Reciprocal Symmetry in State/Signal Systems

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Abstract

The notion of reciprocity is well-known in circuit theory: if a linear passive time-invariant circuit does not contain any gyrators, then it is reciprocal in the standard input/state/output sense, i.e., the impedance and conductance transfer functions are congruent to their adjoints. Here we extend this notion to state/signal systems.

Keywords

Reciprocity, circuit theory, state/signal system.

In the state space approach to circuit theory one often models the relationship between the port voltages u and currents i by a system of the type

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bi(t), \\ u(t) &= Cx(t) + Di(t), \qquad t \geq 0, \end{aligned} \tag{1}$$

where $x(t)$ is the internal state of the system (the charges in the capacitors and the currents in the coils). The impedance (transfer) matrix of this system is given by $\mathfrak{D}(\lambda) = C(\lambda - A)^{-1}B + D$, $\lambda \in \mathbb{C}$. It is known that if the circuit does not contain any gyrators, then the impedance matrix is congruent to the impedance matrix of the adjoint system

$$\begin{aligned} \dot{x}_*(t) &= A^*x_*(t) + C^*u_*(t), \\ i_*(t) &= B^*x_*(t) + D^*u_*(t), \qquad t \geq 0, \end{aligned} \tag{2}$$

in the sense that

$$\mathfrak{D}(\lambda) = C(\lambda - A)^{-1}B + D = \Psi\big[C^*(\lambda - A^*)^{-1}B^* + D^*\big]\Psi, \tag{3}$$

where Ψ is the unitary matrix which defines the power product

$$e(t) = 2\Re(u(t), \Psi i(t)), \qquad t \geq 0.$$

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When the continuous time system (1) is mapped into discrete time by the Cayley transform, then the resulting system still has the same symmetry. We call $\mathfrak{D}$ Ψ-*reciprocal* whenever (3) holds.

As is well-known, if the system in (1) is minimal and balanced or simple and conservative, then the reciprocal symmetry implies that the main operator A is signature similar to its adjoint, i.e., $A^* = \mathcal{I}A\mathcal{I}$ for some signature matrix $\mathcal{I} = \mathcal{I}^* = \mathcal{I}^{-1}$. The converse is also true: If A is signature similar to its adjoint, and if $\Psi^*\mathfrak{D}$ is *strictly* positive real, then $\mathfrak{D}$ is Ψ-reciprocal. The same statement holds for the corresponding discrete time system.

If the system is not lossless, then it is still possible to find a simple conservative realization of the impedance matrix, but the state space in this realization is then infinite-dimensional. However, also in this infinite-dimensional case the situation remains the same: if the transfer function $\mathfrak{D}$ is Ψ-reciprocal, then the main operator A of every simple conservative realization of $\mathfrak{D}$ is signature similar to its adjoint, and if $\mathfrak{D}$ has a realization whose main operator is signature similar to its adjoint (conservative or not) and $\Psi^*\mathfrak{D}$ is strictly positive real, then $\mathfrak{D}$ is Ψ-reciprocal.

In this talk we discuss the influence of the reciprocal symmetry when one replaces the input/state/output (i/s/o) setting described above by the state/signal (s/s) setting. To keep the presentation short we restrict the discussion to the discrete time conservative case.

In the state/signal setting all the inputs and outputs are combined into one vector signal. In the discrete time setting the standard i/s/o system is replace by the s/s (graph) system

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n = 0, 1, 2, \dots. \tag{4}$$

where V is as closed subspace of $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ called the *generating subspace*; here $\mathcal{X}$ is the state space (a Hilbert space) and $\mathcal{W}$ is the signal space. To model the passivity of the system we introduce a Kreĭn space inner product in the signal space $\mathcal{W}$, and defines the system to be passive if V is a maximal nonnegative subspace of the Kreĭn *node space* $\mathfrak{K} := \left[\begin{smallmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$. It is *conservative* if $V = V^{[\perp]}$. The signal space of the (causal) adjoint system is $\mathcal{W}_* := -\mathcal{W}$, and its generating subspace V_* is the maximal nonnegative subspace

$$V_* = \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & \mathcal{I}_{(\mathcal{W}_*, \mathcal{W})} \end{bmatrix} V^{[\perp]} \tag{5}$$

of the adjoint nodes space $\mathfrak{K}_* := \left[\begin{smallmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W}_* \end{smallmatrix}\right]$; here $\mathcal{I}_{(\mathcal{W}_*, \mathcal{W})}$ is the identity mapping from $\mathcal{W}$ to $\mathcal{W}_*$.

In the s/s setting the transfer function is replaced by a behavior. These behaviors appear in both time and frequency domain versions, but for simplicity we here focus on only one version, namely the full time domain behavior, which is the subspace of $\ell^2(\mathbb{Z}; \mathcal{W})$ consisting of all the signal components w of all full trajectories (x, w) of Σ on $\mathbb{Z}$ satisfying $x(n) \to 0$ as $n \to -\infty$. In the passive case $\mathfrak{W}$ is a shift invariant maximal nonnegative subspace of $\ell^2(\mathbb{Z}; \mathcal{W})$. The corresponding behavior $\mathfrak{W}_*$ of the adjoint system turns out to be a reflection of the orthogonal companion to $\mathfrak{W}$ in $\ell^2(\mathbb{Z};\mathcal{W})$. Reciprocity means in the s/s setting that there exists a skew-adjoint involution $\mathcal{J} = -\mathcal{J}^* = \mathcal{J}^{-1}$ such that $\mathfrak{W} = \mathcal{J}\mathcal{R}\mathfrak{W}^{[\perp]}$, where $\mathcal{R}$ is the reflection operator $(\mathcal{R}w)(k) = w(-k-1)$, $k \in \mathbb{Z}$. In this case we call $\mathfrak{W}$ $\mathcal{J}$-reciprocal.

Before stating our main theorem about state/signal reciprocity we need two more notions. The *reachable subspace* of Σ is the closed linear span in $\mathcal{X}$ of all states $x(n)$ of all trajectories (x,w) on $\mathbb{Z}^+$ with zero initial state $x(0) = 0$. A s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ is (internally) *simple* if the closed linear span of the reachable subspaces of Σ and Σ_* is the whole state space. It is (externally) *pure* if its behavior is strictly positive in $\ell^2(\mathbb{Z};\mathcal{W})$. (The corresponding i/s/o impedance condition for purity is that the impedance function multiplied by Ψ^* is strictly positive real, and the corresponding i/s/o scattering condition is that the scattering function is strictly contractive.)

Theorem 0.1. *Let $\mathfrak{W}$ be a passive behavior on the signal space $\mathcal{W}$.*

1. *If $\mathfrak{W}$ is $\mathcal{J}$-reciprocal for some skew-adjoint involution $\mathcal{J}$ in $\mathcal{W}$, then there exists a simple conservative realization $\Sigma = (V;\mathcal{X};\mathcal{W})$ of $\mathfrak{W}$ satisfying*

$$V = \begin{bmatrix} 0 & \mathcal{I} & 0 \\ \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{J} \end{bmatrix} V^{[\perp]} \tag{6}$$

   for some signature operator $\mathcal{I}$.

2. *If $\Sigma = (V;\mathcal{X};\mathcal{W})$ is a conservative realization of $\mathfrak{W}$ satisfying* (6) *for some signature operator $\mathcal{I}$ and some skew-adjoint involution $\mathcal{J}$, then $\mathfrak{W}$ is $\mathcal{J}$-reciprocal.*

Reciprocal i/s/o systems setting are discussed, e.g., in a finite-dimensional setting in [Wil72], [OJ85], [ABGR90], and [LR95], and in an infinite-dimensional setting in [Fuh75], [Obe96], [GO99], and [AADR02]. More details about state/signal systems can be found in [AS05, AS07a, AS07b, AS07c, AS09a, AS09b] and [Sta06]. Continuous time state/signal systems have been studied in [KS09].

References

[AADR02] Daniel Alpay, Tomas Ya. Azizov, Aad Dijksma, and James Rovnyak, *Colligations in Pontryagin spaces with a symmetric characteristic function*, Linear operators and matrices, Oper. Theory Adv. Appl., vol. 130, Birkhäuser, Basel, 2002, pp. 55–82.

[ABGR90] Daniel Alpay, Joseph A. Ball, Israel Gohberg, and Leiba Rodman, *Realization and factorization for rational matrix functions with symmetries*, Extension and interpolation of linear operators and matrix functions, Oper. Theory Adv. Appl., vol. 47, Birkhäuser, Basel, 1990, pp. 1–60. MR MR1120273 (92j:47031)

[AS05] Damir Z. Arov and Olof J. Staffans, *State/signal linear time-invariant systems theory. Part I: Discrete time systems*, The State Space Method, Generalizations and Applications (Basel Boston Berlin), Operator Theory: Advances and Applications, vol. 161, Birkhäuser-Verlag, 2005, pp. 115–177.

[AS07a] ———, *State/signal linear time-invariant systems theory. Part II: Passive discrete time systems*, Internat. J. Robust Nonlinear Control **17** (2007), 497–548.

[AS07b] ______, *State/signal linear time-invariant systems theory. Part III: Transmission and impedance representations of discrete time systems*, Operator Theory, Structured Matrices, and Dilations, Tiberiu Constantinescu Memorial Volume (Bucharest Romania), Theta Foundation, 2007, Available from American Mathematical Society, pp. 101–140.

[AS07c] ______, *State/signal linear time-invariant systems theory. Part IV: Affine representations of discrete time systems*, Complex Anal. Oper. Theory **1** (2007), 457–521.

[AS09a] ______, *A Kreĭn space coordinate free version of the de Branges complementary space*, J. Funct. Anal. **256** (2009), 3892–3915.

[AS09b] ______, *Two canonical passive state/signal shift realizations of passive discrete time behaviors*, J. Funct. Anal. (2009), To appear.

[Fuh75] Paul A. Fuhrmann, *On J-symmetric restricted shifts*, Proc. Amer. Math. Soc. **51** (1975), 421–426.

[GO99] Aurelian Gheondea and Raimund J. Ober, *Completely J-positive linear systems of finite order*, Math. Nachr. **203** (1999), 75–101.

[KS09] Mikael Kurula and Olof J. Staffans, *Well-posed state/signal systems in continuous time*, Complex Anal. Oper. Theory (2009), To appear.

[LR95] Peter Lancaster and Leiba Rodman, *Minimal symmetric factorizations of symmetric real and complex rational matrix functions*, Proceedings of the Workshop "Nonnegative Matrices, Applications and Generalizations" and the Eighth Haifa Matrix Theory Conference (Haifa, 1993), vol. 220, 1995, pp. 249–282.

[Obe96] Raimund Ober, *System theoretic aspects of completely symmetric systems*, Recent Developments in Operator Theory and its Applications (Winnipeg, MB, 1994) (Basel Boston Berlin), Operator Theory: Advances and Applications, vol. 87, Birkhäuser-Verlag, 1996, pp. 233–262.

[OJ85] Philippe C. Opdenacker and Edmond A. Jonckheere, *LQG balancing and reduced LQG compensation of symmetric passive systems*, Internat. J. Control **41** (1985), 73–109.

[Sta06] Olof J. Staffans, *Passive linear discrete time-invariant systems*, Proceedings of the International Congress of Mathematicians, Madrid, 2006, 2006, pp. 1367–1388.

[Wil72] Jan C. Willems, *Dissipative dynamical systems Part II: Linear systems with quadratic supply rates*, Arch. Rational Mech. Anal. **45** (1972), 352–393.