

# Scattering and Impedance Passive and Conservative Systems

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# Scattering Passive and Conservative Systems

This talk is mainly about linear time-invariant i/s/o (input/state/output) systems whose dynamics is described by an equation of the type

$$\Sigma : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (1)$$

It has a

**state space**  $\mathcal{X}$  (a Hilbert space),

**input space**  $\mathcal{U}$  (a Hilbert space),

**output space**  $\mathcal{Y}$  (a Hilbert space).

At the moment I only assume that  **$S$  is a closed linear operator with dense domain**. (More assumptions on  $S$  will be added later.)

# Scattering Passive or Conservative Systems

First I talk about scattering passive and conservative systems. These can be defined in at least three different ways:

- 1 We suppose that  $S$  is a system node, and add algebraic conditions on  $S$  and its adjoint  $S^*$  to make this system node scattering passive or conservative.
- 2 Instead of imposing conditions on  $S$  we impose conditions on the set of solutions of (1) which force the system to become scattering passive or conservative.
- 3 Instead of either of the above, we interpret  $S$  as an scattering interpretation of a state/signal system which we know to be passive or conservative.

# System Nodes

## Definition

Let  $S$  be an operator  $S: \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \text{Dom}(S) \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ . We partition  $S$  into  $S = \begin{bmatrix} A \& B \\ C \& D \end{bmatrix}$ , and define the operator  $A$  by  $Ax = A \& B \begin{bmatrix} x \\ 0 \end{bmatrix}$  for all  $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \text{Dom}(S)$ . Then  $S$  is a **system node on  $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$**  if it satisfies the following conditions:

- 1  $S$  is closed (as an operator  $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ ).
- 2  $A \& B$  is closed (as an operator  $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ ).
- 3  $A$  generates a  $C_0$  semigroup.
- 4 For every  $u \in \mathcal{U}$  there exists a  $x \in \mathcal{X}$  such that  $\begin{bmatrix} x \\ u \end{bmatrix} \in \text{Dom}(S)$ .

See, e.g., (Sta05, Lemma 4.7.7).

# Components and Properties of a System Node

- 1 The domain of  $S$  is dense in  $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ .
- 2 The operator  $A\&B$  can be extended to an operator  $\begin{bmatrix} A_{-1} & B \end{bmatrix} \in \mathcal{L}(\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}; \mathcal{X}_{-1})$ , where  $\mathcal{X}_{-1}$  is the standard “extrapolation space” constructed from  $A$ . For all  $\lambda \in \rho(A)$  we have  $(\lambda - A_{-1})^{-1}Bu \in \mathcal{X}$  and  $\begin{bmatrix} (\lambda - A_{-1})^{-1}B \\ 1_{\mathcal{U}} \end{bmatrix} \in \text{Dom}(S)$ .
- 3 The operator  $Ax = A\&B \begin{bmatrix} x \\ 0 \end{bmatrix}$  with domain  $\text{Dom}(S) \cap \begin{bmatrix} \mathcal{X} \\ 0 \end{bmatrix}$  is called the **main operator** of  $S$ .
- 4 The operator  $Cx = C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}$  with domain  $\text{Dom}(S) \cap \begin{bmatrix} \mathcal{X} \\ 0 \end{bmatrix}$  is called the **observation operator** of  $S$ .
- 5 The operator  $B$  above is called the **control operator** of  $S$ .
- 6 The function  $\widehat{\mathcal{D}}(\lambda) = C\&D \begin{bmatrix} (\lambda - A_{-1})^{-1}B \\ 1_{\mathcal{U}} \end{bmatrix}$  is called the **transfer function** of  $S$ .
- 7 The **adjoint**  $S^*$  of a system node  $S$  is also a system node.
- 8  $A^d = A^*$ ,  $B^d = C^*$ ,  $C^d = B^*$ , and  $\widehat{\mathcal{D}}^d(\lambda) = \widehat{\mathcal{D}}^*(\bar{\lambda})$ .

# Scattering Passive and Conservative System Nodes

The **algebraic** condition for a system node  $S$  to be **scattering passive** is the following:

$$2\Re\langle A\&B \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, x_0 \rangle_X + |C\&D \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}|_Y^2 \leq |u_0|_U^2, \quad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \text{Dom}(S). \quad (2)$$

The **algebraic** conditions for a system node  $S$  to be **scattering conservative** are the following:

$$\begin{aligned} 2\Re\langle A\&B \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, x_0 \rangle_X + |C\&D \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}|_Y^2 &= |u_0|_U^2, & \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \text{Dom}(S), \\ 2\Re\langle [A\&B]^d \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, x_0 \rangle_X + |[C\&D]^d \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}|_Y^2 &= |y_0|_Y^2, & \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \text{Dom}(S^*). \end{aligned} \quad (3)$$

There is some redundancy in (3). See (MSW06).

# Trajectories of (1)

Recall the original equation:

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+. \quad (1)$$

## Definition

Let  $S: \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \text{Dom}(S) \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$  be a closed operator.

- 1 A triple  $(x, u, y)$  is called a **classical solution** of (1) on  $\mathbb{R}^+$  if  $x \in C^1([0, \infty); X)$ ,  $u \in C([0, \infty); U)$ ,  $y \in C([0, \infty); Y)$ , and (1) holds.
- 2 A triple  $(x, u, y)$  is called a **generalized solution** of (1) on  $\mathbb{R}^+$  if  $x \in C(\mathbb{R}^+; X)$ ,  $u \in L^2_{\text{loc}}(\mathbb{R}^+; U)$ ,  $y \in L^2_{\text{loc}}(\mathbb{R}^+; Y)$ , and there exists a sequence  $(x_k, u_k, y_k)$  of classical solutions of (1) on  $\mathbb{R}^+$  such that  $x_k \rightarrow x$  in  $C(\mathbb{R}^+; X)$ ,  $u_k \rightarrow u$  in  $L^2_{\text{loc}}(\mathbb{R}^+; U)$  and  $y_k \rightarrow y$  in  $L^2_{\text{loc}}(\mathbb{R}^+; Y)$ .

# Scattering Passivity in Terms of Trajectories

## Theorem

$S: \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \text{Dom}(S) \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$  is a *scattering passive system node if and only if*  $S$  is closed and the set of classical solutions of (1) on  $\mathbb{R}^+$  satisfy the following four conditions:

- 1 For every  $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \text{Dom}(S)$  there *exists a classical solution*  $(x, u, y)$  of (1) on  $\mathbb{R}^+$  with  $x(0) = x_0$  and  $u(0) = u_0$ .
- 2 The set of all initial states  $x(0)$  of all classical solutions  $(x, u, y)$  on  $\mathbb{R}^+$  is *dense* in  $\mathcal{X}$ .
- 3 The set of all  $u \in C(\mathbb{R}^+; \mathcal{U})$  with  $u(0) = 0$  for which there exists a classical solution  $(x, u, y)$  of (1) on  $\mathbb{R}^+$  with  $x(0) = 0$  is *dense* in  $L_{\text{loc}}^2(\mathbb{R}^+; \mathcal{U})$ .
- 4 All classical solutions  $(x, u, y)$  satisfy the *power inequality*

$$\frac{d}{dt} \|x(t)\|_{\mathcal{X}}^2 + \|y(t)\|_{\mathcal{Y}}^2 \leq \|u(t)\|_{\mathcal{U}}^2, \quad t \in \mathbb{R}^+. \quad (4)$$



# Discussion

The above theorem was proved in (KS09) and (Kur10).

- **In principle** this enables us to prove that  $S$  is a (scattering passive) system node by just studying the set of classical solutions of (1).
- **In practice** this does not help much: The study of classical trajectories is not easier than the study of the operator  $S$  itself.

This leads us to the **third alternative**: We interpret (1) as a **scattering interpretation of a state/signal system**.

# State/Signal Interpretation

We rewrite the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (1)$$

in the graph form

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ u(t) \\ y(t) \end{bmatrix} \in V, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \quad (5)$$

where

$$V := \begin{bmatrix} A & B \\ 1_x & 0 \\ 0 & 1_u \\ C & D \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}. \quad (6)$$

# Power Inequality

Next we combine  $u(t)$  and  $y(t)$  into one single vector

$w(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$ , and define the following indefinite **scattering type Kreĭn space inner product in the signal space  $\mathcal{W}$** :

$$[w_1, w_2]_{\mathcal{W}} = \left[ \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \right]_{\mathcal{W}} = (u_1, u_2)_{\mathcal{U}} - (y_1, y_2)_{\mathcal{Y}}. \quad (7)$$

Then the power inequality

$$\frac{d}{dt} \|x(t)\|_{\mathcal{X}}^2 + \|y(t)\|_{\mathcal{Y}}^2 \leq \|u(t)\|_{\mathcal{U}}^2, \quad t \in \mathbb{R}^+. \quad (8)$$

can be rewritten in the form

$$- (\dot{x}(t), x(t))_{\mathcal{X}} - (x(t), \dot{x}(t))_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} \geq 0. \quad (9)$$

# The Node Space $\mathfrak{K}$

This motivates us to introduce the following indefinite **Kreĭn space inner product in the node space**  $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ :

$$\left[ \begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]_{\mathfrak{K}} = -(z_1, x_2)\mathcal{X} - (x_1, z_2)\mathcal{X} + [w_1, w_2]_{\mathcal{W}}. \quad (10)$$

Then we can rewrite (8) in the form

$$\left[ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}, \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \right]_{\mathfrak{K}} \geq 0.$$

The characterization by means of classical solutions said that every  $(x(0), u(0), y(0))$  can be the initial data of a classical solution.

Taking  $t = 0$  and using (5) we find that

$$\left[ \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right]_{\mathfrak{K}} \geq 0, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V,$$

i.e.,  $V$  is a nonnegative subspace of the Kreĭn node space  $\mathfrak{K}$ .

# State/Signal Passivity and Conservativity

## Theorem

Let  $S: \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \text{Dom}(S) \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$  be a closed operator, and define the *generating subspace*  $V$  as above with the *scattering inner product* (7) in the signal space  $\mathcal{W}$ . Then

- 1  $S$  is a *scattering passive system node* if and only if  $V$  is *maximal nonnegative* in the node space  $\mathfrak{K}$ , i.e., it is nonnegative, and it is not strictly contained in any other nonnegative subspace.
- 2  $S$  is a *scattering conservative system node* if and only if  $V$  is *Lagrangian*, i.e.,  $V = V^{\perp}$ .

$$V^{\perp} = \left\{ \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in \mathfrak{K} \mid \left[ \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right] = 0 \text{ for all } \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}.$$

This result turns out to be significant! It is often possible to *prove directly* that  $V$  is maximal nonnegative or Lagrangian

# Impedance Passive and Conservative Systems

It is time to move on to **impedance** passive or conservative systems. These can be defined in three different ways:

- We suppose that  $S$  is a **system node**, and add algebraic conditions on  $S$  and its adjoint  $S^*$  to make this system node impedance passive or conservative. **This is not a good choice.**
- Instead of imposing conditions on  $S$  we impose **conditions on the set of solutions** of (1) which force the system to become impedance passive or conservative. **This is, in principle OK, but not that helpful.**
- Instead of either of the above, we interpret  $S$  as an **impedance interpretation of a state/signal system** which we know to be passive or conservative. **This leads to the “right” result.**

The problem with the first approach is that, in contrast to the scattering case, **there is no underlying physical reason why an impedance passive system should be generated by a system node.**

# Impedance Passive and Conservative System Nodes

The **algebraic** condition for a system node  $S = \begin{bmatrix} A&B \\ C&D \end{bmatrix}$  with  $\mathcal{Y} = \mathcal{U}$  to be **impedance passive** is the following:

- The operator  $\begin{bmatrix} A&B \\ -C&D \end{bmatrix}$  is maximal dissipative, i.e.,

$$\left( \begin{bmatrix} A&B \\ -C&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} A&B \\ -C&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right)_{\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}} \leq 0, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \text{Dom}(S),$$

and  $\begin{bmatrix} A&B \\ -C&D \end{bmatrix}$  is not strictly contained in any other operator which satisfies the same condition.

The **algebraic** condition for a system node  $S$  with  $\mathcal{Y} = \mathcal{U}$  to be **impedance conservative** is the following:

- The operator  $\begin{bmatrix} A&B \\ -C&D \end{bmatrix}$  is skew-adjoint, i.e.,

$$\begin{bmatrix} A&B \\ -C&D \end{bmatrix}^* = - \begin{bmatrix} A&B \\ -C&D \end{bmatrix}.$$

See (Sta02). **However, an impedance passive system need not be induced by a system node.**

# Impedance Passivity in Terms of Trajectories

One reasonable **definition in terms of trajectories** of impedance passivity of  $S: [\mathcal{X}] \supset \text{Dom}(S) \rightarrow [\mathcal{U}]$  would be to require  $S$  to satisfy:

- 1 For every  $[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}] \in \text{Dom}(S)$  there **exists a classical solution**  $(x, u, y)$  of (1) on  $\mathbb{R}^+$  with  $x(0) = x_0$  and  $u(0) = u_0$ .
- 2 The set of all initial states  $x(0)$  of all classical solutions  $(x, u, y)$  on  $\mathbb{R}^+$  is **dense** in  $\mathcal{X}$ .
- 3 The set of all  $u + y \in C(\mathbb{R}^+; \mathcal{U})$  with  $u(0) = 0 = y(0)$  for which there exists a classical solution  $(x, u, y)$  of (1) on  $\mathbb{R}^+$  with  $x(0) = 0$  is **dense** in  $L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U})$ .
- 4 All classical solutions  $(x, u, y)$  of (1) on  $\mathbb{R}^+$  satisfy the **power inequality**

$$\frac{d}{dt} \|x(t)\|_{\mathcal{X}}^2 \leq 2\Re(u(t), y(t))_{\mathcal{U}}, \quad t \in \mathbb{R}^+. \quad (11)$$



# Discussion

- 1 The above definition would lead to a more or less correct notion.
- 2 However, that characterization by itself is not very useful.
- 3 But there is still **another alternative**: We interpret (1) as an **impedance interpretation of a state/signal system**.

# State/Signal Interpretation

Once more we rewrite the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (1)$$

in the graph form

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ u(t) \\ y(t) \end{bmatrix} \in V, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \quad (5)$$

where

$$V := \begin{bmatrix} A & B \\ 1_x & 0 \\ 0 & 1_u \\ C & D \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}. \quad (6)$$

# Power Inequality

We then combine  $u(t)$  and  $y(t)$  into one single vector  $w(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$  and define the following indefinite **impedance type Kreĭn space inner product in the signal space  $\mathcal{W}$** :

$$[w_1, w_2]_{\mathcal{W}} = \left[ \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \right]_{\mathcal{W}} = (u_1, y_2)_{\mathcal{U}} + (y_1, u_2)_{\mathcal{U}}. \quad (12)$$

Then the power inequality

$$\frac{d}{dt} \|x(t)\|_{\mathcal{X}}^2 \leq 2\Re(u(t), y(t))_{\mathcal{U}}, \quad t \in \mathbb{R}^+, \quad (13)$$

can be rewritten in the form

$$- (\dot{x}(t), x(t))_{\mathcal{X}} - (x(t), \dot{x}(t))_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} \geq 0. \quad (14)$$

# The Node Space $\mathfrak{K}$

This motivates us to introduce the following indefinite **Kreĭn space inner product in the node space**  $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ :

$$\left[ \begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]_{\mathfrak{K}} = -(z_1, x_2)\mathcal{X} - (x_1, z_2)\mathcal{X} + [w_1, w_2]_{\mathcal{W}}. \quad (10)$$

Then we can rewrite (13) in the form

$$\left[ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}, \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \right]_{\mathfrak{K}} \geq 0.$$

The characterization by means of classical solutions said that every  $(x(0), u(0), y(0))$  can be the initial data of a classical solution.

Taking  $t = 0$  and using (5) we find that

$$\left[ \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right]_{\mathfrak{K}} \geq 0, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V,$$

i.e.,  $V$  is a nonnegative subspace of the Kreĭn node space  $\mathfrak{K}$ .

## Theorem

Define the generating subspace  $V$  as above with the *impedance inner product (12)* in the signal space. Then

- 1  $V$  is *maximal nonnegative* in the node space  $\mathfrak{K}$  if and only if the operator  $\begin{bmatrix} A&B \\ -C&D \end{bmatrix}$  is *maximal dissipative*.
- 2  $V$  is *Lagrangian* in the node space  $\mathfrak{K}$  if and only if the operator  $\begin{bmatrix} A&B \\ -C&D \end{bmatrix}$  is *skew-adjoint*.

## Theorem

Define the generating subspace  $V$  as above with the *scattering inner product (7)* in the signal space  $\mathcal{W}$ . Then

- 1  $S$  is a *scattering passive* system node if and only if  $V$  is *maximal nonnegative* in the node space  $\mathfrak{K}$ .
- 2  $S$  is a *scattering conservative* system node if and only if  $V$  is a *Lagrangian* subspace of the node space  $\mathfrak{K}$ .

# Discussion

- The only difference between these two results is that we use **different indefinite inner products in the signal space  $\mathcal{W}$** .
- It is possible to convert the scattering type inner product into an impedance inner product by doing a  $45^\circ$  rotation (followed by a reflection): if we define

$$e = \frac{1}{\sqrt{2}}(u + y), \quad f = \frac{1}{\sqrt{2}}(u - y), \quad (15)$$

then

$$(e, f)_{\mathcal{U}} + (f, e)_{\mathcal{U}} = \|u\|_{\mathcal{U}}^2 - \|y\|_{\mathcal{U}}^2. \quad (16)$$

- This transformation is called the **external Cayley transform**. It is its **own inverse**

$$u = \frac{1}{\sqrt{2}}(e + f), \quad y = \frac{1}{\sqrt{2}}(e - f), \quad (17)$$

and it converts a **scattering type inner product into an impedance type inner product** and the other way around.

# Discussion

This makes it possible to **use the impedance setting** to prove that the generating subspace  $V$  is **maximal nonnegative**, and then go back to a scattering setting to **get a scattering conservative system node**.

# Theorem on Impedance $\rightarrow$ Scattering

## Theorem

Let  $\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}$  be maximal dissipative in  $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ . Define

$$V_{\text{sca}} := \begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 & 0 \\ 0 & 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} A\&B \\ 1_{\mathcal{X}} & 0 \\ 0 & 1_{\mathcal{U}} \\ C\&D \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}. \quad (18)$$

Then  $V_{\text{sca}}$  can be written in the graph form

$$V_{\text{sca}} := \begin{bmatrix} [A\&B]_{\text{sca}} \\ 1_{\mathcal{X}} & 0 \\ 0 & 1_{\mathcal{U}} \\ [C\&D]_{\text{sca}} \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}, \quad (19)$$

where  $S_{\text{sca}}$  is a scattering passive system node.



## Some Formulas

$$E_{\text{imp}} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C\&D]_{\text{imp}} \end{bmatrix} \right). \quad (20)$$

$$S_{\text{sca}} = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \left( \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2}I \end{bmatrix} + \begin{bmatrix} [A\&B]_{\text{imp}} \\ 0 & 0 \end{bmatrix} \right) E_{\text{imp}}^{-1}. \quad (21)$$

$$E_{\text{sca}} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C\&D]_{\text{sca}} \end{bmatrix} \right), \quad (22)$$

$$S_{\text{imp}} = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \left( \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2}I \end{bmatrix} + \begin{bmatrix} [A\&B]_{\text{sca}} \\ 0 & 0 \end{bmatrix} \right) E_{\text{sca}}^{-1}. \quad (23)$$

# Theorem from “Hot Air” paper

## Theorem

Let  $S_{\text{imp}} = \begin{bmatrix} [A\&B]_{\text{imp}} \\ [C\&D]_{\text{imp}} \end{bmatrix}$ , and suppose that  $T := \begin{bmatrix} [A\&B]_{\text{imp}} \\ -[C\&D]_{\text{imp}} \end{bmatrix}$  is maximal dissipative. Then the operator  $E_{\text{imp}}$  from (20) is injective on  $\mathcal{D}(S_{\text{imp}})$ . We denote its range by  $\mathcal{D}(S_{\text{sca}})$  and we define  $S_{\text{sca}}$  (with domain  $\mathcal{D}(S_{\text{sca}})$ ) by (21). Then  $S_{\text{sca}}$  is a scattering passive system node, and  $E_{\text{sca}}^{-1} = E_{\text{imp}}$ .

We denote by  $A_{\text{sca}}$ ,  $B_{\text{sca}}$  and  $C_{\text{sca}}$  the semigroup generator, the control operator and the observation operator of  $S_{\text{sca}}$ , and we denote by  $\widehat{\mathcal{D}}_{\text{sca}}$  its transfer function.

The operator  $S_{\text{imp}}$  can be recovered from  $S_{\text{sca}}$  via the formulas (22)–(23).

The system node  $S_{\text{sca}}$  is scattering conservative if and only if  $T$  is skew-adjoint.

# Computation of the Transfer Function

It is also easy to compute the resolvent and the transfer function from the formula, valid for all  $\Re s > 0$ ,

$$\begin{aligned} & \begin{bmatrix} (s - A_{\text{sca}})^{-1} & (s - A_{\text{sca}})^{-1} B_{\text{sca}} \\ C_{\text{sca}} (s - A_{\text{sca}})^{-1} & 1_{\mathcal{U}} + \widehat{\mathfrak{D}}_{\text{sca}}(s) \end{bmatrix} \\ &= \begin{bmatrix} 1_{\mathcal{U}} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \left( \begin{bmatrix} s & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} - \begin{bmatrix} [A \& B]_{\text{imp}} \\ -[C \& D]_{\text{imp}} \end{bmatrix} \right)^{-1} \begin{bmatrix} 1_{\mathcal{U}} & 0 \\ 0 & \sqrt{2} \end{bmatrix}. \end{aligned}$$

# Conclusion

The above theorem is actually quite useful.

- Many problems in mathematical physics come **naturally formulated in impedance form**:
  - The standard decomposition of signals in the analysis of electrical circuits is in pairs of **currents and voltages**. This is an impedance type decomposition of the interaction signals.
  - In partial differential equations the boundary conditions often come in pairs of conditions in such a way that the inner product between these in a suitable boundary space describes the power entering (or leaving) the system through the boundary. This is an **impedance type decomposition of the boundary data**.
- The impedance setting is often **algebraically simpler** than the scattering setting, as long as there is no need to worry about well-posedness.
- The preceding theorem enables us to prove **scattering passivity or conservativity** directly **from the impedance analysis**.

# Application

In the “Hot air” application that George was talking about  $S_{\text{imp}}$  is the restriction to  $\text{Dom}(S_{\text{imp}})$  of the operator

$$\left[ \begin{array}{cc|c} 0 & -L & 0 \\ L^* & G & \frac{1}{\sqrt{2}}K^* \\ \hline 0 & \frac{1}{\sqrt{2}}K & 0 \end{array} \right].$$

After changing the sign of the output row we get a maximal dissipative operator. The external Cayley transform  $S_{\text{sca}}$  of  $S_{\text{imp}}$  is restriction to its domain of the operator

$$\left[ \begin{array}{cc|c} 0 & -L & 0 \\ L^* & G - \frac{1}{2}K^*K & K^* \\ \hline 0 & -K & 1_{\mathcal{U}} \end{array} \right].$$

This gives us  $[A\&B]_{\text{sca}}$ ,  $[C\&D]_{\text{sca}}$ ,  $A_{\text{sca}}$ , and  $C_{\text{sca}}$ .

# Computation of the Transfer Function

To compute the resolvent and the transfer function we need to invert, for all  $s$  with  $\Re s > 0$ ,

$$\begin{aligned}
 & \left[ \begin{array}{cc|c} s & L & 0 \\ -L^* & s - G & -\frac{1}{\sqrt{2}}K^* \\ \hline 0 & \frac{1}{\sqrt{2}}K & 1 \end{array} \right] \\
 = & \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ -\frac{1}{s}L^* & 1 & -\frac{1}{\sqrt{2}}K^* \\ \hline 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{cc|c} s & 0 & 0 \\ 0 & P(s) & 0 \\ \hline 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{cc|c} 1 & \frac{1}{s}L & 0 \\ 0 & 1 & 0 \\ \hline 0 & \frac{1}{\sqrt{2}}K & 1 \end{array} \right],
 \end{aligned}$$

where

$$P(s) = s + \frac{1}{2}K^*K + \frac{1}{s}L^*L - G$$

is a boundedly invertible operator  $E_0 \rightarrow E'_0$ . Here  $E_0 = \text{Dom}(K) = \text{Dom}(L) = \text{Dom}(G)$  and  $E'_0$  is the dual space.

# Computation of the Transfer Function

By inverting the above identity we get

$$\begin{aligned}
 & \begin{bmatrix} (s - A_{\text{sca}})^{-1} & (s - A_{\text{sca}})^{-1} B_{\text{sca}} \\ C_{\text{sca}}(s - A_{\text{sca}})^{-1} & 1 + \widehat{\mathcal{D}}_{\text{sca}}(s) \end{bmatrix} \\
 &= \left[ \begin{array}{cc|c} 1 & -\frac{1}{s}L & 0 \\ 0 & 1 & 0 \\ \hline 0 & -K & \sqrt{2} \end{array} \right] \left[ \begin{array}{cc|c} \frac{1}{s} & 0 & 0 \\ 0 & V(s) & 0 \\ \hline 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ \frac{1}{s}L^* & 1 & K^* \\ \hline 0 & 0 & \sqrt{2} \end{array} \right] \\
 &= \left[ \begin{array}{cc|c} \frac{1}{s} - \frac{1}{s^2}LV(s)L^* & -\frac{1}{s}LV(s) & -\frac{1}{s}LV(s)K^* \\ \frac{1}{s}V(s)L^* & V(s) & V(s)K^* \\ \hline -\frac{1}{s}KV(s)L^* & -KV(s) & 2 - KV(s)K^* \end{array} \right],
 \end{aligned}$$

where  $V(s) = P(s)^{-1}$ . This gives us  $(s - A_{\text{sca}})^{-1}$ ,  $B_{\text{sca}}$ , and  $\widehat{\mathcal{D}}_{\text{sca}}(s)$ .

# State/Signal Formulation

Above I have discussed **scattering** and **impedance** systems separately, and shown that impedance systems can always be converted into scattering systems. **What about the converse?** To get a full understanding of the situation we must modify the underlying assumptions: Instead of starting with a scattering node or an impedance node and rewriting them into graph form we should **start with a state/signal node**:

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (24)$$

This state/signal system has a **Hilbert state space**  $\mathcal{X}$  to which the state  $x(t)$  belongs, and a **Kreĭn signal space**  $\mathcal{W}$  to which the interaction signal  $w(t)$  belongs, but **we do not distinguish between inputs and output**: both of these are part of the “signal”  $w(t)$ .



# General Setup of State/Signal Node

Recall the equation describing the dynamics:

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (24)$$

The **generating subspace**  $V$  is a closed subspace of the **node space**  $\mathfrak{R}$ , which is equipped with the inner product

$$\left[ \begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]_{\mathfrak{R}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}. \quad (10)$$

# Simplifying assumption:

I make the following simplifying assumption:

$$\text{If } \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V, \text{ then } z = 0.$$

This assumption was redundant in the cases described earlier when  $V$  was the graph of a closed operator  $S$ , and it can be removed “without loss of generality”. It says that  $\dot{x}(t)$  is determined uniquely by  $x(t)$  and  $w(t)$ .

## Definition

The state/signal system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is

- 1 **passive** if  $V$  is a maximal nonnegative subspace of  $\mathfrak{K}$ ;
- 2 **conservative** if  $V$  is a Lagrangian subspace of  $\mathfrak{K}$ .

# Fundamental Decompositions

- A **fundamental decomposition** of the signal space  $\mathcal{W}$  is of the type  $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ , where  $\mathcal{U}$  is uniformly positive,  $-\mathcal{Y}$  is uniformly negative, and  $\mathcal{U}$  and  $\mathcal{Y}$  are orthogonal to each other in  $\mathcal{W}$ .
- We let  $\mathcal{U}$  and  $\mathcal{Y}$  inherit Hilbert space inner products from  $\mathcal{W}$  and  $-\mathcal{W}$ , respectively.
- Then the inner product in  $\mathcal{W}$  can be written in the **scattering form**

$$[w_1, w_2]_{\mathcal{W}} = \left[ \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \right]_{\mathcal{W}} = (u_1, u_2)_{\mathcal{U}} - (y_1, y_2)_{\mathcal{Y}}. \quad (7)$$

- There exist **infinitely many** such fundamental decompositions when  $\mathcal{W}$  is indefinite (which is the usual case).  $\mathcal{W} > 0$  means that the system has no output, and  $\mathcal{W} < 0$  means that the system has no input.

# Scattering Representations

A passive state/signal system has many **scattering representations**. These are the system node representations corresponding to some **fundamental decomposition** of the signal space.

## Theorem

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a passive state/signal system, and let  $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$  be a **fundamental decomposition** of  $\mathcal{W}$ . Then  **$V$  is the graph of a scattering passive system node  $S$  on  $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$** . Moreover,  $\Sigma$  is conservative if and only if  $S$  is conservative.

# Use of Scattering Representations

- To show that  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is a **passive** state/signal system it suffices to show that there is **one fundamental decomposition**  $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$  such that  $V$  is the graph of a scattering passive system node.
- However, there are also **other methods** that can be used to show that  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is passive or conservative (such as the **impedance setting**).
- Once we know that  $\Sigma$  is passive (or conservative), we conclude that **every fundamental decomposition** corresponds to a passive (or conservative) system node.

# Lagrangian Decompositions of the Signal Space

- Impedance representation arise from **Lagrangian decompositions** of the signal space. These are decompositions of the type  $\mathcal{W} = \mathcal{E} \dot{+} \mathcal{F}$ , where  $\mathcal{E}^{[\perp]} = \mathcal{E}$  and  $\mathcal{F}^{[\perp]} = \mathcal{F}$ .
- The subspaces  $\mathcal{E}$  and  $\mathcal{F}$  do not inherit unique inner products from  $\mathcal{W}$ , since the inner product in  $\mathcal{W}$  is degenerate on  $\mathcal{E}$  and  $\mathcal{F}$ , i.e.,  $[w, w]_{\mathcal{W}} = 0$  for every  $w \in \mathcal{E}$  and  $w \in \mathcal{F}$ .
- However, they do inherit the topology of  $\mathcal{W}$ , and that topology gives some **non-unique inner products** in  $\mathcal{E}$  and  $\mathcal{F}$  that are unique only up to equivalence of the corresponding norms.
- With the appropriate choices of norms in  $\mathcal{E}$  and  $\mathcal{F}$  the inner product in  $\mathcal{W}$  can be written in the form

$$\left[ \begin{bmatrix} e_1 \\ f_1 \end{bmatrix}, \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \right]_{\mathcal{W}} = (e_1, \Psi f_1)_{\mathcal{E}} + (\Psi f_1, e_2)_{\mathcal{E}}, \quad \begin{bmatrix} e_1 \\ f_1 \end{bmatrix}, \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \in \mathcal{W},$$

for some unitary operator  $\Psi: \mathcal{F} \rightarrow \mathcal{E}$ .

# Lagrangian Decompositions Do Not Always Exist

- Unfortunately, **Lagrangian decompositions do not always exist.**
- A **necessary condition** for the existence of a Lagrangian decomposition is that the **positive and negative dimensions of  $\mathcal{W}$  are the same** (possibly infinite).
- Even if a Lagrangian decomposition does exist,  **$V$  need not be the graph of an operator** with respect to this decomposition; it could also be the graph of a **non-densely defined maximal dissipative relation**.
- However, if  $V$  happens to be the graph of an operator  $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ , then  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is passive or conservative **if and only if  $\begin{bmatrix} A & B \\ -C & D \end{bmatrix}$  is maximal dissipative or skew-adjoint.**

# The True Story

The correct interpretation of our first impedance  $\rightarrow$  scattering result is the following:

- The two subspaces  $V_{\text{sca}}$  and  $V_{\text{imp}}$  are not images of each other (as I incorrectly explained earlier), but they are one and the same maximal nonnegative subspace  $V$ .
- The only difference between the scattering and impedance cases are that they correspond to two different decompositions of the signal space.
- The signals and trajectories all the time stay the same. We just split the signal in inputs and output in two different ways.

In our original example:

$$\text{Impedance input} = \begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix};$$

$$\text{Impedance output} = \begin{bmatrix} 0 \\ \mathcal{U} \end{bmatrix};$$

$$\text{Scattering input} = \begin{bmatrix} 1_{\mathcal{U}} \\ 1_{\mathcal{U}} \end{bmatrix} \mathcal{U};$$

$$\text{Scattering output} = \begin{bmatrix} 1_{\mathcal{U}} \\ -1_{\mathcal{U}} \end{bmatrix} \mathcal{U}.$$



# Scattering Versus Impedance Decompositions

**Impedance** decompositions are usually **canonical**:

- In electrical circuits **current** and **voltage** are **natural physical variables**, and their inner product of these is the **power** entering the system.
- In partial differential equations **Dirichlet** and **Neumann** traces are natural variables. Suitable products of these give the power entering the system.
- Often physical systems have an extra built-in **symmetry**, called the **reciprocal** symmetry in the case of electrical circuits, which **fix the Lagrangian decomposition uniquely** (the system is reciprocal with respect to exactly one Lagrangian decomposition). (The “Hot air” system is reciprocal if  $G$  is self-adjoint.)
- Algebraically impedance systems are often **simpler** than scattering systems (as illustrated by the “Hot air” paper).
- But Lagrangian decompositions **do not always exist**.

# Scattering Versus Impedance Decompositions

**Scattering** decompositions are highly non-unique and **not canonical**:

- In the external Cayley transform applied to an electrical circuit we add currents and voltages to each other. This is physically not possible, since they have different physical dimensions. To do this we have to **choose some arbitrary normalization constant** (such as a  $1\Omega$  resistance). This normalization constant is **completely arbitrary**.
- The same problem arises in partial differential equations: In order to add a magnetic field to an electric field we have to choose some arbitrary normalization constant.
- Algebraically impedance systems are **more complicated** than scattering systems.
- But every passive state/signal system has a scattering representation. In this sense **scattering systems are more general**.

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