Reciprocal Symmetry in State/Signal Systems

Damir Z. Arov
Division of Mathematical Analysis
Institute of Physics and Mathematics
South-Ukrainian Pedagogical University

Olof Staffans
Department of Mathematics
Åbo Akademi University
http://web.abo.fi/~staffans/

Outline

- J-Conservative Discrete Time Input/State/Output Systems
- External and Internal I/S/O Reciprocity
- Conservative State/Signal Systems
- External and Internal S/S Reciprocity

Discrete Time-Invariant I/S/O Systems

A linear discrete-time-invariant i/s/o (input/state/output) system is of the form

$$\Sigma_{i/s/o}: \begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \quad x(0) = x_0, \\ y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+. \end{cases} \tag{1}$$

A, B, C, D, are bounded linear operators and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

the input $u(n) \in \mathcal{U}$ = the input space,
the state $x(n) \in \mathcal{X}$ = the state space,
the output $y(n) \in \mathcal{Y}$ = the output space (all Hilbert spaces).

A trajectory = a triple of sequences (u, x, y) satisfying (1).

Forward J-Conservative I/S/O System

$\Sigma_{i/s/o}$ is forward J-conservative if all trajectories satisfy

$$\|x(n+1)\|_{\mathcal{X}}^2 = \|x(n)\|_{\mathcal{X}}^2 + \left\langle \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, J \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} \right\rangle_{\mathcal{Y} \oplus \mathcal{U}}, \quad n \in \mathbb{Z}^+.$$

Here

$$j(u, y) = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle_{\mathcal{Y} \oplus \mathcal{U}}.$$

is the supply rate induced by the signature operator $J = J^* = J^{-1}$.

Adjoint I/S/O System

The (causal) adjoint system is given by

$$\Sigma^*_{i/s/o} : \begin{cases} x_*(n+1) = A^* x_*(n) + C^* y_*(n), & n \in \mathbb{Z}^+, \qquad x_*(0) = x_{*0}, \\ u_*(n) = B^* x_*(n) + D^* y_*(n), & n \in \mathbb{Z}^+. \end{cases} \tag{2}$$

The adjoint system is forward J_*-conservative if all the trajectories satisfy

$$\|x_*(n+1)\|^2_{\mathcal{X}} = \|x_*(n)\|^2_{\mathcal{X}} + \left\langle \begin{bmatrix} u_*(n) \\ y_*(n) \end{bmatrix}, J_* \begin{bmatrix} u_*(n) \\ y_*(n) \end{bmatrix} \right\rangle_{\mathcal{U} \oplus \mathcal{Y}}, \; n \in \mathbb{Z}^+.$$

Here $J_* = \begin{bmatrix} 0 & -1_{\mathcal{U}} \\ 1_{\mathcal{Y}} & 0 \end{bmatrix} J \begin{bmatrix} 0 & -1_{\mathcal{Y}} \\ 1_{\mathcal{U}} & 0 \end{bmatrix}$ defines the adjoint supply rate.

Simple J-conservative System

$\Sigma_{i/s/o}$ is J-conservative if $\Sigma_{i/s/o}$ is forward J-conservative and $\Sigma^*_{i/s/o}$ is forward J_*-conservative.

Simple J-conservative System

$\Sigma_{i/s/o}$ is J-conservative if $\Sigma_{i/s/o}$ is forward J-conservative and $\Sigma^*_{i/s/o}$ is forward J_*-conservative.

The reachable subspace $\mathfrak{R}$ of $\Sigma_{i/s/o}$ is the closed linear span of all the values $x(n)$, $n \geq 0$, as (u, x, y) varies over all trajectories of $\Sigma_{i/s/0}$ with $x_0 = 0$.

The unobservable subspace $\mathfrak{U}$ of $\Sigma_{i/s/o}$ is the set of all initial states $x(0)$ of all "unobservable" trajectories $(0, x, 0)$ of $\Sigma_{i/s/o}$ (i.e., both u and y are identically zero).

$\Sigma_{i/s/o}$ is simple if the closed linear span of $\mathfrak{R}$ and $\mathfrak{U}^\perp$ is all of $\mathcal{X}$.

Simple J-conservative System

$\Sigma_{i/s/o}$ is J-conservative if $\Sigma_{i/s/o}$ is forward J-conservative and $\Sigma^*_{i/s/o}$ is forward J_*-conservative.

The reachable subspace $\mathfrak{R}$ of $\Sigma_{i/s/o}$ is the closed linear span of all the values $x(n)$, $n \geq 0$, as (u, x, y) varies over all trajectories of $\Sigma_{i/s/0}$ with $x_0 = 0$.

The unobservable subspace $\mathfrak{U}$ of $\Sigma_{i/s/o}$ is the set of all initial states $x(0)$ of all "unobservable" trajectories $(0, x, 0)$ of $\Sigma_{i/s/o}$ (i.e., both u and y are identically zero).

$\Sigma_{i/s/o}$ is simple if the closed linear span of $\mathfrak{R}$ and $\mathfrak{U}^\perp$ is all of $\mathcal{X}$.

Theorem 1. An simple J-conservative i/s/o system $\Sigma_{i/s/o}$ is uniquely determined, up to a unitary similarity transformation in its state space, by its transfer function (defined in some neighborhood of the origin)[1]

$$\mathfrak{D}(z) := zC(1 - zA)^{-1}B + D.$$

[1] The same statement is true true for the balanced minimal realization.

Outline

- J-Conservative Discrete Time Input/State/Output Systems
- External and Internal I/S/O Reciprocity
- Conservative State/Signal Systems
- External and Internal S/S Reciprocity

Externally Reciprocal Impedance Systems

By an impedance supply rate we mean the following: There is a unitary operator $\Psi : \mathcal{U} \to \mathcal{Y}$ (= a "unit resistance") such that the supply rate (power product) is given by $j_{\mathrm{imp}}(u, y) = 2\Re(y, \Psi u)$. The signature operator is $J_{\mathrm{imp}} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$, and the dual signature operator is $J_* = \left[\begin{smallmatrix} 0 & \Psi^* \\ \Psi & 0 \end{smallmatrix}\right]$.

The impedance (= transfer) function $\mathfrak{D}$ is always a Ψ-Nevanlinna (= positive real) function in the unit disk, i.e., $\Psi^*\mathfrak{D}(z) + \mathfrak{D}(z)^*\Psi \geq 0$ for all $z \in \mathbb{D}$.

Externally Reciprocal Impedance Systems

By an impedance supply rate we mean the following: There is a unitary operator $\Psi\colon \mathcal{U} \to \mathcal{Y}$ (= a "unit resistance") such that the supply rate (power product) is given by $j_{\mathrm{imp}}(u, y) = 2\Re(y, \Psi u)$. The signature operator is $J_{\mathrm{imp}} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$, and the dual signature operator is $J_* = \left[\begin{smallmatrix} 0 & \Psi^* \\ \Psi & 0 \end{smallmatrix}\right]$.

The impedance (= transfer) function $\mathfrak{D}$ is always a Ψ-Nevanlinna (= positive real) function in the unit disk, i.e., $\Psi^*\mathfrak{D}(z) + \mathfrak{D}(z)^*\Psi \geq 0$ for all $z \in \mathbb{D}$.

If φ satisfies, in addition,

$$\varphi(z) = \Psi\varphi^*(\overline{z})\Psi, \quad z \in \mathbb{D},$$

where $\Psi\colon \mathcal{U} \to \mathcal{Y}$ is unitary, then we call φ is Ψ-reciprocal, and say that $\Sigma_{i/s/o}$ is externally reciprocal.

Externally Reciprocal Impedance Systems

By an impedance supply rate we mean the following: There is a unitary operator $\Psi : \mathcal{U} \to \mathcal{Y}$ (= a "unit resistance") such that the supply rate (power product) is given by $j_{\rm imp}(u,y) = 2\Re(y, \Psi u)$. The signature operator is $J_{\rm imp} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$, and the dual signature operator is $J_* = \left[\begin{smallmatrix} 0 & \Psi^* \\ \Psi & 0 \end{smallmatrix}\right]$.[2]

The impedance (= transfer) function $\mathfrak{D}$ is always a Ψ-Nevanlinna (= positive real) function in the unit disk, i.e., $\Psi^*\mathfrak{D}(z) + \mathfrak{D}(z)^*\Psi \geq 0$ for all $z \in \mathbb{D}$.

If φ satisfies, in addition,

$$\varphi(z) = \Psi\varphi^*(\overline{z})\Psi, \quad z \in \mathbb{D},$$

where $\Psi : \mathcal{U} \to \mathcal{Y}$ is unitary, then we call φ is Ψ-reciprocal, and say that $\Sigma_{i/s/o}$ is externally reciprocal.

[2] $\Sigma_{i/s/o}$ is impedance conservative $\Leftrightarrow \begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & C^*\Psi \\ \Psi^*C & \Psi^*D + D^*\Psi \end{bmatrix}$..

Internally Reciprocal Impedance Systems

(External) reciprocity is a very common property:

- If $\dim \mathcal{U} = \dim \mathcal{Y} = 1$, and φ is real on the real axis, then φ is reciprocal.
- The impedance function (transfer function from current to voltage) of every passive electrical circuit which does not contain any gyrators is reciprocal.

Internally Reciprocal Impedance Systems

(External) reciprocity is a very common property:

- If $\dim \mathcal{U} = \dim \mathcal{Y} = 1$, and φ is real on the real axis, then φ is reciprocal.
- The impedance function (transfer function from current to voltage) of every passive electrical circuit which does not contain any gyrators is reciprocal.

Theorem 2. A pure (= strictly positive real) Nevanlinna function $\mathfrak{D}$ is Ψ-reciprocal if and only if the (essentially unique) simple conservative realization $\Sigma_{i/s/o}$ of $\mathfrak{D}$ is internally reciprocal (= signature similar to its adjoint) in the sense that there exists a signature operator $\mathcal{I} = \mathcal{I}^* = \mathcal{I}^{-1}$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathcal{I} & 0 \\ 0 & \Psi \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} \mathcal{I} & 0 \\ 0 & \Psi \end{bmatrix} \quad (\Rightarrow A = \mathcal{I}^* A^* \mathcal{I}).$$

Internally Reciprocal Impedance Systems

(External) reciprocity is a very common property:

- If $\dim \mathcal{U} = \dim \mathcal{Y} = 1$, and φ is real on the real axis, then φ is reciprocal.
- The impedance function (transfer function from current to voltage) of every passive electrical circuit which does not contain any gyrators is reciprocal.

Theorem 2. A pure (= strictly positive real) Nevanlinna function $\mathfrak{D}$ is Ψ-reciprocal if and only if the (essentially unique) simple conservative realization $\Sigma_{i/s/o}$ of $\mathfrak{D}$ is internally reciprocal (= signature similar to its adjoint) in the sense that there exists a signature operator $\mathcal{I} = \mathcal{I}^* = \mathcal{I}^{-1}$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathcal{I} & 0 \\ 0 & \Psi \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} \mathcal{I} & 0 \\ 0 & \Psi \end{bmatrix} \quad (\Rightarrow A = \mathcal{I}^* A^* \mathcal{I}).$$

Thus, external reciprocity of a pure impedance function $\Leftrightarrow$ internal reciprocity of the simple conservative realization.[3]

[3] The same statement is true true for the balanced minimal realization.

Other Supply Rates

Analogous results are true for other supply rates as well (such as scattering and transmission).

Reciprocal i/s/o systems setting are discussed, e.g., in a finite-dimensional setting in

[Wil72], [OJ85], [ABGR90], and [LR95],

and in an infinite-dimensional setting in

[Fuh75], [Obe96], [GO99], and [AADR02].

Other Supply Rates

Analogous results are true for other supply rates as well (such as scattering and transmission).

Reciprocal i/s/o systems setting are discussed, e.g., in a finite-dimensional setting in

[Wil72], [OJ85], [ABGR90], and [LR95],

and in an infinite-dimensional setting in

[Fuh75], [Obe96], [GO99], and [AADR02].

Claim: The simplest way to treat a genaral supply rate is to replace the input/state/output system $\Sigma_{i/s/o}$ by a state/signal signal (s/s) system.

Outline

- J-Conservative Discrete Time Input/State/Output Systems
- External and Internal I/S/O Reciprocity
- Conservative State/Signal Systems
- External and Internal S/S Reciprocity

Signal Space and Energy Balance

We start by combining the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ into one signal space $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. This signal space has a natural Kreĭn space inner product obtained from the signature operator J in the supply rate j, namely

$$\left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]_{\mathcal{W}} = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y' \\ u' \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}.$$

The forward J-energy balance equation becomes (with $w(n) = \left[\begin{smallmatrix} y(n) \\ u(n) \end{smallmatrix}\right]$)

$$\|x(n+1)\|_{\mathcal{X}}^2 = \|x(n)\|_{\mathcal{X}}^2 + [w(n), w(n)]_{\mathcal{W}}, \qquad n \in \mathbb{Z}^+,$$

or equivalently,

$$-(x(n+1), x(n+1))_{\mathcal{X}} + (x(n), x(n))_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+.$$

Graph Representation of I/S/O System

The basic i/s/o relation

$$\Sigma_{i/s/o}: \begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \qquad x(0) = x_0, \\ \quad y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+. \end{cases} \tag{1}$$

can be written in graph form

$$\Sigma_{s/s}: \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0. \tag{3}$$

Graph Representation of I/S/O System

The basic i/s/o relation

$$\Sigma_{i/s/o}:\begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \qquad x(0) = x_0, \\ \qquad y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+. \end{cases} \tag{1}$$

can be written in graph form[4]

$$\Sigma_{s/s}:\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0. \tag{4}$$

4

$$V = \left\{ \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \begin{matrix} z = Ax + Bu, \\ y = Cx + Du, \end{matrix} \; w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \; x \in \mathcal{X}, \; u \in \mathcal{U} \right\}.$$

State/Signal System: Summary

The dynamics of a discrete time state/signal system Σ is defined by

$$\Sigma : \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0, \tag{4}$$

where V is the generating subspace of the node space $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

By a trajectory of Σ we mean a pair of sequences (x, w) satisfying (4).

We call x the state component and w the signal component of the trajectory.

Σ is well-posed if (4) defines a "reasonable dynamics".[5]

---

[5] For every $x_0 \in \mathcal{X}$ there is a trajectory with $x(0) = x_0$, and this trajectory depends continuously on x_0 and the signal part $w(\cdot)$.

Forward Conservativity of State/Signal Node

The forward energy balance

$$-(x(n+1), x(n+1))_{\mathcal{X}} + (x(n), x(n))_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+, \tag{5}$$

tells us to use the following natural (indefinite) Kreĭn space inner product in $\mathfrak{K}$:

$$\left[\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]\right]_{\mathfrak{K}} = -(z_1, z_2)_{\mathcal{X}} + (x_1, x_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}. \tag{6}$$

It is easy to see that (5) holds for all trajectories of Σ if and only if

$$\left[\left[\begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right]\right]_{\mathfrak{K}} = 0 \quad \forall \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V.$$

In other words, (5) holds if and only if V is a neutral subspace of $\mathfrak{K}$ with the inner product (6).

Conservativity of State/Signal Node

$$V^{[\perp]} = \left\{ \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in \mathfrak{K} \;\middle|\; \left[\begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right]_{\mathfrak{K}} = \forall \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}.$$

$\Sigma_{s/s}$ is forward conservative $\Leftrightarrow V \subset V^{[\perp]}$.

The "adjoint system" is forward conservative $\Leftrightarrow V^{[\perp]} \subset V$.

Define: $\Sigma_{s/s}$ is conservative if $V = V^{[\perp]}$.

If $\Sigma_{s/s}$ is conservative, then it is automatically well-posed.

Conservativity of State/Signal Node

$$V^{[\perp]} = \left\{ \left[\begin{smallmatrix} z_* \\ x_* \\ w_* \end{smallmatrix}\right] \in \mathfrak{K} \,\middle|\, \left[\left[\begin{smallmatrix} z_* \\ x_* \\ w_* \end{smallmatrix}\right], \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \right]_{\mathfrak{K}} = \forall \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \right\}.$$

$\Sigma_{s/s}$ is forward conservative $\Leftrightarrow V \subset V^{[\perp]}$.

The "adjoint system" is forward conservative $\Leftrightarrow V^{[\perp]} \subset V$.

Define: $\Sigma_{s/s}$ is conservative if $V = V^{[\perp]}$.

If $\Sigma_{s/s}$ is conservative, then it is automatically well-posed.

The reachable subspace $\mathfrak{R}$ and the unobservable subspace $\mathfrak{U}$ are defined in the same way as for i/s/o systems.

$\Sigma_{i/s/o}$ is simple if the closed linear span of $\mathfrak{R}$ and $\mathfrak{U}^{\perp}$ is all of $\mathcal{X}$.

The Behavior of a State/Signal Systems

In s/s theory the transfer function of an i/s/o system is replaced by the (frequency domain) behavior of the s/s system.

behavior of s/s system $\simeq$ graph of the transfer function of a i/s/o system.

More precisely, the behavior is the subspace of all H^2-functions $\widehat{w}(\cdot)$ on $\mathbb{D}$ which satisfy

$$\begin{bmatrix} \frac{1}{z}\hat{x}(z) \\ \hat{x}(z) \\ \widehat{w}(z) \end{bmatrix} \in V, \qquad z \in \mathbb{D}, \tag{7}$$

for some analytic function $\hat{x}(z)$.

Interpretation: $\hat{x}(z)$ is the Z-transform of the state part and $\widehat{w}(z)$ is the Z-transform of the signal part of a trajectory (x, w) of $\Sigma_{s/s}$ with $x(0) = 0$ and $w(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{W})$.

The Behavior of a State/Signal Systems

In s/s theory the transfer function of an i/s/o system is replaced by the (frequency domain) behavior of the s/s system.

behavior of s/s system $\simeq$ graph of the transfer function of a i/s/o system.

More precisely, the behavior is the subspace of all H^2-functions $\widehat{w}(\cdot)$ on $\mathbb{D}$ which satisfy

$$\begin{bmatrix} \frac{1}{z}\hat{x}(z) \\ \hat{x}(z) \\ \widehat{w}(z) \end{bmatrix} \in V, \qquad z \in \mathbb{D}, \tag{7}$$

for some analytic function $\hat{x}(z)$.

Interpretation: $\hat{x}(z)$ is the Z-transform of the state part and $\widehat{w}(z)$ is the Z-transform of the signal part of a trajectory (x, w) of $\Sigma_{s/s}$ with $x(0) = 0$ and $w(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{W})$.

Denote: $\mathfrak{W}$ = behavior of $\Sigma_{s/s}$ and $\mathfrak{W}(z) := \{\widehat{w}(z) \mid \widehat{w}(\cdot) \in \mathfrak{W}\}$, $z \in \mathbb{D}$.

Passive Behaviors

It turns out that

- the behavior $\mathfrak{W}$ of a conservative s/s system is a maximal nonnegative shift-invariant subspace of $H^2(\mathbb{D};\mathcal{W})$

with respect to the indefinite inner product inherited from the Kreĭn space $\mathcal{W}$ (shift-invariance means that it is invariant under multiplication with z).

Passive Behaviors

It turns out that

- the behavior $\mathfrak{W}$ of a conservative s/s system is a maximal nonnegative shift-invariant subspace of $H^2(\mathbb{D};\mathcal{W})$

with respect to the indefinite inner product inherited from the Kreĭn space $\mathcal{W}$ (shift-invariance means that it is invariant under multiplication with z).

- Passive behavior $=$ a maximal nonnegative shift-invariant subspace of $H^2(\mathbb{D};\mathcal{W})$.

- Strictly passive behavior $=$ a maximal strictly positive shift-invariant subspace of $H^2(\mathbb{D};\mathcal{W})$.

References

More details about state/signal systems can be found in [AS05, AS07a, AS07b, AS07c, AS09a, AS09b] and [Sta06].

Continuous time state/signal systems have been studied in [KS09, Kur09].

Outline

- J-Conservative Discrete Time Input/State/Output Systems
- External and Internal I/S/O Reciprocity
- Conservative State/Signal Systems
- External and Internal S/S Reciprocity

External and Internal Reciprocity

The state/signal analogue of external reciprocity, i.e., reciprocity of the transfer function, is the following:

A passive behavior $\mathfrak{W}$ is J-reciprocal if $\mathcal{J} = -\mathcal{J}^* = \mathcal{J}^{-1}$ is a skew-adjoint involution in the signal space $\mathcal{W}$ and $\mathfrak{W}(z) = \mathcal{J}\mathfrak{W}(\overline{z})^{[\perp]}$, $z \in \mathbb{D}$. (In the impedance i/s/o case we may take $\mathcal{J} = \begin{bmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}$.)

A conservative s/s system $\Sigma_{s/s}$ is externally reciprocal if the behavior $\mathfrak{W}$ of $\Sigma_{s/s}$ is J-reciprocal for some skew-adjoint involution $\mathcal{J}$.

External and Internal Reciprocity

The state/signal analogue of external reciprocity, i.e., reciprocity of the transfer function, is the following:

A passive behavior $\mathfrak{W}$ is J-reciprocal if $\mathcal{J} = -\mathcal{J}^* = \mathcal{J}^{-1}$ is a skew-adjoint involution in the signal space $\mathcal{W}$ and $\mathfrak{W}(z) = \mathcal{J}\mathfrak{W}(\overline{z})^{[\perp]}$, $z \in \mathbb{D}$. (In the impedance i/s/o case we may take $\mathcal{J} = \left[\begin{smallmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$.)

A conservative s/s system $\Sigma_{s/s}$ is externally reciprocal if the behavior $\mathfrak{W}$ of $\Sigma_{s/s}$ is J-reciprocal for some skew-adjoint involution $\mathcal{J}$.

A conservative s/s system $\Sigma_{s/s}$ is internally reciprocal if it is internally signature similar to its adjoint, i.e., there exists a signature operator $\mathcal{I} = \mathcal{I}^* = \mathcal{I}^{-1}$ in the state space $\mathcal{X}$ and a boundedly invertible operator $\mathcal{J} \in \mathcal{B}(\mathcal{W})$ such that

$$V = \left[\begin{smallmatrix} 0 & \mathcal{I} & 0 \\ \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{J} \end{smallmatrix}\right] V^{[\perp]}.$$

Connection Between External and Internal Reciprocity

Theorem 3. Let $\mathfrak{W}$ be a passive behavior on the signal space $\mathcal{W}$.

(i) If $\mathfrak{W}$ is $\mathcal{J}$-reciprocal for some skew-adjoint involution $\mathcal{J}$ in $\mathcal{W}$, then the (essentially unique) simple conservative realization $\Sigma = (V; \mathcal{X}; \mathcal{W})$ of $\mathfrak{W}$ satisfies

$$V = \begin{bmatrix} 0 & \mathcal{I} & 0 \\ \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{J} \end{bmatrix} V^{[\perp]} \tag{8}$$

for some signature operator $\mathcal{I}$. (Here $V^{[\perp]} = V$ since Σ is conservative.[6])

(ii) If $\Sigma = (V; \mathcal{X}; \mathcal{W})$ is a conservative realization of $\mathfrak{W}$ satisfying (8) for some signature operator $\mathcal{I}$ and some skew-adjoint involution $\mathcal{J}$, then $\mathfrak{W}$ is $\mathcal{J}$-reciprocal.

[6]The same claim is true for minimal passive balanced systems (in which case $V^{[\perp]} \neq V$).

Further Questions

Further Questions

- Both $\mathcal{I}$ and $\mathcal{J}$ are determined uniquely by V. Exactly to what extent is $\mathcal{J}$ determined uniquely by $\mathfrak{W}$?

Further Questions

- Both $\mathcal{I}$ and $\mathcal{J}$ are determined uniquely by V. Exactly to what extent is $\mathcal{J}$ determined uniquely by $\mathfrak{W}$?

- $\mathcal{J}$ defines a continuous non-degenerate anti-symmetric bilinear form in $\mathcal{W}$ (which corresponds to the reactive power in classical circuit theory).

Further Questions

- Both $\mathcal{I}$ and $\mathcal{J}$ are determined uniquely by V. Exactly to what extent is $\mathcal{J}$ determined uniquely by $\mathfrak{W}$?

- $\mathcal{J}$ defines a continuous non-degenerate anti-symmetric bilinear form in $\mathcal{W}$ (which corresponds to the reactive power in classical circuit theory).

- There is a one-to-one correspondence between the set of all skew-adjoint involutions $\mathcal{J}$ in $\mathcal{W}$ and all Lagrangian decompositions $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$ of $\mathcal{W}$. In particular, a necessary condition for reciprocity is that $\dim_+ \mathcal{W} = \dim_- \mathcal{W}$ (the input and output dimensions must be the same).

Further Questions

- Both $\mathcal{I}$ and $\mathcal{J}$ are determined uniquely by V. Exactly to what extent is $\mathcal{J}$ determined uniquely by $\mathfrak{W}$?

- $\mathcal{J}$ defines a continuous non-degenerate anti-symmetric bilinear form in $\mathcal{W}$ (which corresponds to the reactive power in classical circuit theory).

- There is a one-to-one correspondence between the set of all skew-adjoint involutions $\mathcal{J}$ in $\mathcal{W}$ and all Lagrangian decompositions $\mathcal{W} = \mathcal{F} \dot{+} \mathcal{E}$ of $\mathcal{W}$. In particular, a necessary condition for reciprocity is that $\dim_+ \mathcal{W} = \dim_- \mathcal{W}$ (the input and output dimensions must be the same).

- This leads to a connection to the theory of port-Hamiltonian systems!

References

[AADR02] Daniel Alpay, Tomas Ya. Azizov, Aad Dijksma, and James Rovnyak, Colligations in Pontryagin spaces with a symmetric characteristic function, Linear operators and matrices, Oper. Theory Adv. Appl., vol. 130, Birkhäuser, Basel, 2002, pp. 55–82.

[ABGR90] Daniel Alpay, Joseph A. Ball, Israel Gohberg, and Leiba Rodman, Realization and factorization for rational matrix functions with symmetries, Extension and interpolation of linear operators and matrix functions, Oper. Theory Adv. Appl., vol. 47, Birkhäuser, Basel, 1990, pp. 1–60. MR MR1120273 (92j:47031)

[AS05] Damir Z. Arov and Olof J. Staffans, State/signal linear time-invariant systems theory. Part I: Discrete time systems, The State Space Method, Generalizations and Applications (Basel Boston Berlin), Operator Theory: Advances and Applications, vol. 161, Birkhäuser-Verlag, 2005, pp. 115–177.

[AS07a] ———, State/signal linear time-invariant systems theory. Part II: Passive discrete time systems, Internat. J. Robust Nonlinear Control **17** (2007), 497–548.

[AS07b] ———, State/signal linear time-invariant systems theory. Part III: Transmission and impedance representations of discrete time systems, Operator Theory, Structured Matrices, and Dilations, Tiberiu Constantinescu Memorial Volume (Bucharest Romania), Theta Foundation, 2007, Available from American Mathematical Society, pp. 101–140.

[AS07c] ———, State/signal linear time-invariant systems theory. Part IV: Affine representations of discrete time systems, Complex Anal. Oper. Theory **1** (2007), 457–521.

[AS09a] ———, A Kreĭn space coordinate free version of the de Branges complementary space, J. Funct. Anal. **256** (2009), 3892–3915.

[AS09b] ———, Two canonical passive state/signal shift realizations of passive discrete time behaviors, J. Funct. Anal. (2009), To appear.

[Fuh75] Paul A. Fuhrmann, On J-symmetric restricted shifts, Proc. Amer. Math. Soc. **51** (1975), 421–426.

[GO99] Aurelian Gheondea and Raimund J. Ober, Completely J-positive linear systems of finite order, Math. Nachr. **203** (1999), 75–101.

[KS09] Mikael Kurula and Olof J. Staffans, Well-posed state/signal systems in continuous time, Complex Anal. Oper. Theory (2009), To appear.

[Kur09] Mikael Kurula, Passive continuous time state/signal systems, In preparation, 2009.

[LR95] Peter Lancaster and Leiba Rodman, Minimal symmetric factorizations of symmetric real and complex rational matrix functions, Proceedings of the Workshop "Nonnegative Matrices, Applications and Generalizations" and the Eighth Haifa Matrix Theory Conference (Haifa, 1993), vol. 220, 1995, pp. 249–282.

[Obe96] Raimund Ober, System theoretic aspects of completely symmetric systems, Recent Developments in Operator Theory and its Applications (Winnipeg, MB, 1994) (Basel Boston Berlin), Operator Theory: Advances and Applications, vol. 87, Birkhäuser-Verlag, 1996, pp. 233–262.

[OJ85] Philippe C. Opdenacker and Edmond A. Jonckheere, LQG balancing and reduced LQG compensation of symmetric passive systems, Internat. J. Control **41** (1985), 73–109.

[Sta06] Olof J. Staffans, Passive linear discrete time-invariant systems, Proceedings of the International Congress of Mathematicians, Madrid, 2006, 2006, pp. 1367–1388.

[Wil72] Jan C. Willems, Dissipative dynamical systems Part II: Linear systems with quadratic supply rates, Arch. Rational Mech. Anal. **45** (1972), 352–393.