

# The Reciprocal Symmetry in State/Signal Systems in Continuous Time

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**Abstract**—The notion of reciprocity is well-known in circuit theory: if a linear passive time-invariant circuit does not contain any gyrators, then it is reciprocal in the standard input/state/output sense, i.e., the impedance and conductance transfer functions are congruent to their adjoints. Here we extend this notion to the class of all (possibly infinite-dimensional) state/signal systems in continuous time.

## I. INTRODUCTION

In the state space approach to circuit theory one often models the relationship between the port voltages  $u$  and currents  $i$  by a system of the type

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bi(t), \\ u(t) &= Cx(t) + Di(t), \quad t \geq 0, \end{aligned} \quad (1)$$

where  $x(t)$  is the internal state of the system (the charges in the capacitors and the currents in the coils). The impedance (transfer) matrix of this system is given by  $\mathfrak{D}(\lambda) = C(\lambda - A)^{-1}B + D$ ,  $\lambda \in \mathbb{C}$ . It is known that if the circuit does not contain any gyrators, then the impedance matrix is congruent to the impedance matrix of the adjoint system

$$\begin{aligned} \dot{x}_*(t) &= A^*x_*(t) + C^*u_*(t), \\ i_*(t) &= B^*x_*(t) + D^*u_*(t), \quad t \geq 0, \end{aligned} \quad (2)$$

in the sense that

$$\mathfrak{D}(\lambda) = C(\lambda - A)^{-1}B + D = \Psi[B^*(\lambda - A^*)^{-1}C^* + D^*]\Psi, \quad (3)$$

where  $\Psi$  is the unitary matrix which defines the power product

$$e(t) = 2\Re(u(t), \Psi i(t)), \quad t \geq 0.$$

We call  $\mathfrak{D}$   $\Psi$ -reciprocal whenever (3) holds.

As is well-known, if the system in (1) is passive balanced and minimal, then the reciprocal symmetry implies that the main operator  $A$  is signature similar to its adjoint, i.e.,  $A^* = \mathcal{I}A\mathcal{I}$  for some signature matrix  $\mathcal{I} = \mathcal{I}^* = \mathcal{I}^{-1}$ . A partial converse is also true: If  $A$  is signature similar to its adjoint, and if  $\Psi^*\mathfrak{D}$  is strictly positive real, then  $\mathfrak{D}$  is  $\Psi$ -reciprocal.

In the finite-dimensional case a passive balanced minimal system is (simple and) conservative if and only if the system is lossless, i.e., if it does not contain any resistors. If the system is not lossless, then it is still possible to find a simple conservative realization of the impedance matrix, but the

state space in this realization is then infinite-dimensional. However, also in this infinite-dimensional case the situation remains essentially the same: if the transfer function  $\mathfrak{D}$  is  $\Psi$ -reciprocal, then the main operator  $A$  of every simple conservative realization of  $\mathfrak{D}$  is signature similar to its adjoint, and if  $\mathfrak{D}$  has a conservative realization (that need not be simple) whose main operator is signature similar to its adjoint and  $\Psi^*\mathfrak{D}$  is strictly positive real, then  $\mathfrak{D}$  is  $\Psi$ -reciprocal.

In this talk we discuss reciprocal symmetry in the case where one replaces the input/state/output setting described above by the so called state/signal setting described below. In this setting all the inputs and outputs are combined into one vector signal, but the state still remains a separate component of the system.

## II. PASSIVE STATE/SIGNAL SYSTEMS

A linear passive *continuous time invariant state/signal system*  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  consists of a Hilbert *state space*  $\mathcal{X}$ , a Kreĭn (signal) space  $\mathcal{W}$ , and a *generating subspace*  $V$  of the *node space*  $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}$  with following two properties:

- (i)  $V$  does not contain any vector of the form  $\begin{bmatrix} z \\ 0 \end{bmatrix} \in V$  with  $z \neq 0$ ;
- (ii)  $V$  is a maximal nonnegative subspace of the Kreĭn node space  $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}$  equipped with the indefinite inner product

$$\left[ \begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]_{\mathfrak{K}} = -(z_1, x_2) - (x_1, z_2) + [w_1, w_2]_{\mathcal{W}}. \quad (4)$$

Nonnegativity of  $V$  means that that

$$\left[ \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right]_{\mathfrak{K}} = -2\Re(z, x)_{\mathcal{X}} + [w, w]_{\mathcal{W}} \geq 0, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V, \quad (5)$$

and maximality means that  $V$  is not strictly contained in any other nonnegative subspace of  $\mathfrak{K}$ .

The notion of a *classical trajectory* of  $\Sigma$  is defined in terms of the generating subspace  $V$ : by a classical trajectory of  $\Sigma$  on  $I$  we mean a pair of functions  $\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C^1(I; \mathcal{X}) \\ C(I; \mathcal{W}) \end{bmatrix}$  satisfying

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad t \in I, \quad (6)$$

where  $\dot{x}(t) = \frac{d}{dt}x(t)$ . By a (*generalized*) *trajectory* of  $\Sigma$  on  $I$  we mean a pair of functions  $\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C(I; \mathcal{X}) \\ L_{\text{loc}}^2(I; \mathcal{W}) \end{bmatrix}$  which can be approximated by a sequence of classical trajectories  $\begin{bmatrix} x_n \\ w_n \end{bmatrix}$  in such a way that  $x_n \rightarrow x$  in  $\mathcal{X}$  locally uniformly on  $I$ , and  $w_n \rightarrow w$  in  $L_{\text{loc}}^2(I; \mathcal{W})$ .

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By combining (6) and (5) we find that all classical trajectories  $\begin{bmatrix} x \\ w \end{bmatrix}$  of  $\Sigma$  on some time interval  $I$  satisfy the inequality

$$\left[ \begin{array}{c} \dot{x}(t) \\ x(t) \\ w(t) \end{array}, \begin{array}{c} \dot{x}(t) \\ x(t) \\ w(t) \end{array} \right]_{\mathfrak{R}} = -\frac{d}{dt} \|x(t)\|_{\mathcal{X}}^2 + [w(t), w(t)]_{\mathcal{W}} \quad (7)$$

$$\geq 0, \quad t \in I.$$

Integrating this inequality over a finite subinterval  $[t_1, t_2] \subset I$  we get the equivalent inequality

$$\|x(t_2)\|_{\mathcal{X}}^2 \leq \|x(t_1)\|_{\mathcal{X}}^2 + \int_{t_1}^{t_2} [w(s), w(s)]_{\mathcal{W}} ds, \quad (8)$$

$$t_1 \leq t_2 \in I.$$

By continuity of the integral, (8) remains valid for all generalized trajectories of  $\Sigma$  on  $I$ . In many applications the inequality (7) holds in the form the *Lagrangian identity* (or *Green's identity*)

$$\left[ \begin{array}{c} \dot{x}(t) \\ x(t) \\ w(t) \end{array}, \begin{array}{c} \dot{x}(t) \\ x(t) \\ w(t) \end{array} \right]_{\mathfrak{R}} = -\frac{d}{dt} \|x(t)\|_{\mathcal{X}}^2 + [w(t), w(t)]_{\mathcal{W}} \quad (9)$$

$$= 0, \quad t \in I.$$

Consequently (8) then holds for all generalized trajectories of  $\Sigma$  in the form of the identity

$$\|x(t_2)\|_{\mathcal{X}}^2 = \|x(t_1)\|_{\mathcal{X}}^2 + \int_{t_1}^{t_2} [w(s), w(s)]_{\mathcal{W}} ds, \quad (10)$$

$$t_1 \leq t_2 \in I.$$

In this case we have equality in (5). In Kreĭn space terminology, this means that  $V \subset V^{\perp}$ , where

$$V^{\perp} = \left\{ \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in \mathfrak{R} \mid \left[ \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right]_{\mathfrak{R}} = 0 \text{ for all } \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}$$

is the orthogonal companion to  $V$ , and we then say that  $\Sigma$  *energy preserving*. If instead  $V^{\perp} \subset V$ , then we call  $\Sigma$  *co-energy preserving*, and if  $V^{\perp} = V$ , then we call  $\Sigma$  *conservative*.

The *reachable subspace*  $\mathfrak{R}$  of  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is the closed linear span in  $\mathcal{X}$  of all states  $x(t)$  of all trajectories  $(x, w)$  on  $\mathbb{R}^+$  with zero initial state  $x(0) = 0$ . The *unobservable subspace*  $\mathfrak{U}$  is the set of all initial states  $x(0)$  of all unobservable trajectories on  $\mathbb{R}^+$ , i.e., trajectories  $\begin{bmatrix} x \\ w \end{bmatrix}$  whose signal part  $w$  is identically zero. A passive state/signal system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is *controllable* if  $\mathfrak{R} = \mathcal{X}$ , it is *observable* if  $\mathfrak{U} = \{0\}$ , and it is *simple* if the linear span of  $\mathfrak{R}$  and  $\mathfrak{U}^{\perp}$  is dense in  $\mathcal{X}$ .

### III. THE FULL AND FUTURE BEHAVIORS OF A STATE/SIGNAL SYSTEM

In the state/signal setting the transfer function is replaced by a behavior. These behaviors appear in both time and frequency domain versions, but for simplicity we here focus on the time domain version. By the (time domain) *full behavior*  $\mathfrak{W}^{\Sigma}$  of a passive state/signal system  $\Sigma$  we mean the set of all signal parts  $w$  of all trajectories  $\begin{bmatrix} x \\ w \end{bmatrix}$  of  $\Sigma$  on  $\mathbb{R}$  satisfying  $\lim_{t \rightarrow -\infty} x(t) = 0$  and  $w \in L^2(\mathbb{R}; \mathcal{W})$ . This is a closed subspace of  $L^2(\mathbb{R}; \mathcal{W})$  which has three characteristic properties:

- A)  $\mathfrak{W}^{\Sigma}$  is both left-shift and right-shift invariant.
- B)  $\mathfrak{W}^{\Sigma}$  is a maximal nonnegative subspace of the Kreĭn space  $K^2(\mathbb{R}; \mathcal{W})$ . As a topological vector space  $K^2(\mathbb{R}; \mathcal{W})$  coincides with  $L^2(\mathbb{R}; \mathcal{W})$ , but the inner product in  $K^2(\mathbb{R}; \mathcal{W})$  is the (indefinite) Kreĭn space inner product

$$[w_1(\cdot), w_2(\cdot)]_{K^2(\mathbb{R}; \mathcal{W})} = \int_{-\infty}^{\infty} [w_1(s), w_2(s)]_{\mathcal{W}} ds. \quad (11)$$

- C)  $\mathfrak{W}^{\Sigma}$  is *causal*. Causality can be characterized in different ways. One such characterization is that if we define  $\mathfrak{W}_+^{\Sigma}$  to be the subspace of all functions  $w \in \mathfrak{W}^{\Sigma}$  which vanish identically on  $\mathbb{R}^-$ , then  $\mathfrak{W}_+^{\Sigma}$  is a maximally nonnegative subspace of  $K^2(\mathbb{R}^+; \mathcal{W})$ . Here  $K^2(\mathbb{R}^+; \mathcal{W})$  is defined in the same way as  $K^2(\mathbb{R}; \mathcal{W})$  with  $\mathbb{R}$  replaced by  $\mathbb{R}^+$ .

The set  $\mathfrak{W}_+^{\Sigma}$  defined above is called the *future behavior* of  $\Sigma$ . It consists of all the signal parts  $w$  of all trajectories  $\begin{bmatrix} x \\ w \end{bmatrix}$  of  $\Sigma$  on  $\mathbb{R}^+$  satisfying  $x(0) = 0$  and  $w \in L^2(\mathbb{R}^+; \mathcal{W})$ .

### IV. STATE/SIGNAL REALIZATIONS OF PASSIVE BEHAVIORS

Motivated by the above discussion we call a subspace  $\mathfrak{W}$  of  $K^2(\mathbb{R}; \mathcal{W})$  a *passive full behavior* on the signal space  $\mathcal{W}$  if  $\mathfrak{W}$  has properties A)–C) above. Thus, the full behavior induced by a passive state/signal system is a *passive full behavior*.

The so called *inverse problem* is the following: Given a passive full behavior  $\mathfrak{W}$  on a (Kreĭn) signal space  $\mathcal{W}$ , one is asked to construct a passive state/signal system  $\Sigma$  whose full behavior  $\mathfrak{W}^{\Sigma}$  is equal to  $\mathfrak{W}$ . We call such a state/signal system a *realization* of  $\mathfrak{W}$ . Indeed, it is shown in [AKS10] that to each passive full behavior there exist (infinitely many) passive state/signal realizations of any given passive behavior  $\mathfrak{W}$ . It is even possible to require a realization to have some additional properties. Three special state/signal realizations have constructed in [AKS10]: a) the first one is observable and co-energy preserving, b) the second is controllable and energy preserving, and c) the third is simple and conservative.

Each realization with one of the properties a)–c) is determined uniquely by the given passive full behavior  $\mathfrak{W}$  up to a unitary similarity transformation in the state space. In other words, if we have two realizations  $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$  and  $\Sigma_2 = (V_2; \mathcal{X}_2, \mathcal{W})$  of the same passive full behavior, and both  $\Sigma_1$  and  $\Sigma_2$  belong to the same class of systems which have properties a), b), or c), then there exists a unitary operator  $\mathcal{V}: \mathcal{X}_1 \rightarrow \mathcal{X}_2$  such that

$$V_2 = \begin{bmatrix} \mathcal{V} & 0 & 0 \\ 0 & \mathcal{V} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_1.$$

### V. SKEW-ADJOINT INVOLUTIONS ON A KREĬN SPACE

By a *skew-adjoint involution*  $\mathcal{J}$  on a Kreĭn space  $\mathcal{W}$  we mean a bounded linear operator  $\mathcal{J}$  on  $\mathcal{W}$  with a bounded

inverse satisfying the conditions

$$\mathcal{J} = -\mathcal{J}^* = \mathcal{J}^{-1}.$$

A necessary and sufficient condition for the existence of such an operator is that the positive and negative dimensions of  $\mathcal{W}$  are the same. In particular,  $\mathcal{W}$  cannot be a Hilbert space (of positive dimension). Another equivalent condition is that  $\mathcal{W}$  has a so called *Lagrangian decomposition*, i.e., a direct sum decomposition  $\mathcal{W} = \mathcal{E} \dot{+} \mathcal{F}$  where both  $\mathcal{E}$  and  $\mathcal{F}$  are Lagrangian subspaces of  $\mathcal{W}$  (i.e.,  $\mathcal{E} = \mathcal{E}^{[\perp]}$  and  $\mathcal{F} = \mathcal{F}^{[\perp]}$ ). Indeed, given such a decomposition, if we define  $Je = -e$  for all  $e \in \mathcal{E}$  and  $Jf = f$  for all  $f \in \mathcal{F}$ , then  $J$  is a skew-adjoint involution on  $\mathcal{W}$ . Conversely, if  $\mathcal{J}$  is a skew-adjoint involution on  $\mathcal{W}$  with  $\dim \mathcal{W} \neq 0$ , then the eigenvalues of  $\mathcal{J}$  are  $\pm 1$ , and if we let  $\mathcal{E}$  and  $\mathcal{F}$  be the eigenspaces of  $\mathcal{J}$  corresponding to the eigenvalues  $-1$  and  $+1$ , respectively, then  $\mathcal{W} = \mathcal{E} \dot{+} \mathcal{F}$  is a Lagrangian decomposition of  $\mathcal{W}$ . Thus, there is a one-to-one correspondence between the set of all skew-adjoint involutions on  $\mathcal{W}$  and the set of all ordered Lagrangian decompositions  $\mathcal{W} = \mathcal{E} \dot{+} \mathcal{F}$  of  $\mathcal{W}$ .

## VI. RECIPROCAL STATE/SIGNAL SYSTEMS

In the case of a passive state/signal system  $\Sigma = (V, \mathcal{X}, \mathcal{W})$  the quadratic form induced by the Kreĭn space inner product  $[\cdot, \cdot]_{\mathcal{W}}$  in the signal space  $\mathfrak{W}$  is often called the *power product*, since the value of this quadratic form applied to the signal  $w(t)$  is equal to the power entering the system from the surroundings at time  $t$ . More precisely, this is the *active power*. In the reciprocal case the active power is complemented by the *reactive power*  $[\cdot, \mathcal{J} \cdot]_{\mathcal{W}}$ , where  $\mathcal{J}$  is a skew-adjoint involution on  $\mathcal{W}$ . Whereas the active power is an indefinite nondegenerate *symmetric* quadratic form on  $\mathcal{W}$ , the reactive power is a nondegenerate *skew-symmetric* quadratic form on  $\mathcal{W}$ .

In the state/signal setting the reciprocity of a passive full behavior  $\mathfrak{W}$  amounts to the existence of a skew-adjoint involution  $\mathcal{J}$  such that  $\mathfrak{W} = \mathcal{J}\mathfrak{A}\mathfrak{W}^{[\perp]}$ , where  $\mathfrak{A}$  is the reflection operator  $(\mathfrak{A}w)(t) = w(-t)$ ,  $t \in \mathbb{R}$ , and  $\mathfrak{W}^{[\perp]}$  is the orthogonal companion of  $\mathfrak{W}$  with respect to the inner product (11). In this case we call  $\mathfrak{W}$   *$\mathcal{J}$ -reciprocal*.

**Theorem 1.** *Let  $\mathfrak{W}$  be a passive behavior on the signal space  $\mathcal{W}$ .*

- (i) *If  $\mathfrak{W}$  is  $\mathcal{J}$ -reciprocal for some skew-adjoint involution  $\mathcal{J}$  in  $\mathcal{W}$ , then there exists a simple conservative realization  $\Sigma = (V; \mathcal{X}; \mathcal{W})$  of  $\mathfrak{W}$  satisfying*

$$V = \begin{bmatrix} -\mathcal{I} & 0 & 0 \\ 0 & \mathcal{I} & 0 \\ 0 & 0 & \mathcal{J} \end{bmatrix} V^{[\perp]} \quad (12)$$

*for some signature operator  $\mathcal{I}$ .*

- (ii) *Conversely, if  $\Sigma = (V; \mathcal{X}; \mathcal{W})$  is a (not necessarily simple) conservative realization of  $\mathfrak{W}$  and (12) holds for some signature operator  $\mathcal{I}$  and some skew-adjoint involution  $\mathcal{J}$ , then  $\mathfrak{W}$  is  $\mathcal{J}$ -reciprocal.*

Reciprocal input/state/output systems in a finite-dimensional setting are discussed in, e.g., [Wil72], [OJ85], [ABGR90], and [LR95], and in an infinite-dimensional

setting in [Fuh75], [Obe96], [GO99], and [AADR02]. More details about state/signal systems can be found in [AS05]–[AS10] (discrete time systems) and [KS09], [Kur10], and [AKS10] (continuous time systems).

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