A Minimax Formulation of the Infinite-Dimensional Nehari Problem

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Abstract

We formulate a minimax game which is equivalent to the Nehari problem in the sense that this minimax game is well-posed if and only if the Hankel norm of a given operator is less than a prescribed constant γ. This game and the dual game provide us with physical interpretations of the Riccati operators that are commonly used in the solution of the Nehari problem.

1 Introduction

Let $\mathcal{D}$ be a bounded linear causal time-invariant mapping from $L^2(\mathbf{R};U)$ to $L^2(\mathbf{R};Y)$, where U and Y are Hilbert spaces. The Nehari problem is connected to the Hankel norm of $\mathcal{D}$, which is defined as (the notations are explained in the next section)

$$\|\pi_+\mathcal{D}\pi_-\| = \sup_{u\in L^2(\mathbf{R}^-;U)} \|\mathcal{D}\pi_-u\|_{L^2(\mathbf{R}^+;Y)}. \tag{1}$$

The Nehari theorem says in the finite-dimensional case (where the Laplace transform of $\mathcal{D}$ is a rational matrix-valued function) that $\|\pi_+\mathcal{D}\pi_-\| \le \gamma$ if and only if it is possible to find an anti-causal bounded linear time-invariant operator $\mathcal{U}^*$ from $L^2(\mathbf{R};U)$ to $L^2(\mathbf{R};Y)$ such that $\|\mathcal{D}-\mathcal{U}^*\| \le \gamma$; here $\|\mathcal{D}-\mathcal{U}^*\|$ is the operator norm of $\mathcal{D}-\mathcal{U}^*$. We get the "suboptimal" version of the same theorem by replacing "$\le$" by "$<$". In the sequel we study only the simpler, suboptimal case, but we allow the Laplace transform of $\mathcal{D}$ to be an arbitrary (non-rational) H^∞ function. We also allow U and Y to be infinite-dimensional.

One standard proof of the (suboptimal) Nehari theorem is based on a J-spectral factorization, either directly in the frequency domain, or by use of a state space method. This solution is very similar to the solution of the suboptimal full information H^∞ problem. It is well-known that the latter problem also has a game-theoretic interpretation, but to the best of our knowledge, no such interpretation has been available for the Nehari problem up to now. Our game-theoretic interpretation makes it possible to study the Nehari problem with a technique that is almost identical to the technique used for the full information H^∞ problem in [14, 15].

2 The Minimax Game

We use a combination of frequency domain and state space methods and start with a given realization of $\mathcal{D}$ as the input/output map of a stable well-posed linear system $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ (such a realization always exists; see [10, Theorem 4.3] or [12, Definition 2.10]).

We denote $\mathbf{R}^+ = [0,\infty)$, $\mathbf{R}^- = (-\infty,0)$,

$$(\pi_J u)(s) = \begin{cases} u(s), & s\in J,\\ 0, & s\notin J,\end{cases} \quad \text{for all } J\subset\mathbf{R},$$
$$\pi_+u = \pi_{\mathbf{R}^+}, \qquad \pi_-u = \pi_{\mathbf{R}^-},$$
$$(\tau^t u)(s) = u(t+s), \quad -\infty < t,s < \infty.$$

Definition 1 *Let U, X, and Y be Hilbert spaces. A* stable well-posed linear system Ψ *on (U,X,Y) is a quadruple $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ of continuous linear operators satisfying the following conditions:*

(i) $t\mapsto \mathcal{A}^t$ is a bounded strongly continuous semigroup of operators on X;

(ii) $\mathcal{B}\colon L^2(\mathbf{R};U)\to X$ satisfies $\mathcal{A}^t\mathcal{B}u = \mathcal{B}\tau^t\pi_-u$, for all $u\in L^2(\mathbf{R};U)$ and all $t\in\mathbf{R}^+$;

(iii) $\mathcal{C}\colon X\to L^2(\mathbf{R};Y)$ satisfies $\mathcal{C}\mathcal{A}^t x = \pi_+\tau^t\mathcal{C}x$, for all $x\in X$ and all $t\in\mathbf{R}^+$;

(iv) $\mathcal{D}\colon L^2(\mathbf{R};U)\to L^2(\mathbf{R};Y)$ satisfies $\tau^t\mathcal{D}u = \mathcal{D}\tau^t u$, $\pi_-\mathcal{D}\pi_+u = 0$, and $\pi_+\mathcal{D}\pi_-u = \mathcal{C}\mathcal{B}u$, for all $u\in L^2(\mathbf{R};U)$ and all $t\in\mathbf{R}$.

The different components of Ψ are called as follows: U is the input space, X is the state space, Y is the output space, $\mathcal{A}$ is the semigroup, $\mathcal{B}$ is the input map, $\mathcal{B}\mathcal{B}^*$ is the controllability Gramian, $\mathcal{C}$ is the output map, $\mathcal{C}^*\mathcal{C}$ is the observability Gramian, and $\mathcal{D}$ is the input-output map.

Here the condition $\tau^t\mathcal{D} = \mathcal{D}\tau^t$ in (iv) says that $\mathcal{D}$ is *time-invariant*, the condition $\pi_-\mathcal{D}\pi_+ = 0$ says that $\mathcal{D}$ is *causal*, and the condition $\pi_+\mathcal{D}\pi_- = \mathcal{C}\mathcal{B}$ says that the Hankel operator of $\mathcal{D}$ can be factored into the product $\mathcal{C}\mathcal{B}$. We denote the set of all bounded linear time-invariant operators from $L^2(\mathbf{R};U)$ into $L^2(\mathbf{R};Y)$ by $TI(U;Y)$, and $TIC(U;Y)$ stands for operators that are, in addition, causal. We abbreviate $TIC(U;U)$ to $TIC(U)$. It is not difficult to show that the adjoint of an operator in $TIC(U;Y)$ is an anti-causal operator in $TI(Y;U)$ (i.e., $\pi_+\mathcal{D}\pi_- = 0$). For more details, explanations, and examples, we refer the reader to [9]–[10], [11]–[15], [16]–[19] and the references therein.

Usually the *state* x and *output* y of a well-posed linear system are defined so that they correspond to the solution to a *Cauchy problem* (nonhomogeneous initial value problem) of the type

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \in \mathbf{R}^+, \\ x(0) &= x_0, \end{aligned} \tag{2}$$

which means that x and y are given by

$$\begin{aligned} x(t) &= \mathcal{A}^t x_0 + \mathcal{B}\tau^t\pi_+ u, \quad t \in \mathbf{R}^+, \\ y &= \mathcal{C}\,x_0 + \mathcal{D}\pi_+ u. \end{aligned} \tag{3}$$

Here $\mathcal{A}^t$ is the semigroup generated by A (it maps the initial state $x_0 \in H$ into the final state $x(t) \in H$), $\mathcal{B}\tau^t\pi_+$ is the map from the input $u \in L^2(\mathbf{R}^+;U)$ to the final state $x(t)$, $\mathcal{C}$ is the map from the initial state x_0 to the output $y \in L^2(\mathbf{R}^+;Y)$, and $\mathcal{D}\pi_+$ is the input-output map from u to y.

In the two player game that we construct for the Nehari problem, we add an extra term and use a different notion of state and output. This has to do with the fact that we are interested in the Hankel operator $\pi_+\mathcal{D}\pi_-$ of $\mathcal{D}$, which maps $u \in L^2(\mathbf{R}^-;U)$ into $y \in L^2(\mathbf{R}^+;Y)$, i.e, we are primarily interested in the values of u on $\mathbf{R}^-$ and the values of y on $\mathbf{R}^+$. The system that we use in the Nehari problem can be described in differential form by

$$\begin{aligned} x'(t) &= Ax(t) + B\pi_- u(t), \\ y(t) &= Cx(t) + \pi_- v(t) + D\pi_- u(t), \quad t \in \mathbf{R}, \\ x(0+) &= x(0-) + x_0, \end{aligned} \tag{4}$$

or in integral form by

$$\begin{aligned} x(t) &= \mathcal{B}\tau^t\pi_- u, && t \in \mathbf{R}^-, \\ x(t) &= \mathcal{A}^t x_0 + \mathcal{B}\tau^t\pi_- u, && t \in \mathbf{R}^+, \\ \pi_- y &= \pi_- v + \pi_-\mathcal{D}\pi_- u, \\ \pi_+ y &= \mathcal{C}x_0 + \pi_+\mathcal{D}\pi_- u. \end{aligned} \tag{5}$$

These equations contain two inputs, namely the previous $u \in L^2(\mathbf{R}^-;U)$ and the additional $v \in L^2(\mathbf{R}^-;Y)$. Both of these inputs vanish on $\mathbf{R}^+$, whereas both x and y are allowed to be nonzero on all of $\mathbf{R}$. Clearly, by interpreting $\mathcal{A}^t x_0$ to be zero for $t \in \mathbf{R}^-$, we can simplify the preceding equations to

$$\begin{aligned} x(t) &= \mathcal{A}^t x_0 + \mathcal{B}\tau^t\pi_- u, \quad t \in \mathbf{R}, \\ y &= \mathcal{C}x_0 + \pi_- v + \mathcal{D}\pi_- u \end{aligned} \tag{6}$$

(observe that by Definition 1(iii) the term $\mathcal{C}x_0$ vanishes on $\mathbf{R}^-$).

Fix some $\gamma > 0$. For each $x_0 \in H$, $v \in L^2(\mathbf{R}^-;Y)$ and $u \in L^2(\mathbf{R}^-;U)$ we define the *cost*

$$\begin{aligned} Q(x_0, v, u) &= \int_{-\infty}^{\infty} \|y(t)\|_Y^2\,dt - \gamma^2 \int_{-\infty}^{\infty} \|u(t)\|_U^2\,dt \\ &= \int_{-\infty}^{\infty} \|y(t)\|_Y^2\,dt - \gamma^2 \int_{-\infty}^{0} \|u(t)\|_U^2\,dt. \end{aligned} \tag{7}$$

The related minimax problem is to first minimize $Q(x_0,v,u)$ for each fixed $x_0 \in H$ and $u \in L^2(\mathbf{R}^-;U)$ with respect to $v \in L^2(\mathbf{R}^-;Y)$ to get

$$Q^{\min}(x_0,u) = \min_{v \in L^2(\mathbf{R}^-;Y)} Q(x_0,v,u), \tag{8}$$

and then to maximize $Q^{\min}(x_0,u)$ for each fixed $x_0 \in H$ with respect to $u \in L^2(\mathbf{R}^-;U)$ to get

$$Q^{\mathrm{opt}}(x_0) = \max_{u \in L^2(\mathbf{R}^-;U)} Q^{\min}(x_0,u). \tag{9}$$

The minimization step is actually trivial in the sense that the obvious solution is to take

$$v^{\min}(x_0,u) = -\pi_-\mathcal{D}\pi_- u;$$

this will make $\pi_- y^{\min}(x_0,u) = 0$ and

$$\begin{aligned} Q^{\min}(x_0,u) &= \int_0^{\infty} \|y(t)\|_Y^2\,dt - \gamma^2 \int_{-\infty}^{0} \|u(t)\|_U^2\,dt \\ &= \|\mathcal{C}x_0 + \pi_+\mathcal{D}\pi_- u\|^2_{L^2(\mathbf{R}^+;Y)} \\ &\quad - \gamma^2\|u\|^2_{L^2(\mathbf{R}^-;U)}. \end{aligned} \tag{10}$$

3 Equivalent Nehari Conditions

As the following theorem shows, there are a number of conditions that are equivalent to the Nehari condition $\|\pi_+\mathcal{D}\pi_-\| < \gamma$. (Below $\mathcal{L}(U;Y)$ stands for the set of all bounded linear operators $U \to Y$.)

Theorem 1 *Referring to the list of conditions given below, we have (IV) ⇒ (V) ⇒ (VI) ⇒ (I) ⇒ (II) ⇒ (III). If both U and Y are finite-dimensional and $\mathcal{D}$ is of the form*

$$(\mathcal{D}u)(t) = Du(t) + \int_{-\infty}^{t} E(t-s)u(s)\,ds, \tag{11}$$

where $D \in \mathcal{L}(U;Y)$ and $E \in L^1(\mathbf{R}^+;\mathcal{L}(U;Y))$ (i.e., the system has an L^1 impulse response), then (III) $\Rightarrow$ (IV). Thus, conditions (I)–(VI) are equivalent if both U and Y are finite-dimensional and $\mathcal{D}$ is of the form (11).

(I) *The Hankel norm $\|\pi_+\mathcal{D}\pi_-\|$ of $\mathcal{D}$ satisfies $\|\pi_+\mathcal{D}\pi_-\| < \gamma$;*

(II) *The spectral radius of the product of the controllability and observability Gramians is less than γ^2.*

(III) *For each $x_0 \in H$ and $u \in L^2(\mathbf{R}^-;U)$, the function $v \mapsto Q(x_0,v,u)$ is uniformly convex on $L^2(\mathbf{R}^-;Y)$, and, for each $x_0 \in H$, the function $u \mapsto Q^{\min}(x_0,u) = \min_{v\in L^2(\mathbf{R}^-;Y)} Q(x_0,v,u)$ is uniformly concave on $L^2(\mathbf{R}^-;U)$;*

(IV) *There exists an operator $\mathcal{X} = \left[\begin{smallmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22}\end{smallmatrix}\right] \in TIC(Y \times U)$ satisfying*

$$\begin{aligned}\begin{bmatrix} I & \mathcal{D} \\ 0 & I\end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I\end{bmatrix} \begin{bmatrix} I & \mathcal{D} \\ 0 & I\end{bmatrix} \\ = \begin{bmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22}\end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I\end{bmatrix} \begin{bmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22}\end{bmatrix}^*,\end{aligned}$$

such that $\mathcal{X}$ has an inverse in $TIC(Y \times U)$ and $\mathcal{X}_{11}$ has an inverse in $TIC(Y)$.

(V) *There exists a so called central anti-causal operator $\mathcal{U}^* \in TI(U;Y)$ such that the operator norm of $\mathcal{D} - \mathcal{U}^* \in TI(U;Y)$ satisfies $\|\mathcal{D} - \mathcal{U}^*\| < \gamma$;*

(VI) *There exists an anti-causal operator $\mathcal{U}^* \in TI(U;Y)$ such that the operator norm of $\mathcal{D} - \mathcal{U}^* \in TI(U;Y)$ satisfies $\|\mathcal{D} - \mathcal{U}^*\| < \gamma$;*

In the finite-dimensional case Theorem 1 is, of course, well known (except that we have not seen the minimax condition (III) used in this connection before); see, e.g., [1] and the comments in [8, Section 10.7]. In the case where both U and Y are finite-dimensional and $\mathcal{D}$ is of the form (11) our infinite-dimensional version is essentially the same as the one found in [2] (again with the exception of (III)). See also [3, 4, 5, 6, 7].

The operator $\mathcal{X}$ in (IV) is called *a J_1-co-spectral factor* of $\left[\begin{smallmatrix} I & \mathcal{D} \\ 0 & I\end{smallmatrix}\right]^* J_\gamma \left[\begin{smallmatrix} I & \mathcal{D} \\ 0 & I\end{smallmatrix}\right]$, where $J_1 = \left[\begin{smallmatrix} I & 0 \\ 0 & -I\end{smallmatrix}\right]$ and $J_\gamma = \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I\end{smallmatrix}\right]$. The extra invertibility condition on $\mathcal{X}_{11}$ is not part of the definition of a spectral factor; this condition is an essential addition to (IV). The operator $\mathcal{U}^*$ in (V) and (VI) is called a suboptimal *Nehari extension* of $\mathcal{D}$. If we denote the inverse of $\mathcal{X}$ in (IV) by $\mathcal{Y}$, then the central suboptimal Nehari extension $\mathcal{U}^*$ in (V) is given by $\mathcal{U}^* = (\mathcal{X}_{11}^*)^{-1}\mathcal{X}_{21}^* = -\mathcal{Y}_{21}^*(\mathcal{Y}_{22}^*)^{-1}$. The word "central" refers to the fact that this compensator corresponds to the zero value of the parameter $\mathcal{V}$ in the parameterization of all possible compensators $\mathcal{U}^*$ in (VI) given in Theorem 3 below.

4 A Realization of the Spectral Factor

In the finite-dimensional case (and and some infinite-dimensional ones), one usually connects conditions (I)–(VI) to the existence of the solutions of two Riccati equations. These Riccati equations play an important role in the construction of state space realizations of the spectral factor $\mathcal{X}$ and its inverse $\mathcal{Y}$. We are able to derive similar realizations in the infinite-dimensional case directly from the original system and from the spectral factor and its inverse, without involving the Riccati equations. (By a *realization* of $\mathcal{X}$ or $\mathcal{Y}$ we mean a well-posed linear system with input and output space equal to $Y \times U$ whose input-output map is $\mathcal{X}$ or $\mathcal{Y}$.)

Theorem 2 *The spectral factor $\mathcal{X}$ in (IV) has the realization*

$$\begin{bmatrix} \mathcal{A} & \begin{bmatrix}\mathcal{H}_1 & \mathcal{H}_2\end{bmatrix} \\ \begin{bmatrix}\mathcal{C} \\ \mathcal{K}_0\end{bmatrix} & \begin{bmatrix}\mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22}\end{bmatrix}\end{bmatrix},$$

and its inverse $\mathcal{Y}$ has the realization

$$\begin{bmatrix} \mathcal{A} & \begin{bmatrix}\mathcal{H}_0 & \mathcal{B}\end{bmatrix} \\ \begin{bmatrix}\mathcal{K}_1 \\ \mathcal{K}_2\end{bmatrix} & \begin{bmatrix}\mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \mathcal{Y}_{21} & \mathcal{Y}_{22}\end{bmatrix}\end{bmatrix},$$

where the input and output maps above are given by

$$\begin{aligned} \mathcal{H}_0 &= -\mathcal{B}\mathcal{D}^*\pi_-, & \mathcal{K}_0 &= \pi_+\mathcal{D}^*\mathcal{C}, \\ \mathcal{H}_1 &= \mathcal{B}\mathcal{Y}_{12}^*\pi_-, & \mathcal{K}_1 &= \pi_+\mathcal{X}_{11}^*\mathcal{C}, \\ \mathcal{H}_2 &= -\mathcal{B}\mathcal{Y}_{22}^*\pi_-, & \mathcal{K}_2 &= -\pi_+\mathcal{X}_{12}^*\mathcal{C}. \end{aligned}$$

These input and output maps have been constructed according to the recipes given in [13, Lemmas 4.10 and 4.12]. Observe, in particular, the occurrance of the original output map $\mathcal{C}$ in the first realization, and the original input map $\mathcal{B}$ in the second. We remark that it is possible to get two alternative realizations of $\mathcal{X}$ and $\mathcal{Y}$ by inverting the two systems given above; these inverted systems are similarity transformed versions of the realizations presented here.

5 Parameterization of all Suboptimal Anti-Causal Nehari Extensions

The appropriate parameterization of all anti-causal suboptimal Nehari extensions obeys the familiar formula:

Theorem 3 *Assuming (IV), we can parameterize all anti-causal suboptimal Nehari extensions (i.e., all anti-causal solutions to $\|\mathcal{D} - \mathcal{U}^*\| < \gamma$) as*

$$\begin{aligned} \mathcal{U}^* &= (\mathcal{X}_{11}^* - \mathcal{V}^*\mathcal{X}_{12}^*)^{-1}(\mathcal{X}_{21}^* - \mathcal{V}^*\mathcal{X}_{22}^*) \\ &= -(\mathcal{Y}_{11}^*\mathcal{V}^* + \mathcal{Y}_{21}^*)(\mathcal{Y}_{12}^*\mathcal{V}^* + \mathcal{Y}_{22}^*)^{-1}, \end{aligned} \tag{12}$$

where $\mathcal{V} \in TIC(Y;U)$ satisfies $\|\mathcal{V}\| < 1$ but is otherwise arbitrary. By taking $\|\mathcal{V}\| \le 1$ instead of $\|\mathcal{V}\| < 1$, we get a parameterization of all Nehari extensions satisfying $\|\mathcal{D} - \mathcal{U}^*\| \le \gamma$.

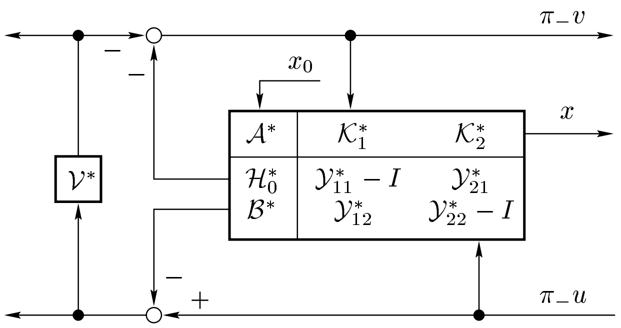

Figure 1: Parameterization of suboptimal Nehari extensions $\mathcal{U}^* : \pi_-u \mapsto \pi_-v$

It is possible to get an anti-causal state space realization of this parameterization that is very similar to the corresponding causal realization of the parameterization of the solution to the full information H^∞ problem. Compare the anti-causal Figure 1 to the corresponding causal Figure 13 in [15]. In this figure the suboptimal Nehari extension $\mathcal{U}^*$ is the input-output map from π_-u to π_-v.

6 The Riccati Equation

Our investigation of to what extent (I)–(VI) are equivalent to the solvability of the two standard Nehari Riccati equations, is still incomplete. However, it is easy to show that the optimal cost $Q^{\mathrm{opt}}(x_0)$ can be written in the form

$$Q^{\mathrm{opt}}(x_0) = \langle x_0, \Pi x_0 \rangle,$$

where the Riccati operator Π is given by

$$\Pi = \mathcal{C}^*(\gamma^2 I - \mathcal{C}\mathcal{B}\mathcal{B}^*\mathcal{C}^*)^{-1}\mathcal{C} = \mathcal{C}^*\mathcal{C}(\gamma^2 I - \mathcal{B}\mathcal{B}^*\mathcal{C}^*\mathcal{C})^{-1}$$

(it follows from (II) that $\gamma^2 I - \mathcal{B}\mathcal{B}^*\mathcal{C}^*\mathcal{C}$ is invertible). The same operator Π appears in the finite-dimensional theory as the solution to one of the two Riccati equations for this problem. See, for example, [4], where this operator is denoted by X. If we, instead, apply the same minimax approach with $\mathcal{D}$ replaced by its adjoint $\mathcal{D}^*$, then we get the operator (denoted by W in [4]) which is known to be the solution of the second of the two Nehari Riccati equations.

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