuation equation, then the q-observable part has many interesting properties. We collect some of these in the following lemma.

Lemma 4.6. *Let (q, K) be a solution of the optimal continuation equation of* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

1. *For the same initial state $z \in \mathcal{N}(q)^\perp$ and input $\{u_n\}_{n\in\mathbb{Z}^+}$, the state and output of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and its q-observable part are related by $x_n^o = P_{\mathcal{N}(q)^\perp} x_n$, $y_n^o = y_n$ (for all $n \in \mathbb{Z}^+$).*
2. *The finite-time reachable states of the q-observable part are exactly the elements of $P_{\mathcal{N}(q)^\perp} \Xi_-$.*

3. *The elements of* $P_{\mathcal{N}(q)^\perp}\Xi_+$ *are exactly the finite cost states for the q-observable part.*
4. *The future minimizing operator of the q-observable part is* $\mathcal{I}_f|_{P_{\mathcal{N}(q)^\perp}\Xi_+}$.
5. *If* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the finite future incremental cost condition, then so does its q-observable part.*
6. *If* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the finite future cost condition, then so does its q-observable part.*

Proof. 1. This is a special case of Theorem B.1 with $\mathcal{V} = \mathcal{N}(q)$ and $\mathcal{W} = \mathcal{X}$. Lemma 2.13 and the fact that $\mathcal{N} \subset \ker C$ show that the assumptions of Theorem B.1 are satisfied.
2. This is vitually identical to the proof of Lemma B.16 and is therefore omitted.
3. and 4. It follows from part 1 of this lemma that the finite cost states for the q-observable part are exactly those elements of $\mathcal{N}(q)^\perp$ that are finite cost states for the original node (i.e. the elements of $\Xi_+ \cap \mathcal{N}(q)^\perp$) and that in this case the optimal control and the optimal output are equal for both nodes. From Lemma B.9 with $\mathcal{V} = \mathcal{N}(q)$ and $\mathscr{W} = \Xi_+$ we obtain $P_{\mathcal{N}(q)^\perp}\Xi_+ = \Xi_+ \cap \mathcal{N}(q)^\perp$. Note that the assumption $\mathcal{V} \subset \mathscr{W}$ is satisfied since $\mathcal{N}(q) \subset D(q)$ and $D(q) \subset \Xi_+$ by Lemma 2.12. The desired conclusions follow.
5. If the original node satisfies the finite future incremental cost condition, then $\Xi_- \subset \Xi_+$. Obviously this implies that $P_{\mathcal{N}(q)^\perp}\Xi_- \subset P_{\mathcal{N}(q)^\perp}\Xi_+$. The assertion then follows from parts 2 and 3 of this lemma.
6. It follows from $\Xi_+ = \mathcal{X}$ that $P_{\mathcal{N}(q)^\perp}\Xi_+ = \mathcal{N}(q)^\perp$, which proves the assertion.
□

Next we define what it means for (components of) a node to be bounded with respect to a nonnegative symmetric sesquilinear form.

Definition 4.7. Let q be a nonnegative symmetric sesquilinear form in $\mathcal{X}$.

- An operator $T \in \mathcal{L}(\mathcal{X})$ is called *bounded with respect to q* if $D(q)$ is T-invariant and there exists a $M \geq 0$ such that $q(Tz, Tz) \leq Mq(z, z)$ for all $z \in D(q)$.
- An operator $T \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ is called *bounded with respect to q* if $T\mathcal{U} \subset D(q)$ and there exists a $M \geq 0$ such that $q(Tu, Tu) \leq M\|u\|_{\mathcal{U}}^2$ for all $u \in \mathcal{U}$.
- An operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is called *bounded with respect to q* if there exists a $M \geq 0$ such that $\|Tz\|_{\mathcal{Y}}^2 \leq Mq(z, z)$ for all $z \in D(q)$.

As the next lemma shows, the optimal continuation equation and the control Riccati equation imply that certain operators are bounded with respect to the solution of the equation.

Lemma 4.8. *If* (q, K) *is a solution of the optimal continuation equation of* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, *then* K, C *and* $A + BK$ *are bounded with respect to* q.
If (q, s, K) *is a solution of the control Riccati equation of* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, *then* A, B, C *and* K *are bounded with respect to* q.

Proof. From the optimal continuation equation we obtain, for all $z \in D(q)$, $\|Kz\|^2 \leq q(z,z)$, which implies that K is bounded with respect to q. Similarly we obtain that $C+DK$ is bounded with respect to q. Since DK is bounded with respect to q, it follows from this that C is. From the optimal continuation equation we also immediately obtain that $A+BK$ is bounded with respect to q.

Now assume that (q,s,K) is a solution of the control Riccati equation. The q-boundedness of K and C follows from the first part of the lemma and the fact that a solution of the control Riccati equation is a solution of the optimal continuation equation (Lemmas 3.9 and 3.13). That A is bounded with respect to q follows from (3.3) with $u=0$, which gives $q(Az,Az) \leq q(z,z)$. That B is bounded with respect to q follows from (3.3) with $z=0$, which gives $q(Bu,Bu) \leq s(u)$ and using that s is bounded. $\square$

For a nonnegative symmetric sesquilinear form q in $\mathcal{X}$ it is easily seen that

$$\langle z_1, z_2\rangle_g := \langle z_1, z_2\rangle_{\mathcal{X}} + q(z_1,z_2)$$

is an inner product on $D(q)$. We will call this the *graph inner product* induced by q.

Lemma 4.9. *Let (q,s,K) be a solution of the control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then A, B and C are bounded with respect to the graph norm induced by q on $D(q)$.*

Proof. This follows immediately from Lemma 4.8 and the fact that A, B, C are bounded with respect to the $\mathcal{X}$-norm. For example for A we have $q(Az,Az) \leq M_1 q(z,z)$ and $\|Az\|^2_{\mathcal{X}} \leq M_2\|z\|^2_{\mathcal{X}}$, so

$$q(Az,Az) + \|Az\|^2_{\mathcal{X}} \leq \max\{M_1,M_2\}\left(q(z,z) + \|z\|^2_{\mathcal{X}}\right),$$

and the proofs for B and C are similar. $\square$

Lemma 4.9 enables us to make the following definition.

Definition 4.10. Assume that the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has a solution (q,s,K) to its control Riccati equation. Then the restriction of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to $D(q) \times \mathcal{U}$ and with codomain $D(q) \times \mathcal{Y}$, where $D(q)$ is equipped with the graph norm in both instances, is called the *graph node* induced by (q,s,K).

We note that the graph inner product and the graph node were used in DaPrato and Delfour [5], [6] as well.

The following lemma gives some properties of the graph node.

Lemma 4.11. *Let (q,s,K) be a solution of the control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.*

1. *For the same initial state $z \in D(q)$ and input $\{u_n\}_{n\in\mathbb{Z}^+}$, the state and the output of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and its graph node are equal.*
2. *The finite-time reachable states of the graph node are the elements of Ξ_-.*
3. *The triple (q,s,K) is a classical solution of the control Riccati equation of the graph node.*

Proof. 1. This is obvious from the domain inclusions in the definition of the control Riccati equation.
2. This follows from part 1 of this lemma and the fact that $\Xi_- \subset D(q)$.
3. The equation (3.3) obviously holds for the graph node since it holds for the original node. The sesquilinear form q is defined on the whole of $D(q)$ and is bounded in the graph norm: $q(z,z) \leq \|z\|_g^2 = \|z\|_{\mathcal{X}}^2 + q(z,z)$. So we have a classical solution of the control Riccati equation. □

As Lemma 4.11 shows, the graph node has certain desirable properties. However, it also lacks certain desirable properties (e.g. internal stability of the closed-loop system, which we will discuss in Section 5). For this reason we introduce the concept of a completed q-compression. This basically means that we make q the new inner-product of the state space of the q-observable part and extend by continuity to again obtain a node (see Appendix B for the precise details).

Theorem 4.12. *Let (q, s, K) be a solution of the control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.*

1. *The completed q-compression is well-defined.*
2. *For the same initial state $z \in P_{\mathcal{N}(q)^\perp} D(q)$ and input $\{u_n\}_{n\in\mathbb{Z}^+}$, the state and output of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and its completed q-compression are related by $x_n^q = P_{\mathcal{N}(q)^\perp} x_n$, $y_n^q = y_n$ (for all $n \in \mathbb{Z}^+$).*
3. *The finite-time reachable states of the completed q-compression are exactly the elements of $P_{\mathcal{N}(q)^\perp}\Xi_-$.*
4. *The control Riccati equation of the completed q-compression has a classical solution in which the sesquilinear form in the state space coincides with the inner-product in the state space.*
5. *The future minimizing operator of the completed q_f-compression is an isometry.*
6. *The completed q_f-compression is observable.*

Proof. 1. We check the conditions of Theorem B.14. That q is a closable (even closed) nonnegative symmetric sesquilinear form and that its domain is strongly-invariant follows immediately from the control Riccati equation. That the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is bounded with respect to the semi-norm induced by q on $D(q)$ is guaranteed by Lemma 4.8. Hence Theorem B.14 applies and the completed q-compression of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is well-defined.
2. and 3. follow directly from Lemma B.16.
4. We first note that $P_{\mathcal{N}(q)^\perp} D(q) \subset D(q)$ by Corollary B.10 and that (using $P_{\mathcal{N}(q)^\perp}^2 = P_{\mathcal{N}(q)^\perp}$)

$$q(P_{\mathcal{N}(q)^\perp} z_1, P_{\mathcal{N}(q)^\perp} z_2) = q(z_1, z_2) \ z_1, z_2 \in P_{\mathcal{N}(q)^\perp} D(q). \tag{4.1}$$

Equation (3.3) holds for $z \in P_{\mathcal{N}(q)^\perp} D(q)$ by the above inclusion and A, B, C can be replaced by the corresponding operators of the completed q-compression by the equality (4.1). Since $P_{\mathcal{N}(q)^\perp} D(q)$ is dense in the state space of the completed q-compression and by continuity of the involved operators and sesquilinear forms the equation holds for all z in the state space of the completed q-compression.

So we obtain a classical solution of the control Riccati equation of the completed q-compression. The first component of this solution is a continuous extension of a compression of q, which is the inner-product of the completed q-compression.
5. It follows from part 1 of this lemma that the optimal control and the optimal output for an initial condition in $P_{\mathcal{N}(q)^\perp}D(q)$ are the same for the original node and for its completed q-compression, so the future minimizing operators are the same when restricted to $P_{\mathcal{N}(q)^\perp}D(q)$. Since the norm in the state space for the completed q_f-compression is $\|\mathcal{I}_f z\|_{l^2(\mathbb{Z}^+,\mathcal{U}\times\mathcal{Y})}$ for z in the dense subspace $P_{\mathcal{N}(q)^\perp}D(q)$, it follows that the future minimizing operator of the completed q_f-compression is an isometry.
6. It follows from Lemma 4.4 that the kernel of the optimal cost sesquilinear form of the completed q_f-compression equals the unobservable subspace of the completed q_f-compression. The kernel of the optimal cost sesquilinear form is equal to the kernel of the future minimizing operator. Since by part 5 of this lemma the kernel of the future minimizing operator is trivial, it follows that the completed q_f-compression is observable. □

An unfortunate thing about the completed q-compression from Theorem 4.12 is that it is in general not controllable. We take care of this problem in the following lemma by considering a restriction of q instead of q itself.

Theorem 4.13. *Let (q, s, K) be a solution of the control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Denote $q|_{\Xi_-}$ by q^-.*

1. *The completed q^--compression is well-defined.*
2. *For the same initial state $z \in P_{\mathcal{N}(q)^\perp}\Xi_-$ and input $\{u_n\}_{n\in\mathbb{Z}^+}$, the state and output of a node and its completed q^--compression are related by $x_n^q = P_{\mathcal{N}(q)^\perp}x_n$, $y_n^q = y_n$ (for all $n \in \mathbb{Z}^+$).*
3. *The finite-time reachable states of the completed q^--compression are exactly the elements of $P_{\mathcal{N}(q)^\perp}\Xi_-$.*
4. *The control Riccati equation of the completed q^--compression has a classical solution in which the sesquilinear form in the state space coincides with the inner-product in the state space.*
5. *The completed q^--compression is controllable.*
6. *The future minimizing operator of the completed q_f^--compression is an isometry.*
7. *The completed q_f^--compression is observable.*

Proof. 1. We check the conditions of Theorem B.14. That q^- is a closable nonnegative symmetric sesquilinear form follows from the fact that it is a restriction of the closed nonnegative symmetric sesquilinear form q. That its domain, Ξ_-, is strongly invariant is obvious. That the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is bounded with respect to the semi-norm induced by q^- on $D(q^-)$ is guaranteed by Lemma 4.8. Hence Theorem B.14 applies and the completed q^--compression of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is well-defined.
2. and 3. follow directly from Lemma B.16.

4. follows in the same way as the corresponding statement in Theorem 4.12, now using that $P_{\mathcal{N}(q)^\perp}\Xi_- \subset D(q)$ by Corollary B.10 and $\Xi_- \subset D(q)$.
5. Controllability follows since the finite-time reachable states (which are the elements of $P_{\mathcal{N}(q)^\perp}\Xi_-$ by part 3 of this lemma) are by definition of the state space dense.
6. Follows in exactly the same way as the corresponding statement in Theorem 4.12.
7. Observability follows in exactly the same way as the corresponding statement in Theorem 4.12. □

5. The closed-loop system

In this section we study the closed-loop system (1.2). We first establish for which unbounded feedback operators K the problem is well-defined. Next we show that the future optimal cost feedback operator minimizes the closed-loop input-output norm among the allowed feedback operators. We then discuss stability properties of the closed-loop system. To obtain good stability properties we have to consider completed q-compressions (see Appendix B).

Instead of the system (1.2) we initially study the more general system

$$\begin{aligned} x_{n+1} &= (A+BK)x_n + BEw_n, && n \in \mathbb{Z}^+, \\ y_n &= (C+DK)x_n + DEw_n, && n \in \mathbb{Z}^+, \\ u_n &= Kx_n + Ew_n, && n \in \mathbb{Z}^+, \\ x_0 &= z, \end{aligned} \tag{5.1}$$

where $E\colon \mathcal{W} \to \mathcal{U}$ is a bounded linear operator and $K : D(K) \subset \mathcal{X} \to \mathcal{U}$ is a linear operator with a domain that is A-invariant and that contains the image of B.

In the next theorem we first show that with the above assumptions on K the input-output map of (5.1) is well-defined. We then show that the future optimal feedback operator minimizes both the $\mathcal{L}(l^1(\mathbb{Z}^+,\mathcal{W}), l^2(\mathbb{Z}^+,\mathcal{Y}\times\mathcal{U}))$ and the $\mathcal{L}(l^2(\mathbb{Z}^+,\mathcal{W}), l^2(\mathbb{Z}^+,\mathcal{Y}\times\mathcal{U}))$ norm of this input-output map and how this minimal norm can be expressed in terms of the optimal sensitivity sesquilinear form.

Theorem 5.1. 1. *For $K : D(K) \subset \mathcal{X} \to \mathcal{U}$ with a domain that is A-invariant and that contains the image of B, the map from $\{w_n\}_{n\in\mathbb{Z}^+}$ to $\{[y_n; u_n]\}_{n\in\mathbb{Z}^+}$ in* (5.1) *(with $z = 0$) is well-defined on the sequences with finite support.*

2. *Assume that the finite future incremental cost condition holds. Then the future optimal feedback operator K_f minimizes both the $\mathcal{L}(l^1(\mathbb{Z}^+,\mathcal{W}), l^2(\mathbb{Z}^+,\mathcal{Y}\times\mathcal{U}))$ and the $\mathcal{L}(l^2(\mathbb{Z}^+,\mathcal{W}), l^2(\mathbb{Z}^+,\mathcal{Y}\times\mathcal{U}))$ norm of the map from $\{w_n\}_{n\in\mathbb{Z}^+}$ to $\{[y_n; u_n]\}_{n\in\mathbb{Z}^+}$ in* (5.1) *(with $z = 0$), where K ranges over all linear maps $D(K) \subset \mathcal{X} \to \mathcal{U}$ with a domain that is A-invariant and that contains the image of B. These minimum norms both equal the square root of $\sup_{\|v\|=1} s_f(Ev, Ev)$, where s_f is the optimal sensitivity sesquilinear form.*

Proof. 1. Since $D(K)$ is A-invariant and contains the image of B, it follows that $(C + DK)(A + BK)^n BE$, $K(A + BK)^n BE$ are well-defined for $n \geq 0$. It follows that the map from $\{w_n\}_{n\in\mathbb{Z}^+}$ to $\{[y_n; u_n]\}_{n\in\mathbb{Z}^+}$ in (5.1) is well-defined on the sequences with finite support.
2. We first show that $\sup_{\|v\|=1} s_f(Ev, Ev)$ is a lower bound for the the square of the $\mathcal{L}(l^p(\mathbb{Z}^+, \mathcal{W}), l^2(\mathbb{Z}^+, \mathcal{Y} \times \mathcal{U}))$ norm for any $p \geq 1$. In (5.1) take $z = 0$ and w as $w_0 = v$, $w_k = 0$ if $k > 0$. Note that the l^p norm of w is $\|v\|_{\mathcal{W}}$, which is independent of p. Also note that with this choice of w, the trajectories of (5.1) with initial condition zero are the same as those for (1.1) with $u_n = Kx_n$ and initial condition $z = BEv$ shifted by one time unit. It follows that with this choice we have

$$\begin{aligned}\|u\|^2_{l^2(\mathbb{Z}^+,\mathcal{U})} + \|y\|^2_{l^2(\mathbb{Z}^+,\mathcal{Y})} \quad &= \quad \|Ev\|^2 + \|DEv\|^2 + \sum_{k=1}^{\infty} \|u_k\|^2_{\mathcal{U}} + \|y_k\|^2_{\mathcal{Y}} \\ &\geq \quad \|Ev\|^2 + \|DEv\|^2 + q_f(BEv, BEv) = s_f(Ev, Ev).\end{aligned}$$

Taking the supremum over all $v \in \mathcal{W}$ with norm 1 gives the desired result.

Next we show that if (q, s, K) is a solution of the control Riccati equation, then $\sup_{\|v\|=1} s(Ev, Ev)$ is an upper bound for the square of the $\mathcal{L}(l^2(\mathbb{Z}^+, \mathcal{W}), l^2(\mathbb{Z}^+, \mathcal{Y}\times \mathcal{U}))$ norm of the closed-loop system with the feedback operator K. By Lemma 3.16 with $z = 0$ and $v_k = Ew_k$ we have

$$\sum_{k=0}^{n} \|u_k\|^2_{\mathcal{U}} + \|y_k\|^2_{\mathcal{Y}} \leq \sum_{k=0}^{n} s(Ew_k, Ew_k). \tag{5.2}$$

It follows that we have

$$\sum_{k=0}^{n} \|u_k\|^2_{\mathcal{U}} + \|y_k\|^2_{\mathcal{Y}} \leq M \sum_{k=0}^{n} \|w_k\|^2_{\mathcal{U}},$$

where $M := \sup_{\|w\|=1} s(Ew, Ew)$. This establishes that $\sup_{\|w\|=1} s(Ew, Ew)$ is an upper bound for the square of the $\mathcal{L}(l^2(\mathbb{Z}^+, \mathcal{W}), l^2(\mathbb{Z}^+, \mathcal{Y} \times \mathcal{U}))$ norm.

We conclude that the square root of $\sup_{\|w\|=1} s_f(w, w)$ is the smallest possible $\mathcal{L}(l^2(\mathbb{Z}^+, \mathcal{W}), l^2(\mathbb{Z}^+, \mathcal{Y} \times \mathcal{U}))$ norm and that this minimum is reached by the feedback operator K_f.

Next we note that $\|w\|_{l^2(\mathbb{Z}^+,\mathcal{W})} \leq \|w\|_{l^1(\mathbb{Z}^+,\mathcal{W})}$, so that $\sup_{\|w\|=1} s(w, w)$ is also an upper bound for the $\mathcal{L}(l^1(\mathbb{Z}^+, \mathcal{W}), l^2(\mathbb{Z}^+, \mathcal{Y}\times\mathcal{U}))$ norm if K is chosen such that (q, s, K) is a solution of the control Riccati equation. Combining this with the earlier established lower-bound, it follows that the feedback operator K_f also minimizes this norm. □

Since $K : D(K) \to \mathcal{U}$ will in general be an unbounded operator on the state space $\mathcal{X}$, the operator

$$\left[\begin{array}{c|c} A + BK & B \\ \hline C + DK & D \\ K & I \end{array}\right] \tag{5.3}$$

will in general not be a node with state space $\mathcal{X}$. If (q, s, K) is a solution of the control Riccati equation, then (5.3) is a node with as state space $D(q)$ equipped with the graph norm. We call this the *graph closed-loop node*. If (q, K) is only a solution of the optimal continuation equation, then (5.3) is not necessarily a node, but the operator

$$\left[\begin{array}{c|c} A+BK & 0 \\ \hline C+DK & 0 \\ K & 0 \end{array}\right] \tag{5.4}$$

is a node with as state space $D(q)$ equipped with the graph norm. We call this the *partial graph closed-loop system*.

The so defined nodes do not behave vey well in general, but they are a convenient starting point for the procedure outlined in Appendix B which does produce nodes with very good properties.

Definition 5.2. The node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with trajectories given by (1.1) is called

- *strongly stable* if, for each initial condition and input zero, the state x_n converges to zero as n goes to infinity.
- *Input stable* if the sequence of maps from the input $\{u_k\}_{k=1,\dots,n}$ to x_n (with initial condition zero) is uniformly bounded (in n) in the $\mathcal{L}(l^2(\mathbb{Z}^+,\mathcal{U}),\mathcal{X})$ norm.
- *output stable* if the map from the initial state to the output is bounded from $\mathcal{X}$ into $l^2(\mathbb{Z}^+,\mathcal{Y})$.
- *input-output stable* if the map from the input $\{u_k\}_{k\in\mathbb{Z}^+}$ to the output (for initial condition zero) is bounded from $l^2(\mathbb{Z}^+,\mathcal{U})$ into $l^2(\mathbb{Z}^+,\mathcal{Y})$.

Theorem 5.3.
1. *Let (q, K) be a solution of the optimal continuation equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then the corresponding partial graph closed-loop node has a completed q-compression. This completed q-compression is output stable.*
2. *The completed q_f-compression of the partial graph closed-loop node is strongly stable and output stable.*
3. *Let (q, s, K) be a solution of the control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then the corresponding graph closed-loop node has a completed q-compression. This completed q-compression is input stable, output stable and input-output stable.*
4. *Assume that the finite future incremental condition holds. Then the completed q_f-compression of the graph closed-loop node is strongly stable, input stable, output stable and input-output stable.*

Proof. 1. We check the conditions of Theorem B.14. We obviously have the q is a closable (even bounded) nonnegative symmetric sesquilinear form in the state space of the partial graph closed-loop node. Since its domain is the whole state space, it is obviously strongly invariant. That the partial graph closed-loop node is bounded with respect to the semi-norm induced by q follows from Lemma 4.8. So the partial graph closed-loop node has a well-defined completed q-compression.

Note that, by continuity, the optimal continuation equation holds for all z in the state space. From (2.3) we easily obtain that

$$q(x_n, x_n) + \sum_{k=0}^{n-1} \|y_k\|^2_{\mathcal{Y}} + \|u_k\|^2_{\mathcal{U}} = q(z, z), \tag{5.5}$$

where u, x, y and z are related by (1.2) with $v = 0$. It follows that the map from z to $\{[y; u]\}_{k\in\mathbb{Z}^+}$ is bounded from the state space into $l^2(\mathbb{Z}^+, \mathcal{Y} \times \mathcal{U})$, i.e. that the system is output stable.
2. Output stability follows from part 1 of this lemma (together with Lemma 2.12). From (5.5) with $q = q_f$ we obtain by letting $n \to \infty$ and noting that $\sum_{k=0}^{\infty} \|y_k\|^2_{\mathcal{Y}} + \|u_k\|^2_{\mathcal{U}} = q_f(z, z)$ that $q_f(x_n, x_n) \to 0$. Since q_f is the inner-product in the state space of the completed q_f-compression we have that this node is strongly stable.
3. Existence of the completed q-compression follows in a manner completely analoguous to part 1 of this lemma. All the statements on stability easily follow from Lemma 3.16 once we note that the control Riccati equation by continuity holds for all z in the state space.
4. All these statement except the one about strong stability follow from part 3 of this lemma. The strong stability follows from part 2. □
The closed-loop node from Theorem 5.3 is the closed-loop system (in the classical sense with a bounded feedback operator) of the node from Theorem 4.12. We can similarly arrive at and prove properties of the closed-loop system in the classical sense of the node from Theorem 4.13.

Theorem 5.4. *All the statements of Theorem* 5.3 *still hold if we consider the completed q^--compression (completed q_f^--compression) instead of the completed q-compression (completed q_f-compression).*

Proof. The proof is virtually identical to that of Theorem 5.3 and is therefore omitted. □

Remark 5.5. The closed-loop control Riccati equation (3.2) (or its summed version (3.7)) shows exactly that the completed q-compression of the graph node is scattering energy-preserving (in the sense of e.g. [9]) once we replace the inner product in the input space by the equivalent inner-product $s(v_1, v_2)$. The input, output and input-output stability established in Theorem 5.3 follow from this energy-preservation property.

6. Right factorizations

In this section we consider the relation between right factorizations, the finite future incremental cost condition and solvability of the control Riccati equation.

We use the following notation: H^∞ denotes the Hardy space of uniformly bounded holomorphic functions, $\mathbb{D}$ denotes the unit disc, $[-, -]$ is a row vector and $[-;-]$ is a column vector. The transfer function of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is defined in a neigbourhood of zero by $zC(I - zA)^{-1}B + D$. A node is called a *realization* of a holomorphic function defined in a neigbourhood of zero if that function is the transfer function of the node.

Definition 6.1. Let $G : D(G) \subset \mathbb{C} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ be holomorphic at the origin. A function $[M; N] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathcal{U}, \mathcal{U} \times \mathcal{Y}))$ is called a *right factorization* of G if $M(z)$ is invertible for all z in a neighborhood of the origin and $G(z) = N(z)M(z)^{-1}$ in a neighborhood of the origin.

The following theorem is known.

Theorem 6.2. *The following are equivalent conditions for a node.*
- *The finite future cost condition is satisfied.*
- *The control Riccati equation of the node has a classical solution.*

In this case the transfer function of the node has a right factorization.

Conversely, let $G : D(G) \subset \mathbb{C} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ be holomorphic at the origin. If G has a right factorization, then it has a realization that satisfies the above two equivalent conditions.

Proof. Proofs of all the above can be found in [11, Propositions 6.36, 7.12, 7.13]. Alternatively, the equivalence of the finite future cost condition and the solvability of the control Riccati equation follows as in Curtain and Zwart [4, Chapter 6] and the connection with factorization properties of the transfer function is treated in [12]. □

The following theorem is a significant improvement of the above result.

Theorem 6.3. *Let $G : D(G) \subset \mathbb{C} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ be holomorphic at the origin and let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a realization of G. The following are equivalent conditions.*
- *$\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the finite future incremental cost condition.*
- *The control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has a solution.*
- *G has a right factorization.*

Proof. 1. implies 2. This follows from Lemma 3.14.
2. implies 3. If the control Riccati equation of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has a solution, then the corresponding graph node has a classical solution to its control Riccati equation by Lemma 4.11 (part 3). By part 1 of Lemma 4.11 the graph node has transfer function G. It follows from Theorem 6.2 that G has a right factorization.
3. implies 1. We first recall that the Z-transform of a sequence $h : \mathbb{Z} \to \mathcal{H}$ is defined as

$$\hat{h}(z) = \sum_{k=-\infty}^{\infty} h_k z^k,$$

for those $z \in \mathbb{C}$ for which this power series converges absolutely.
Denote the right factorization of G by $[M; N]$. Let $u \in \mathcal{U}$, we show that Bu has finite cost. Define $r_0 := M(0)^{-1}v$, $r_n = 0$ for $n > 0$, and v and y through their

Z-transforms by $\hat{v}(s) = M(s)\hat{r}(s)$, $\hat{y}(s) = N(s)\hat{r}(s)$. Then, since $r \in l^2(\mathbb{Z}^+;\mathcal{U})$ and $[M;N] \in H^\infty$, we have $v \in l^2(\mathbb{Z}^+;\mathcal{U})$, $y \in l^2(\mathbb{Z}^+;\mathcal{Y})$. Since $[M;N]$ is a right factorization of G, y is the output for the input v and initial condition zero. Since $\hat{v}(0) = M(0)\hat{r}(0) = u$ we have $v_0 = u$. From this it follows that the state corresponding to input v (and with initial condition zero) satisfies $x_1 = Bu$. We now see that the state Bu has finite cost: for the input $(v_1, v_2, \ldots)$ we obtain a finite cost. The result then follows from Lemma 3.4. □
Note that the difference between Theorems 6.2 and 6.3 is that in the latter case the conditions hold for all realizations, not just some specific ones.

Remark 6.4. Even more is true: the right factorization can be chosen to be normalized and weakly coprime. See Smith [14] (for the case that $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional) and Mikkola [10] (for the general case). Using the results in Mikkola [10] it is not too difficult to show that the input-output map of the system (5.1) with $K = K_f$ and $E = S_f^{-1/2}$ provides a normalized weakly right coprime factorization.

Appendix A. Alternative forms of the Riccati equation

In this appendix we present several alternative forms of the control Riccati equation, including ones in terms of operators instead of in terms of sesquilinear forms.

Definition A.1. The triple (q, s, K) is called a (nonnegative) solution of the *Lure control Riccati equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if:

1. q is a closed nonnegative symmetric sesquilinear form in $\mathcal{X}$ whose domain satisfies $AD(q) \subset D(q)$, $B\mathcal{U} \subset D(q)$.
2. s is a bounded nonnegative symmetric sesquilinear form on $\mathcal{U}$.
3. $K : D(q) \to \mathcal{U}$ is a linear operator.
4. For all $z \in D(q)$, $u \in \mathcal{U}$ we have

$$\begin{aligned} q(Az, Az) + \|Cz\|_{\mathcal{Y}}^2 &= q(z, z) + s(Kz, Kz), \\ s(u, u) &= \|u\|_{\mathcal{U}}^2 + \|Du\|_{\mathcal{Y}}^2 + q(Bu, Bu), \\ -s(Kz, u) &= \langle Cz, Du\rangle_{\mathcal{Y}} + q(Az, Bu). \end{aligned} \tag{A.1}$$

The solution is called *classical* when $D(q) = \mathcal{X}$.

Lemma A.2. *The triple (q, s, K) is a solution of the Lure control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if and only if it is a solution of the control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.*

Proof. Expanding (3.3) gives

$$\begin{aligned} q(Az, Az) + q(Bu, Bu) + q(Az, Bu) + q(Bu, Az) + \|Cz\|^2 + \|Du\|^2 \\ + \langle Cz, Du\rangle + \langle Du, Cz\rangle + \|u\|^2 \\ = q(z, z) + s(Kz, Kz) + s(u, u) - s(Kz, u) - s(u, Kz). \end{aligned} \tag{A.2}$$

It is easy to see that (A.1) implies (A.2). Conversely, the first and second equations of (A.1) are (A.2) with $u = 0$ and $z = 0$, respectively. The third equation of (A.1)

follows from the first and second equation of (A.1) together with (A.2) applied to z (for equality of the real parts) and iz (for equality of the imaginary parts). □
In the main article we have worked with sesquilinear forms and not the usual operators as solutions for the Riccati equation. One of the reasons for this is that this makes it easier to handle the nondensely defined case. We now consider the technicalities involved in the operator versions of the Riccati equation. We first define what we mean by a not necessarily densely defined symmetric operator.

Definition A.3. An operator T in $\mathcal{H}$ is called *symmetric* if its range is contained in the closure of its domain and $\langle Th_1, h_2\rangle = \langle h_1, Th_2\rangle$ for all $h_1, h_2 \in D(T)$.

Note that a symmetric operator is densely defined and symmetric when considered as an operator in the closure of its domain. We will denote this associated densely defined symmetric operator by $\check{T}$.

Definition A.4. A symmetric operator T is called *self-adjoint* if $\check{T}$ is self-adjoint (in the usual sense).

Note that T is nonnegative (meaning $\langle Th, h\rangle \geq 0$ for all $h \in D(T)$) if and only if $\check{T}$ is. If T is nonnegative self-adjoint, then we define $T^{1/2} := \check{T}^{1/2}$.

Lemma A.5. *There is a one-to-one correspondence between closed nonnegative symmetric sesquilinear forms t and closed nonnegative self-adjoint linear operators T through $t(x,y) = \langle T^{1/2}x, T^{1/2}y\rangle$, $D(t) = D(T^{1/2})$.*

Proof. For the densely defined case this can be found in Kato [7, Chapter 6]. The non densely defined case follows from the densely defined case by employing $\check{T}$. □

Definition A.6. The triple (Q, S, K) is called a (nonnegative) solution of the *operator control Riccati equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if:

1. Q is a closed nonnegative self-adjoint operator in $\mathcal{X}$ whose domain satisfies $AD(Q^{1/2}) \subset D(Q^{1/2})$, $B\mathcal{U} \subset D(Q^{1/2})$.
2. S is a bounded nonnegative self-adjoint operator on $\mathcal{U}$.
3. $K : D(Q^{1/2}) \to \mathcal{U}$ is a linear operator.
4. For all $z \in D(Q^{1/2})$, $u \in \mathcal{U}$ we have

$$\|Q^{1/2}(Az + Bu)\|^2_{\mathcal{X}} + \|Cz + Du\|^2_{\mathcal{Y}} + \|u\|^2_{\mathcal{U}} = \|Q^{1/2}z\|^2_{\mathcal{X}} + \|S(Kz - u)\|^2_{\mathcal{U}}.$$

The solution is called *classical* when $D(Q) = \mathcal{X}$.

Definition A.7. The triple (Q, S, K) is called a (nonnegative) solution of the *operator Lure control Riccati equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if:

1. Q is a closed nonnegative self-adjoint operator in $\mathcal{X}$ whose domain satisfies $AD(Q^{1/2}) \subset D(Q^{1/2})$, $B\mathcal{U} \subset D(Q^{1/2})$.
2. $S : \mathcal{U} \to \mathcal{U}$ is a linear operator.
3. $K : D(Q^{1/2}) \to \mathcal{U}$ is a linear operator.

4. For all $z \in D(Q^{1/2})$ we have

$$\begin{aligned} \|Q^{1/2}Az\|_{\mathcal{X}}^2 + \|Cz\|_{\mathcal{Y}}^2 &= \|Q^{1/2}z\|_{\mathcal{X}}^2 + \|S^{1/2}Kz\|_{\mathcal{U}}^2, \\ S &= I + D^*D + \left(Q^{1/2}B\right)^* Q^{1/2}B, \\ -SK &= \left(Q^{1/2}B\right)^* Q^{1/2}A + D^*C. \end{aligned}$$

The solution is called *classical* when $D(Q) = \mathcal{X}$.

Note that the operator $Q^{1/2}B$ in the preceding definition is bounded (so its adjoint is well-defined), and that S is bounded and symmetric and has a bounded inverse.

Lemma A.8. *Let the triples (q, s, K) and (Q, S, K) correspond to each other as indicated in Lemma* A.5*. Then the following are equivalent.*

- *(q, s, K) is a solution of the control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$,*
- *(q, s, K) is a solution of the Lure control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$,*
- *(Q, S, K) is a solution of the operator control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$,*
- *(Q, S, K) is a solution of the operator Lure control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.*

Proof. That the control Riccati equation and the Lure control Riccati equation are equivalent was shown in Lemma A.2. The operator versions are simply reformulations. □

Corollary A.9. *Let (q_i, s_i, K_i) $(i = 1, 2)$ be two solutions of the control Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. If $q_1 = q_2$, then $s_1 = s_2$ and $K_1 = K_2$.*

Proof. This is obvious from the operator Lure control Riccati equation since S and K are expressed in terms of Q. □

Corollary A.10. *If the triple (q, s, K) is a classical solution of the control Riccati equation, then there exists a nonnegative symmetric $Q \in \mathcal{L}(\mathcal{X})$ such that*

$$(A^*QB + C^*D)(I + B^*QB + D^*D)^{-1}(B^*QA + D^*C) = A^*QA - Q + C^*C. \quad \text{(A.3)}$$

The sesquilinear form q then is the sesquilinear form corresponding to the operator Q, s is the sesquilinear form corresponding to the operator $S = I + B^*QB + D^*D$ and $K = -S^{-1}(B^*QA + D^*C)$.

Conversely, if there exists a nonnegative symmetric $Q \in \mathcal{L}(\mathcal{X})$ that satisfies (A.3)*, then (q, s, K) as above is a classical solution of the control Riccati equation.*

Proof. This is easily seen from Lemma A.8. □

Remark A.11. The equation (A.3) is the one that usually appears in the literature and not the equations involving sesquilinear forms. When dealing with unbounded solutions, as we do, the formulation in terms of sesquilinear forms however seems more practical.

Appendix B. Extension and compression of systems

In this appendix we study rather general extensions of compressions of a given node. In the main part of this article we require several special instances of this general result. We start with a theorem on compressions.

Theorem B.1. *Assume $\mathcal{V}, \mathcal{W}$ are closed A-invariant subspaces with $\mathcal{V} \subset \mathcal{W}$, that $B\mathcal{U} \subset \mathcal{W}$ and that $\mathcal{V} \subset \ker C$. Then the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has the decomposition*

$$\left[\begin{array}{ccc|c} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & B_2 \\ A_{31} & A_{32} & A_{33} & B_3 \\ \hline C_1 & C_2 & 0 & D \end{array}\right] \tag{B.1}$$

with respect to the decomposition of the state space $\mathcal{X} = \mathcal{W}^\perp \oplus (\mathcal{W} \ominus \mathcal{V}) \oplus \mathcal{V}$.

The compression

$$\Sigma_{\mathcal{S}} := \begin{bmatrix} P_{\mathcal{S}} A|_{\mathcal{S}} & P_{\mathcal{S}} B \\ C|_{\mathcal{S}} & D \end{bmatrix} = \begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}$$

of the node to $\mathcal{S} := \mathcal{W} \ominus \mathcal{V}$ has the following property: for the same initial state $z \in \mathcal{S}$ and input $\{u_n\}_{n \in \mathbb{Z}^+}$, the state and output of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and its compression $\Sigma_{\mathcal{S}}$ are related by $x_n^{\mathcal{S}} = P_{\mathcal{S}} x_n$, $y_n^{\mathcal{S}} = y_n$ (for all $n \in \mathbb{Z}^+$).

Proof. That $B_1 = 0$ follows from the assumption $B\mathcal{U} \subset \mathcal{W}$ and that $C_3 = 0$ from the assumption $C|_{\mathcal{V}} = 0$. That A_{13} and A_{23} are zero follows from the A-invariance of $\mathcal{V}$. That $A_{12} = 0$ follows from the A-invariance of $\mathcal{W}$ (which equals $(\mathcal{W} \ominus \mathcal{V}) \oplus \mathcal{V}$ since $\mathcal{V} \subset \mathcal{W}$).

Since A is lower-triangular, so is A^n (for all $n \in \mathbb{Z}^+$) and moreover $(A^n)_{ii} = (A_{ii})^n$ for $i = 1, 2, 3$. It follows that the state of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ for an initial condition in $\mathcal{S}$ and zero input is $(A_{22})^n x_0 + (A^n)_{32} x_0$, so $P_{\mathcal{S}} x_n = P_{\mathcal{S}} A^n x_0 = (A_{22})^n x_0 = x_n^{\mathcal{S}}$. Similarly it follows that $P_{\mathcal{S}} A^n B = A_{22}^n B_2$, $CA^n = C_2 A_{22}^n$ (on $\mathcal{S}$) and $CA^n B = C_2 A_{22}^n B_2$. The statements about the state and output for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and $\Sigma_{\mathcal{S}}$ follow from these equalities. □

Corollary B.2. *Under the assumptions of Theorem* B.1 *the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and its compression $\Sigma_{\mathcal{S}}$ have the same transfer function.*

Proof. This is true since for each input and zero initial condition the outputs are equal. □

The following lemma on subspaces gives alternative characterizations of the state space $\mathcal{S}$ of the compressed system from Theorem B.1.

Lemma B.3. *Let $\mathcal{V} \subset \mathcal{W}$ be closed subspaces of a Hilbert space. Then*

$$\begin{aligned} \mathcal{W} \ominus \mathcal{V} &= \mathcal{V}^\perp \cap \mathcal{W} \\ &= P_{\mathcal{V}^\perp} \mathcal{W} \\ &= P_{\mathcal{W}} \mathcal{V}^\perp. \end{aligned}$$

Denoting the above subspace by $\mathcal{S}$, we have $P_{\mathcal{S}} = P_{\mathcal{V}^\perp}$ on $\mathcal{W}$.

Proof. All the equalities easily follow from the decomposition of the whole space as $\mathcal{W}^\perp \oplus (\mathcal{W} \ominus \mathcal{V}) \oplus \mathcal{V}$.

The final claim also directly follows from the decomposition of the whole space as $\mathcal{W}^\perp \oplus \mathcal{S} \oplus \mathcal{V}$: $P_{\mathcal{S}}(x_1, x_2, x_3) = (0, x_2, 0)$, $P_{\mathcal{V}^\perp}(x_1, x_2, x_3) = (x_1, x_2, 0)$ and since $(x_1, x_2, x_3) \in \mathcal{W}$ we have $x_1 = 0$. □

The following notion of strong invariance of a subspace appeared in [2] in the more general context of state/signal systems.

Definition B.4. A subspace $\mathscr{X}$ of the state space $\mathcal{X}$ of a node is called *strongly invariant* if, for an initial state $z \in \mathscr{X}$ and input $\{u_n\}_{n \in \mathbb{Z}^+}$, the state x_n of the node is in $\mathscr{X}$ (for all $n \in \mathbb{Z}^+$).

The following lemma gives necessary and sufficient conditions under which a subspace is strongly invariant.

Lemma B.5. *A subspace $\mathscr{X}$ of the state space $\mathcal{X}$ is strongly invariant if and only if $\mathscr{X}$ is an A-invariant subspace such that $B\mathcal{U} \subset \mathscr{X}$.*

Proof. By choosing input zero and $n = 1$ we see that $\mathscr{X}$ must be A-invariant. By choosing the initial state zero and $n = 1$ we see that $B\mathcal{U} \subset \mathscr{X}$ must hold.

By the A-invariance of $\mathscr{X}$ we have $A^n\mathscr{X} \subset \mathscr{X}$ and (using in addition that $B\mathcal{U} \subset \mathscr{X}$) that $A^n B\mathcal{U} \subset \mathscr{X}$. It follows that $\mathscr{X}$ is strongly invariant. □
The following theorem identifies a subspace that is strongly invariant for the compressed system from Theorem B.1.

Theorem B.6. *Let $\mathscr{V} \subset \mathscr{W}$ be A-invariant subspaces. Define $\mathscr{S} := P_{\mathscr{V}^\perp}\mathscr{W}$. Then $\mathscr{S} \subset \mathcal{S}$, the state space of the node $\Sigma_{\mathcal{S}}$ induced by $\overline{\mathscr{V}}$ and $\overline{\mathscr{W}}$ from Theorem* B.1 *and $\mathscr{S}$ is an $A_{\mathcal{S}}$-invariant subspace. If in addition $B\mathcal{U} \subset \mathscr{W}$, then $\mathscr{S}$ is strongly invariant for $\Sigma_{\mathcal{S}}$.*

Proof. We obviously have $P_{\mathscr{V}^\perp}\mathscr{W} \subset P_{\mathscr{V}^\perp}\overline{\mathscr{W}}$, Lemma B.3 then gives $\mathscr{S} \subset \mathcal{S}$.

Using the decomposition (B.1) from Theorem B.1 (applied with $\mathcal{V} = \overline{\mathscr{V}}$, $\mathcal{W} = \overline{\mathscr{W}}$ and the node $\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$) it is easily seen that $P_{\mathscr{V}^\perp}AP_{\mathscr{V}^\perp} = P_{\mathscr{V}^\perp}A$. Using that $\mathscr{W}$ is A-invariant we obtain $P_{\mathscr{V}^\perp}A\mathscr{W} \subset P_{\mathscr{V}^\perp}\mathscr{W}$ and so, by the just established equality of operators, $P_{\mathscr{V}^\perp}AP_{\mathscr{V}^\perp}\mathscr{W} \subset P_{\mathscr{V}^\perp}\mathscr{W}$. This is exactly

$$P_{\mathscr{V}^\perp}A\mathscr{S} \subset \mathscr{S}. \tag{B.2}$$

We have $P_{\mathcal{S}} = P_{\mathscr{V}^\perp}$ on $\overline{\mathscr{W}}$ by Lemma B.3. We also have $A\mathscr{S} \subset A\mathcal{S} \subset \overline{\mathscr{W}}$ (again using the decomposition (B.1) for the last inclusion). It follows from these two facts that (B.2) is $P_{\mathcal{S}}A\mathscr{S} \subset \mathscr{S}$, which exactly says that $\mathscr{S}$ is $A_{\mathcal{S}}$-invariant.

It remains to prove the strong invariance of $\mathscr{S}$ under the additional assumption that $B\mathcal{U} \subset \mathscr{W}$. By Lemma B.5 , to do this it suffices to show that the range of the control operator $P_{\mathcal{S}}B$ of the compression $\Sigma_{\mathcal{S}}$ in Theorem Theorem B.1 is contained in $\mathscr{S}$. However, this follows from Lemma B.3, which says that $P_{\mathcal{S}} = P_{\mathscr{V}^\perp}$

on $\overline{\mathscr{W}}$. $\square$

Lemma B.7. *Let $\mathscr{X}$ be a strongly invariant subspace for the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and t a symmetric sesquilinear form in $\mathcal{X}$ whose domain includes $\mathscr{X}$ and that is positive on $\mathscr{X}$. Assume that the node is bounded with respect to the norm induced by t on $\mathscr{X}$, i.e. that there exist $M_A, M_B, M_C \geq 0$ such that*

$$t(Az, Az) \leq M_A t(z,z),\ \ t(Bu, Bu) \leq M_B \|u\|_{\mathcal{U}}^2,\ \ \|Cz\|_{\mathcal{Y}}^2 \leq M_C t(z,z),$$

for $z \in \mathscr{X}$, $u \in \mathcal{U}$. Then the node has a unique continuous extension to the completion of $\mathscr{X}$ under the norm induced by t.

Proof. This is obvious. $\square$

Lemma B.8. *Let t be a nonnegative symmetric sesquilinear form in $\mathcal{X}$ whose domain is a strongly invariant subspace for the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Assume that the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is bounded with respect to the semi-norm induced by t on its domain. Then the kernel $\mathcal{N}(t)$ of t is A-invariant and is contained in the kernel of C.*

Proof. It follows from the boundedness assumption that $t(z,z) = 0$ implies $t(Az, Az) = 0$ and $Cz = 0$, i.e. that $z \in \mathcal{N}(t)$ implies $Az \in \mathcal{N}(t)$ and $z \in \ker C$. $\square$

As the next lemma shows, part of the conclusion of Lemma B.3 is still valid if the larger space is not assumed to be closed.

Lemma B.9. *Let $\mathcal{V} \subset \mathscr{W}$ be subspaces of a Hilbert space with $\mathcal{V}$ closed. Then $P_{\mathcal{V}^\perp}\mathscr{W} = \mathcal{V}^\perp \cap \mathscr{W}$.*

Proof. Decompose $w \in \mathscr{W}$ as $v + v^\perp$ with $v \in \mathcal{V}$ and $v^\perp \in \mathcal{V}^\perp$. Since $v + v^\perp \in \mathscr{W}$ and $v \in \mathcal{V} \subset \mathscr{W}$ it follows that $v^\perp \in \mathscr{W}$. So $P_{\mathcal{V}^\perp}\mathscr{W} \subset \mathcal{V}^\perp \cap \mathscr{W}$.

If $z \in \mathcal{V}^\perp \cap \mathscr{W}$, then $z = P_{\mathcal{V}^\perp} z \in P_{\mathcal{V}^\perp}\mathscr{W}$. It follows that $P_{\mathcal{V}^\perp}\mathscr{W} \supset \mathcal{V}^\perp \cap \mathscr{W}$. $\square$

Corollary B.10. *Let t be a closed nonnegative symmetric sesquilinear form in $\mathcal{X}$. Then $P_{\mathcal{N}(t)^\perp} D(t) = \mathcal{N}(t)^\perp \cap D(t)$.*

Proof. This follows from Lemma B.9 with $\mathcal{V} = \mathcal{N}(t)$ and $\mathscr{W} = D(t)$. $\square$

Corollary B.11. *Let t be a closable nonnegative symmetric sesquilinear form in $\mathcal{X}$. Then $P_{\mathcal{N}(t)^\perp} D(t)$ is contained on the domain of the closure of t and the closure of t defines an inner-product on this subspace.*

Proof. From Corollary B.10 applied to the closure $\bar{t}$ of t we obtain that $P_{\mathcal{N}(\bar{t})^\perp} D(\bar{t})$ is contained on the domain of the closure of t and that $\bar{t}$ is positive on $P_{\mathcal{N}(\bar{t})^\perp} D(\bar{t})$. Using that $\mathcal{N}(\bar{t})^\perp = \mathcal{N}(t)^\perp$ and $D(t) \subset D(\bar{t})$ gives the desired result. $\square$

Lemma B.12. *Let t be a closable nonnegative symmetric sesquilinear form in $\mathcal{X}$ and $S \in \mathcal{L}(\mathcal{X})$. If the domain of t is S-invariant and S is t-bounded, i.e. there exists a $M > 0$ such that $t(Sz, Sz) \leq Mt(z, z)$, then the domain of the closure of t is S-invariant.*

Proof. The domain of the closure consists of all $x \in \mathcal{X}$ such that there exists a sequence $x_n \in D(t)$ with $x_n \to x$ and $t(x_n - x_m, x_n - x_m) \to 0$ for $n, m \to \infty$ [7, Theorem 6.1.17]. Assume $x \in D(\bar{t})$, the obvious candidate for the relevant sequence is Sx_n, where x_n is a sequence as above. Since S is bounded we have $Sx_n \to Sx$ and since S is t-bounded we have $t(S(x_n - x_m), S(x_n - x_m)) \to 0$. So $Sx \in D(\bar{t})$. □

Lemma B.13. *Let t be a closable nonnegative symmetric sesquilinear form in $\mathcal{X}$. Assume that the domain of t is strongly invariant and that the node is bounded with respect to the semi-norm induced by t. Then the domain of the closure of t is strongly invariant and the node is bounded with respect to the semi-norm induced by the closure of t.*

Proof. The strong invariance follows from the inclusions $B\mathcal{U} \subset D(t) \subset D(\bar{t})$ and Lemma B.12.

The boundedness of the node with respect to $\bar{t}$ is established as follows. For $z \in D(\bar{t})$ pick a sequence $z_n \in D(t)$ with $t(z_n, z_n) \to \bar{t}(z, z)$. Then $t(Az_n, Az_n) \to \bar{t}(Az, Az)$ and by taking limits in the inequality $t(Az_n, Az_n) \leq M_A t(z_n, z_n)$ we obtain $\bar{t}(Az, Az) \leq M_A \bar{t}(z, z)$. Boundedness of C with respect to $\bar{t}$ follows similarly and boundedness of B with respect to $\bar{t}$ is immediate. □

Theorem B.14. *Let t be a closable nonnegative symmetric sesquilinear form in $\mathcal{X}$. Assume that the domain of t is strongly invariant and that the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is bounded with respect to the semi-norm induced by t on its domain. Then all of the conditions of Theorem* B.1 *with $\mathcal{V} = \overline{\mathcal{N}(t)}$ and $\mathcal{W} = \overline{D(t)}$ are met, so the compression $\Sigma_{\mathcal{S}}$ of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to $\mathcal{S} := P_{\mathcal{N}(t)^\perp} \overline{D(t)}$ is well defined. The subspace $\mathscr{S} := P_{\mathcal{N}(t)^\perp} D(t)$ is a $\Sigma_{\mathcal{S}}$ strongly invariant subspace that is contained in the domain of the closure of t and the closure of t defines a norm on $\mathscr{S}$. The restriction of the node $\Sigma_{\mathcal{S}}$ to $\mathscr{S}$ has a unique continuous extension to the completion of $\mathscr{S}$ under the norm induced by the closure of t.*

Proof. We first check the conditions of Theorem B.1. By Lemma B.8 the kernel of t is A-invariant. Since the closure of an A-invariant subspace is A-invariant and $D(t)$ is A-invariant by assumption, it follows that $\mathcal{V}$ and $\mathcal{W}$ are A-invariant. Obviously we have $\mathcal{V} \subset \mathcal{W}$. By assumption we have $B\mathcal{U} \subset D(t)$, so $B\mathcal{U} \subset \mathcal{W}$. That $\mathcal{V} \subset \ker C$ follows from Lemma B.8.

That $\mathscr{S}$ is a $\Sigma_{\mathcal{S}}$ strongly invariant subspace follows from Theorem B.6. That $\mathscr{S}$ is contained in the domain of the closure of t and the closure of t defines a norm on $\mathscr{S}$ follows from Lemma B.11.

The statement on the continuous extension follows from Lemma B.7 applied with the subspace $\mathscr{S}$ and as sesquilinear form the closure of t; the assumptions of this lemma are met by Lemma B.13 and Lemma B.11. □

Definition B.15. The continuous extension of the restriction of the node $\Sigma_{\mathcal{S}}$ to $\mathscr{S}$ guaranteed to exist by Theorem B.14 is called the completed t-compression of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

Lemma B.16. *Let t be a closable nonnegative symmetric sesquilinear form in $\mathcal{X}$. Assume that the domain of t is strongly invariant and that the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is bounded with respect to the semi-norm induced by t on its domain. Then all of the conditions of Theorem* B.1 *with $\mathcal{V} = \overline{\mathcal{N}(t)}$ and $\mathcal{W} = \overline{D(t)}$ are met, so the compression $\Sigma_{\mathcal{S}}$ of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to $\mathcal{S} := P_{\mathcal{N}(t)^\perp}\overline{D(t)}$ is well defined. Define the subspace $\mathscr{S} := P_{\mathcal{N}(t)^\perp} D(t)$.*

For the same initial state $z \in \mathscr{S}$ and input $\{u_n\}_{n\in\mathbb{Z}^+}$, the state and output of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and its completed t-compression are related by $x_n^{\mathcal{S}} = P_{\mathcal{N}(t)^\perp} x_n$, $y_n^{\mathcal{S}} = y_n$ (for all $n \in \mathbb{Z}^+$). Moreover, the finite-time reachable states of the completed t-compression are exactly the elements of $P_{\mathcal{N}(t)^\perp}\Xi_-$.

Proof. The result on the relation between the state and output follows immediately from Theorem B.1 using that $\mathscr{S}$ is strongly invariant for $\Sigma_{\mathcal{S}}$.

Let $w \in P_{\mathcal{N}(t)^\perp}\Xi_-$. Then there exists a $z \in \Xi_-$ such that $w = P_{\mathcal{N}(t)^\perp} z$. Let u be an input that reaches z from initial state zero in a finite time for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then the same input reaches w in a finite time for the completed t-compression by the first part of this lemma. So the elements of $P_{\mathcal{N}(t)^\perp}\Xi_-$ are finite-time reachable states of the completed t-compression.

Assume that w is a finite-time reachable state of the completed t-compression. Let u be an input that reaches w from initial state zero in a finite time for the completed t-compression. Apply this input to $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with input zero and let z be the state at the time corresponding to w. By the first part of this lemma we have $w = P_{\mathcal{N}(t)^\perp} z$. Since $z \in \Xi_-$ it follows that $w \in P_{\mathcal{N}(t)^\perp}\Xi_-$. □

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