Abstracts

Scattering and Impedance Passive and Conservative Systems

Olof J. Staffans
(joint work with George Weiss)

In my talk I discussed systems which are either scattering or impedance passive or conservative. There are several similarities between scattering and impedance systems, but there are also significant differences.

In both cases we are talking about input/state/output systems, which have an input $u(t)$ in a Hilbert input space $\mathcal{U}$, a state $x(t)$ in a Hilbert state space $\mathcal{X}$, and an output $y(t)$ in a Hilbert output space $\mathcal{Y}$. The dynamics of the system is defined by an equation of the type

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t \geq 0, \tag{1}$$

where S is a closed densely defined operator from $\mathcal{X} \oplus \mathcal{U}$ to $\mathcal{X} \oplus \mathcal{Y}$ with domain $\operatorname{dom}(S)$. We say that (x, u, y) is a classical trajectory of (1) if x is continuously differentiable, u and y are continuous and, (1) holds.

In the case of a scattering passive system the classical trajectories satisfy the power inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \leq \|u(t)\|^2_{\mathcal{U}} - \|y(t)\|^2_{\mathcal{Y}}, \qquad t \geq 0, \tag{2}$$

whereas the corresponding inequality for an impedance passive system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \leq (y(t), u(t))_{\mathcal{U}}, \qquad t \geq 0; \tag{3}$$

in the impedance case we assume, in addition, that $\mathcal{Y} = \mathcal{U}$. In the case of conservative systems the two inequalities above are replaced by equalities, and they are required to hold both for the original systems, and for the dual systems that one gets by replacing S by its adjoint S^*.

A scattering passive system is always well-posed, and in the case of a scattering passive system the operator S is a so called *system node*. On the contrary, an impedance passive system need not be well-posed, and it need not even be a system node. However, impedance systems have a very simple characterization of a different type. We can always split the operator S into two parts $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, where $A\&B$ maps $\operatorname{dom}(S)$ into the state space $\mathcal{X}$ and $C\&D$ maps $\operatorname{dom}(S)$ into Y. Impedance passivity is characterized by the fact that the operator $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is maximal dissipative.

There is a simple method that can be used to convert impedance passive systems into scattering passive systems, which in the case of an impedance conservative system results in a scattering conservative system. The idea is the following. If we denote the impedance input by e (for "effort") and the impedance output by f (for "flow"), and if we map each trajectory $(x(t), e(t), f(t))$ of the impedance system $\begin{bmatrix} \dot{x}(t) \\ f(t) \end{bmatrix} = S_{\rm imp} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$ into a new family of functions $(x(t), u(t), y(t))$ by taking

$$u(t) = \tfrac{1}{\sqrt{2}}[e(t) + f(t)],$$
$$y(t) = \tfrac{1}{\sqrt{2}}[e(t) - f(t)],$$

then this family is the set of classical trajectories of a scattering passive system $\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S_{\rm sca} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$. The above mapping is called the *external Cayley transform.* The above idea leads to the following theorem, which is proved in [WS10]:

Theorem 1. *Let* $S_{\rm imp} = \begin{bmatrix} [A\&B]_{\rm imp} \\ [C\&D]_{\rm imp} \end{bmatrix}$ *be an operator mapping its domain* $\mathrm{dom}\,(S_{\rm imp}) \subset X \oplus U$ *into* $X \oplus U$, *such that* $T := \begin{bmatrix} [A\&B]_{\rm imp} \\ -[C\&D]_{\rm imp} \end{bmatrix}$ *(with the same domain) is maximal dissipative. Then the operator*

$$E_{\rm imp} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C\&D]_{\rm imp} \end{bmatrix} \right) \tag{4}$$

is injective on $\mathrm{dom}\,(S_{\rm imp})$. *We denote its range by* $\mathrm{dom}\,(S_{\rm sca})$ *and we define* $S_{\rm sca}$ *(with domain* $\mathrm{dom}\,(S_{\rm sca})$*) by*

$$S_{\rm sca} = \begin{bmatrix} [A\&B]_{\rm sca} \\ [C\&D]_{\rm sca} \end{bmatrix} := \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \left(\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2}I \end{bmatrix} + \begin{bmatrix} [A\&B]_{\rm imp} \\ 0 \quad 0 \end{bmatrix} \right) E_{\rm imp}^{-1}. \tag{5}$$

Then $S_{\rm sca}$ *is a scattering passive system node and* $E_{\rm imp}^{-1} = E_{\rm sca}$ *from* (7).

We denote by $A_{\rm sca}$, $B_{\rm sca}$ *and* $C_{\rm sca}$ *the semigroup generator, the control operator and the observation operator of* $S_{\rm sca}$, *and we denote by* $\widehat{\mathfrak{D}}_{\rm sca}$ *its transfer function. Then, for all* $s \in \mathbb{C}_+$,

(6)
$$\begin{bmatrix} (sI - A_{\rm sca})^{-1} & \frac{1}{\sqrt{2}}(sI - A_{\rm sca})^{-1}B_{\rm sca} \\ \frac{1}{\sqrt{2}}C_{\rm sca}(sI - A_{\rm sca})^{-1} & \frac{1}{2}(I + \widehat{\mathfrak{D}}_{\rm sca}(s)) \end{bmatrix} = \left(\begin{bmatrix} sI & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\rm imp} \\ -[C\&D]_{\rm imp} \end{bmatrix} \right)^{-1}.$$

The operator $S_{\rm imp}$ *can be recovered from* $S_{\rm sca}$ *via the formulas*

$$E_{\rm sca} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C\&D]_{\rm sca} \end{bmatrix} \right), \tag{7}$$

$$S_{\rm imp} = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \left(\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2}I \end{bmatrix} + \begin{bmatrix} [A\&B]_{\rm sca} \\ 0 \quad 0 \end{bmatrix} \right) E_{\rm sca}^{-1}. \tag{8}$$

The system node $S_{\rm sca}$ *is scattering conservative if and only if* T *is skew-adjoint.*

The above theorem can be used to construct many interesting scattering passive or conservative system nodes by applying the external Cayley transform to an impedance passive or conservative system. The proofs of the result mentioned in the abstract by George Weiss in this mini workshop are partially based on the above theorem.

Generally speaking, many equations from mathematical physics come naturally in an impedance formulation. For example, in an electrical system there is a natural division of signals into pairs of voltages and currents, the product of which gives the power. The name "impedance" is, in fact, taken from circuit theory, where it is used for the transfer function from current inputs to voltage outputs. A similar situation occurs in many partial differential equations, where the boundary conditions naturally split into conditions of Dirichlet and Neumann types.

Often the impedance formulation is algebraically simpler to work with than the scattering formulation, but on the other hand, the scattering version of a system has better well-posedness properties.

References

[KS09] Mikael Kurula and Olof J. Staffans, *Well-posed state/signal systems in continuous time*, Complex Anal. Oper. Theory **4** (2009), 319–390.

[Kur10] Mikael Kurula, *On passive and conservative state/signal systems in continuous time*, Integral Equations Operator Theory **67** (2010), 377–424, 449.

[WS10] George Weiss and Olof J. Staffans, *A class of scattering passive linear systems comprising the Maxwell equations*, in preparation, 2010.

Reporter: Sven-Ake Wegner