Passive* and Conservative Infinite-Dimensional Impedance and Scattering Systems (from a Personal Point of View)

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*I use the word **passive** in the same meaning as **dissipative**

Background

My present interest in impedance passive systems is roughly 1–2 years old. It begun in 2000 when I met Prof. **Ruth Curtain**, first at IWOTA2000 and then at the PDPS workshop in the summer of 2001, and she kept bombarding me with questions about **positive real functions and their realizations**.

At that time I was trying to learn what a **conservative system** is, working in what many people refer to as the **scattering setting**, inspired primarily by the talk by Prof. **Damir Z. Arov** given at MTNS98 in Padova (see Arov [1979, 1999]), and discussions with Prof. **George Weiss** and Dr. **Jarmo Malinen**.

An important additional source of inspiration was the 'thin air' paper by **Weiss and Tucsnak [2001]**.

Disclaimer

Today I shall present

My personal view on impedance passive systems.

It has been strongly colored by my background, and my knowledge of the history of this problem is very limited. This means that

my citations to the earlier literature are very incomplete.

I simply do not in all instances know **which results should be credited to whom**. Many of these results have been **discovered** and then **rediscovered**, maybe even several times.

Comments are welcome!

Details

For more details see

Weiss et al. [2001],
Staffans and Weiss [2002a,b],
Malinen et al. [2002], and
Staffans [2002a,b].

Some parts of the general theory are also found in my book manuscript

Staffans [2002c].

The exact references and most of the manuscripts are or will be available (in postscript form) at

http://www.abo.fi/˜staffans/

Abstract

U is a Hilbert space.

- By a $\mathcal{L}(U)$**-valued positive analytic function** on $\mathbb{C}^+$ (= the open right half-plane) we mean an analytic function which satisfies the condition $\widehat{\mathfrak{D}} + \widehat{\mathfrak{D}}^* \geq 0$. This function need not be **proper**, i.e., it need not be bounded on any right half-plane.
- We give a complete answer* to the question under what conditions such a function can be realized as the transfer function of an **(impedance) passive state space system** of a certain type (not necessarily well-posed). By this we mean a system which satisfies a certain **energy inequality**.
- The system is **energy preserving** if this energy inequality is an equality, and it is **conservative** if both the system and its dual are (impedance) energy preserving systems.

*The impulse response **cannot contain a pure derivative**.

Abstract (continues)

- A typical example of an impedance conservative system is a system of hyperbolic type with **collocated sensors and actuators**.
- The **diagonal transform** maps an **impedance** passive (energy preserving, conservative) system into a (well-posed) **scattering** passive (energy preserving, conservative) system.
- The **Cayley transform** maps a **continuous time** passive (energy preserving, conservative) system (impedance or scattering) into a passive (energy preserving, conservative) **discrete time system**.
- If we apply **negative output feedback** to an impedance passive system, then the resulting system is both **well-posed and energy stable**.
- Finally, we study **lossless scattering** systems, i.e., scattering conservative systems whose transfer functions are inner.

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1. CONTINUOUS TIME SYSTEM NODES

Introduction

Many infinite-dimensional linear time-independent continuous-time systems can be described by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

on a triple of Hilbert spaces, namely, the **input** space U, the **state** space X, and the **output** space Y. We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operator A is supposed to be the generator of a strongly continuous semigroup. The **generating operators** A, B and C are often **unbounded**, but D is bounded.

The System Node

By modifying this set of equations slightly we get the class of systems which will be used in this work. In the sequel, we think about the block matrix $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as **one single (unbounded) operator** from $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, and write (1) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0.$$

The operator S completely determines the system. Thus, we may identify the system with such an operator, which we call the **node** of the system.

We split S into it top and bottom rows: $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, where $A\&B\colon \mathcal{D}(S) \to X$ and $C\&D\colon \mathcal{D}(S) \to Y$. Thus

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0.$$

Notation: Rigged Hilbert Spaces

Let A be a closed (unbounded) densely defined operator on the Hilbert space X with a nonempty resolvent set. Denote

$$\mathcal{D}(A) =: X_1 \subset X \subset X_{-1} := [\mathcal{D}(A^*)]^*,$$

where we identify X with its dual. The operator A has a unique extension to a bounded linear operator $X \to X_{-1}$, which we denote by $A_{|X}$ (since its domain is X).

Definition 1 *We call S a* **system node** *on the three Hilbert spaces (U, X, Y) if it satisfies condition (S) below:*

(S) $S := \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \supset \mathcal{D}(S) \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ *is a* **closed** *linear operator. Here $A\&B$ is the restriction to $\mathcal{D}(S)$ of $\begin{bmatrix} A_{|X} & B \end{bmatrix}$, where A is the* **generator of a C_0 semigroup** *on X. The operator B is an arbitrary operator in $\mathcal{L}(U; X_{-1})$, and $C\&D$ is an arbitrary linear operator from $\mathcal{D}(S)$ to Y. In addition, we require that*

$$\mathcal{D}(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \,\middle|\, A_{|X}x + Bu \in X \right\}.$$

Note that $\mathcal{D}(S)$ depends only on A and B, and not on $C\&D$.

The operator $A\&B\colon \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \supset \mathcal{D}(S) = \mathcal{D}(A\&B) \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ is closed, and $C\&D\colon \mathcal{D}(A\&B) \to Y$ is continuous (with respect to the graph norm of $A\&B$).

This definition goes back to **Smuljan [1986]** (who was primarily concerned with the well-posed case, which will be described later).

Terminology

- A is the **main operator** (the **semigroup generator**),
- B is the **control operator**,
- $C\&D$ is the **combined observation/feedthrough operator**,
- The operator

  $$Cx := C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad x \in X_1,$$

  is the **observation operator**.
- The **transfer function** is*

  $$\widehat{\mathfrak{D}}(s) := C\&D \begin{bmatrix} (s - A_{|X})^{-1}B \\ 1 \end{bmatrix}, \qquad s \in \rho(A).$$

  It is defined and analytic on $\rho(A)$ (it follows from Definition 1 that the operator $\left[\begin{smallmatrix} (s-A_{|X})^{-1}B \\ 1 \end{smallmatrix}\right]$ maps U into $\mathcal{D}(S)$).

The system node **need not have a feedthrough operator**!

*We denote the identity operator (on any Hilbert space) by 1.

System Node, Alternative Definition

It is possible to alternatively define a system node by specifying the **main operator** A, the **control operator** B, the **observation operator** C, and the **transfer function** $\widehat{\mathfrak{D}}$ evaluated at some point $\alpha \in \rho(A)$.

- A can be an arbitrary generator of a C_0 semigroup on X,
- B can be an arbitrary operator in $\mathcal{L}(U;X_{-1})$,
- C can be an arbitrary operator in $\mathcal{L}(X_1;Y)$,
- The value of the **transfer function** at a given point $\alpha \in \rho(A)$ can be an arbitrary operator $D \in \mathcal{L}(U;Y)$.

This will lead to a system node of the type described in Definition 1. The transfer function will be given by

$$\widehat{\mathfrak{D}}(s) = D + (\alpha - s)C(s-A)^{-1}(\alpha - A_{|X})^{-1}B, \quad s \in \rho(A).$$

This is the version used by **Salamon [1987]** (who was primarily concerned with the well-posed case, which will be described later).

State Trajectory and Output Function

Every system node induces a 'dynamical system' of a certain type:

Lemma 1 *Let S be a system node on (U,X,Y). Then, for each $x_0 \in X$ and $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+;U)$ with $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \mathcal{D}(S)$, the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0, \qquad (2)$$

has a unique solution (x,y) satisfying $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S)$ for all $t \geq 0$, $x \in C^1(\mathbb{R}^+;X)$, and $y \in C(\mathbb{R}^+;Y)$.

We call x the **state trajectory** and y the **output function** of S with initial time zero, **initial state** x_0, and **input function** u.

This result has been known for ages. **(Who did it first?)**

Taking Laplace transforms in (2) we find that

$$\begin{aligned} \hat{x}(s) &= (s-A)^{-1}x_0 + (s-A_{|X})^{-1}B\hat{u}(s), \\ \hat{y}(s) &= C(s-A)^{-1}x_0 + \widehat{\mathfrak{D}}(s)\hat{u}(s), \end{aligned}$$

for $\Re s$ large enough (as in the finite-dimensional case).

Controllability and Observability

- The **reachable** subspace of S is the closure in X of the set of all possible values of $x(t)$ in Lemma 1 if we take $x_0 = 0$ (and let u and t vary). Its orthogonal complement is the **unreachable subspace**.
- The **unobservable** subspace of S is the closure of the set of all $x_0 \in X_1$ for which the output y in Lemma 1 with initial state x_0 and zero input function u is identically zero. Its orthogonal complement is the **observable subspace**.
- S is **(approximately) controllable** if the reachable subspace is all of X and **(approximately) observable** if the observable subspace is all of X.
- S is **simple** if the **intersection** of the unreachable and unobservable subspaces is $\{0\}$.
- S is **minimal** if it is both controllable and observable (the **union** of the unreachable and unobservable subspaces is $\{0\}$).

Well-Posed System Nodes

Definition 2 *A system node S is* **well-posed** *if the following additional condition holds:*

(WP) *For some $t > 0$ there is a finite constant $K(t)$ such that the solution (x,y) in Lemma 1 satisfies*

$$|x(t)|^2 + \|y\|^2_{L^2(0,t)} \leq K(t)\left(|x_0|^2 + \|u\|^2_{L^2(0,t)}\right).$$

It is **energy stable** *if there is some $K < \infty$ so that, for all $t \in \mathbb{R}^+$,*

$$|x(t)|^2 + \|y\|^2_{L^2(0,t)} \leq K\left(|x_0|^2 + \|u\|^2_{L^2(0,t)}\right).$$

It is not difficult to show that if (WP) holds for **one** $t > 0$, then it holds for **all** $t \geq 0$.

2. SCATTERING PASSIVE AND CONSERVATIVE SYSTEMS.

The following definition is a special case of the definitions in the two classical papers Willems [1972a,b] (although we use a slightly different terminology; our **passive** is the **same as Willems' dissipative**).*

*Another difference is that we have replaced Willems' more general **storage function** $S(x)$ by the quadratic function $|x|^2$. Our setting becomes the same as the setting used by Willems in the second part Willems [1972b] if we simply take the norm in the state space to be $|x|^2 = \sqrt{S(x)}$ (this is possible whenever the storage function is quadratic and strictly positive).

J-Passive and Energy Preserving System Nodes

Definition 3 *Let J be a bounded selfadjoint operator on $\left[\begin{smallmatrix} Y \\ U \end{smallmatrix}\right]$. A system node S on (U,X,Y) is* J**-passive** *if the following condition holds:*

(JP) *For all $t>0$, the solution (x,y) in Lemma 1 satisfies*

$$|x(t)|^2 - |x_0|^2 \le \int_0^t \left\langle \begin{bmatrix} y(s) \\ u(s) \end{bmatrix}, J \begin{bmatrix} y(s) \\ u(s) \end{bmatrix} \right\rangle ds.$$

It is J**-energy preserving** *if the above inequality holds in the form of an equality:*

(JE) *For all $t>0$, the solution (x,y) in Lemma 1 satisfies*

$$|x(t)|^2 - |x_0|^2 = \int_0^t \left\langle \begin{bmatrix} y(s) \\ u(s) \end{bmatrix}, J \begin{bmatrix} y(s) \\ u(s) \end{bmatrix} \right\rangle ds.$$

Different choices of J give different passivity notions.

- $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is known as **scattering**,
- $U = Y = \left[\begin{smallmatrix} V \\ V \end{smallmatrix}\right]$ and $J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is known as **impedance** (**admittance**, **immittance**, **resistance**, **conductance**),
- $U = Y = \left[\begin{smallmatrix} V \\ W \end{smallmatrix}\right]$ and $J = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$ is known as **transmission** (**chain scattering**).

Today I focus on the **scattering** ($J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$) and **impedance** ($J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$) settings.

Definition 4 *A system node S is* **scattering passive*** *if the following condition holds:*

(SP) *For all $t>0$, the solution (x,y) in Lemma 1 satisfies*

$$|x(t)|^2 - |x_0|^2 \le \|u\|^2_{L^2(0,t)} - \|y\|^2_{L^2(0,t)}.$$

It is **scattering energy preserving** *if the above inequality holds in the form of an equality:*

(SE) *For all $t>0$, the solution (x,y) in Lemma 1 satisfies*

$$|x(t)|^2 - |x_0|^2 = \|u\|^2_{L^2(0,t)} - \|y\|^2_{L^2(0,t)}.$$

Finally, S is **scattering conservative** *if both S and the dual system node† S^* are (scattering) energy preserving.*

Thus, **every scattering passive system is well-posed**: the passivity inequality **(SP)** implies the well-posedness inequality **(WP)**.

*In Malinen et al. [2002], Staffans and Weiss [2002a,b], Weiss et al. [2001], Weiss and Tucsnak [2001], etc., these systems are called **dissipative**.

†If S is a system node on (U,X,Y), then its adjoint S^* is a system node on (Y,X,U).

Theorem 1 *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U,X,Y). Then the following conditions are equivalent:*

(i) *S is scattering passive.*

(ii) *For all $t > 0$, the solution (x,y) in Lemma 1 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 \leq |u(t)|_U^2 - |y(t)|_Y^2.$$

(iii) *For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$2\Re\left\langle A\&B\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], x_0\right\rangle_X \leq |u_0|_U^2 - \left|C\&D\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\right|_Y^2.$$

(iv) *For some $\alpha \in \rho(A) \cap \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$), the operator*

$$\begin{bmatrix} \mathbf{A}(\alpha) & \mathbf{B}(\alpha) \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} = \begin{bmatrix} (\overline{\alpha}+A)(\alpha-A)^{-1} & \sqrt{2\Re\alpha}(\alpha-A)^{-1}B \\ \sqrt{2\Re\alpha}C(\alpha-A)^{-1} & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \tag{3}$$

is a contraction. (Here $\mathbb{C}^+$ is the open right half-plane.)

This is (a part of) [Staffans and Weiss, 2002a, Theorem 7.4], and it is also found in **Arov and Nudelman [1996]**.

Scattering conservative system nodes

The corresponding **energy preserving** version also holds: simply replace "$\leq$" by "$=$".

By applying the energy preserving version of Theorem 1 both to the original system node S and to the dual system node S^* we get a set of systems which characterize **scattering conservative system nodes**.

Some equivalent but **simpler conditions** are given in Malinen et al. [2002].

3. IMPEDANCE PASSIVE AND CONSERVATIVE SYSTEMS.

Definition 5 *A system node S on (U,X,U) (note that $Y = U$) is* **impedance passive** *if the following condition holds:*

(IP) *For all $t > 0$, the solution (x,y) in Lemma 1 satisfies*

$$|x(t)|_X^2 - |x_0|_X^2 \leq 2\int_0^t \Re\langle y(t), u(t)\rangle_U dt.$$

It is **impedance energy preserving** *if the above inequality holds in the form of an equality:*

(IE) *For all $t > 0$, the solution (x,y) in Lemma 1 satisfies*

$$|x(t)|_X^2 - |x_0|_X^2 = 2\int_0^t \Re\langle y(t), u(t)\rangle_U dt.$$

Finally, S is **impedance conservative** *if both S and the dual system node S^* are impedance energy preserving.*

Note that in this case **well-posedness is neither guaranteed, nor relevant**.

Some Basic Properties

Lemma 2 *A system node S is impedance passive if and only if the* **dual system node** *S^* is impedance passive.*

This is proved in [Staffans, 2002a, Corollary 4.5].

Theorem 2 *An impedance passive system node is* **well-posed if and only if** *its* **transfer function $\widehat{\mathfrak{D}}$ is bounded** *on some (or equivalently, on every)* **vertical line in** *$\mathbb{C}^+$. When this is the case, the growth bound of the system is zero, and, in particular, $\widehat{\mathfrak{D}}$ is bounded on every right half-plane $\mathbb{C}_\varepsilon^+ = \{s \in \mathbb{C} \mid \Re s > \varepsilon\}$ with $\varepsilon > 0$.*

This is [Staffans, 2002a, Theorem 5.1]. It can be used to show that many systems with **collocated actuators and sensors** are well-posed.

Theorem 3 *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U,X,U). Then the following conditions are equivalent:*

(i) *S is impedance passive.*

(ii) *For all $t > 0$, the solution (x,y) in Lemma 1 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 \le 2\Re\langle y(t), u(t)\rangle_U .$$

(iii) *The system node $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is a* **dissipative** *operator in $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$, i.e., for all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$\Re\left\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \le 0. \tag{4}$$

(iv) *For some $\alpha \in \rho(A) \cap \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$), the operator $\alpha - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is invertible, and*

$$\begin{bmatrix} \mathbf{A}^\times(\alpha) & \mathbf{B}^\times(\alpha) \\ \mathbf{C}^\times(\alpha) & \mathbf{D}^\times(\alpha) \end{bmatrix} = \left(\overline{\alpha} + \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}\right)\left(\alpha - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}\right)^{-1} \tag{5}$$

is a **contraction**.

This is a part of [Staffans, 2002a, Theorem 4.2 and Corollary 4.4]. The corresponding **energy preserving** version also holds: simply replace "$\le$" by "$=$".

Theorem 4 *Let S be system node S on (U,X,U). Then the following conditions are equivalent:*

(i) *S is* **impedance conservative**.

(ii) *For all $t > 0$, the solution (x,y) in Lemma 1 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 = 2\Re\langle y(t), u(t)\rangle_U , \tag{6}$$

and the same identity is true for the adjoint system.

(iii) *The system node $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is* **skew-adjoint**, *i.e.,*

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}^* = -\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}. \tag{7}$$

(iv) *$A^* = -A$, $B^* = C$, and $\widehat{\mathfrak{D}}(\alpha) + \widehat{\mathfrak{D}}(-\overline{\alpha})^* = 0$ for some (or equivalently, for all) $\alpha \in \rho(A)$ (in particular, this identity is true for all α with $\Re\alpha \neq 0$).*

(v) *For some $\alpha \in \rho(A) \cap \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$), the operator $\alpha - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is invertible, and the operator $\left[\begin{smallmatrix} \mathbf{A}^\times(\alpha) & \mathbf{B}^\times(\alpha) \\ \mathbf{C}^\times(\alpha) & \mathbf{D}^\times(\alpha) \end{smallmatrix}\right]$ defined in* (5) *is* **unitary**.

This is [Staffans, 2002a, Theorem 4.7].

4. DISCRETE TIME SYSTEMS.

There is a close connection between the passive continuous time systems that we have considered so far and the corresponding **discrete time systems**. We now have

an **input sequence** $u = \{u_k\}_{k=0}^\infty$,
a **state trajectory** $x = \{x_k\}_{k=0}^\infty$,
an **output sequence** $y = \{y_k\}_{k=0}^\infty$.

The dynamics is described by

$$\begin{bmatrix} x_{k+1} \\ y_k \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}\begin{bmatrix} x_k \\ u_k \end{bmatrix}, \qquad k = 0,1,2,\ldots, \qquad x_0 = \text{given}, \tag{8}$$

where $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right] \in \mathcal{L}(\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right])$.

$\mathbf{A}$ is the **main operator**,
$\mathbf{B}$ is the **control operator**,
$\mathbf{C}$ is the **observation operator**,
$\mathbf{D}$ is the **feedthrough operator**, and

$$\widehat{\mathbf{D}}(z) = \mathbf{C}(z - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}, \qquad z \in \rho(\mathbf{A}).$$

is the **transfer function**. Note that $\mathbf{D} = \widehat{\mathbf{D}}(\infty)$.

Observability, controllability, simplicity, and minimality of a discrete time system is defined in exactly the same way as in continuous time, with continuous time trajectories replaced by discrete time trajectories.

Scattering Passive Systems

The system $\mathbf{\Sigma}$ is **scattering passive** if it is true for all $x_0 \in X$, all input sequences $u_k \in U$, and all $m = 0,1,2,\dots$ that

$$|x_{m+1}|_X^2 - |x_0|_X^2 \le \sum_{k=0}^{m} |u_k|_U^2 - \sum_{k=0}^{m} |y_k|_Y^2.$$

True for all $m = 0,1,2,\dots \Leftrightarrow$ **true for** $m = 0$. Thus

$\mathbf{\Sigma}$ is **scattering passive** $\Leftrightarrow$ $\left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ **is a contraction** from $\left[\begin{smallmatrix}X\\ U\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}X\\ Y\end{smallmatrix}\right]$

$\mathbf{\Sigma}$ is **scattering energy preserving** if we have equality, i.e., $\left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ **is isometric**, and

$\mathbf{\Sigma}$ is **scattering conservative** if both the original system and the dual system are energy preserving, i.e., $\left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ **is unitary**.

The main operator $\mathbf{A}$ of each discrete time scattering passive system is a contraction, so $\widehat{\mathbf{D}}$ is defined and analytic (at least) on $\mathbb{D}^+ = \{z \in \mathbf{C} \mid |z| > 1\} \cup \{\infty\}$. It is well-known that $\widehat{\mathbf{D}}(z)$ is a **contractive analytic function on** $\mathbb{D}^+$ (a **Schur function**), i.e., $\widehat{\mathbf{D}}(z)$ is a contraction for every $z \in \mathbb{D}^+$.

Impedance Passive Systems

The system $\mathbf{\Sigma}$ is **impedance passive** if it is true for all $x_0 \in X$, all input sequences $u_k \in U$, and all $m = 0,1,2,\dots$ that

$$|x_{m+1}|_X^2 - |x_0|_X^2 \le \sum_{k=0}^{m} 2\Re\langle u_k, y_k\rangle_U.$$

True for all $m = 0,1,2,\dots \Leftrightarrow$ **true for** $m = 0$. Thus

$\mathbf{\Sigma}$ is **impedance passive** $\Leftrightarrow$

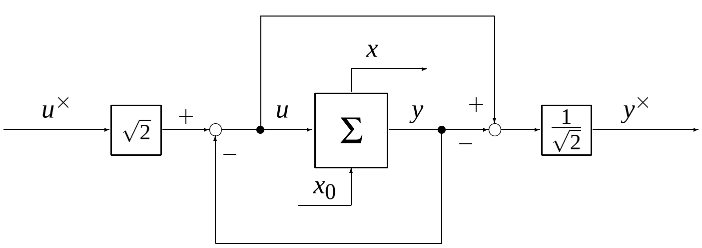
$$\begin{bmatrix}\mathbf{A}^*\\ \mathbf{B}^*\end{bmatrix}\begin{bmatrix}\mathbf{A} & \mathbf{B}\end{bmatrix} \le \begin{bmatrix}1 & \mathbf{C}^*\\ \mathbf{C} & \mathbf{D}+\mathbf{D}^*\end{bmatrix}.$$

$\mathbf{\Sigma}$ is **impedance energy preserving** if we have equality.

$\mathbf{\Sigma}$ is **impedance conservative** if both the original system and the dual system are energy preserving.

The main operator $\mathbf{A}$ of each discrete time impedance passive system is a contraction, so $\widehat{\mathbf{D}}$ is defined and analytic (at least) on $\mathbb{D}^+ = \{z \in \mathbf{C} \mid |z| > 1\} \cup \{\infty\}$. It is well-known that $\widehat{\mathbf{D}}(z)$ is a **positive analytic function on** $\mathbb{D}^+$.

5. THE DIAGONAL TRANSFORM



Suppose that $1+\mathbf{D}$ is invertible. Then we can replace

$$u_k \to u_k^\times = \tfrac{1}{\sqrt{2}}(u_k + y_k)$$
$$y_k \to y_k^\times = \tfrac{1}{\sqrt{2}}(u_k - y_k)$$

to get the new discrete time system

$$\begin{bmatrix}x_{k+1}\\ y_k\end{bmatrix} = \begin{bmatrix}\mathbf{A}^\times & \mathbf{B}^\times\\ \mathbf{C}^\times & \mathbf{D}^\times\end{bmatrix}\begin{bmatrix}x_k\\ u_k\end{bmatrix}, \qquad k = 0,1,2,\dots, \tag{9}$$
$$x_0 = \text{given}.$$

A straightforward computation shows that

$$\mathbf{\Sigma}^\times = \begin{bmatrix}\mathbf{A}^\times & \mathbf{B}^\times\\ \mathbf{C}^\times & \mathbf{D}^\times\end{bmatrix} = \begin{bmatrix}\mathbf{A} - \mathbf{B}(1+\mathbf{D})^{-1}\mathbf{C} & \sqrt{2}\mathbf{B}(1+\mathbf{D})^{-1}\\ -\sqrt{2}(1+\mathbf{D})^{-1}\mathbf{C} & (1-\mathbf{D})(1+\mathbf{D})^{-1}\end{bmatrix}.$$

The transfer function of $\mathbf{\Sigma}^\times$ is $\widehat{\mathbf{D}}^\times(z) = (1-\widehat{\mathbf{D}}(z))(1+\widehat{\mathbf{D}}(z))^{-1}$.

Following Livšic [1973], we call this the **(discrete time) diagonal transform**.

Applying the same transform once more we recover the original system. Thus, the **inverse diagonal transform** $=$ **direct diagonal transform**.

Comments

- The diagonal transform is well-defined if and only if $1+\mathbf{D}$ is invertible.
- The inverse diagonal transform is well-defined if and only if and only if $1+\mathbf{D}^\times$ is invertible.
- The state trajectory $\{x_k^\times\}_{k=0}^\infty$ of (9) coincides with the state trajectory $\{x_k\}_{k=0}^\infty$ of (8) if $x_0^\times = x_0$ and
$$u_k^\times = \frac{1}{\sqrt{2}}(u_k + y_k), \quad k \ge 0.$$
- The corresponding outputs satisfy
$$y_k^\times = \frac{1}{\sqrt{2}}(u_k - y_k), \quad k \ge 0.$$
- These relationships have specifically been chosen in such a way that
$$|u_k^\times|_U^2 - |u_k^\times|_U^2 = 2\Re\langle y_k, u_k\rangle.$$

This immediately implies the following lemma.

Lemma 3 (i) *A discrete time system* $\mathbf{\Sigma} = \left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ *is* **impedance** *passive (energy preserving, conservative) if and only if the* **diagonal transform** *is well-defined and the diagonally transformed system* $\mathbf{\Sigma}^\times = \left[\begin{smallmatrix}\mathbf{A}^\times & \mathbf{B}^\times\\ \mathbf{C}^\times & \mathbf{D}^\times\end{smallmatrix}\right]$ *is* **scattering** *passive (energy preserving, conservative).*

(ii) *A discrete time scattering passive system* $\mathbf{\Sigma}^\times = \left[\begin{smallmatrix}\mathbf{A}^\times & \mathbf{B}^\times\\ \mathbf{C}^\times & \mathbf{D}^\times\end{smallmatrix}\right]$ **is the diagonal transform** *of an impedance passive system* $\mathbf{\Sigma} = \left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ **if and only if $1+\mathbf{D}^\times$ is invertible**.

(iii) *The two systems have* **identical controllability properties***: if one of the two systems* $\mathbf{\Sigma}$ *and* $\mathbf{\Sigma}^\times$ *is controllable (observable, simple, minimal) then so is the other, and their reachable, unreachable, observable, and unobservable subspaces coincide.*

Thus, a discrete time **scattering passive system** can be regarded as a slightly **more general object** than a discrete time **impedance passive system**, since the diagonal transform maps the latter class into a subclass of the former (and it may even have $Y \neq U$ in the scattering case).

Realization Theory

Theorem 5 *Every contractive analytic function on* $\mathbb{D}^+$ *has a simple discrete time scattering conservative realization, which is unique modulo a unitary similarity transform in the state space.*

This is a slightly reformulated version of the results presented in **[Sz.-Nagy and Foiaş, 1970, Section VI.3, pp. 248–259]**.

Theorem 6 *Every positive analytic function on* $\mathbb{D}^+$ *has a simple discrete time impedance conservative realization, which is unique modulo a unitary similarity transform in the state space.*

Proof: Apply the inverse diagonal transform to Theorem 5.

(Who did this first?)

6. THE CAYLEY TRANSFORM

The **diagonal transform** is active **on the outside**, i.e., in the input/output space $\left[\begin{smallmatrix}U\\ U\end{smallmatrix}\right]$. It represents a 45° rotation plus a reflection in the **right hand side** of the scattering energy balance equation

$$|x_{k+1}|^2 - |x_k|^2 = |u_k|^2 - |y_k|^2, \quad k \ge 0.$$

The **Cayley transform** does almost the same thing **on the inside**, i.e., in the state space: It formally represents a 45° rotation in the **left hand side** of the same identity.

However, it has a different interpretation:

it maps **continuous time into discrete time**!

Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a **continuous time** system node, and let $1 \in \rho(A)$. The **Cayley transform** of the node S is the **discrete time** system

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} (1+A)(1-A)^{-1} & \sqrt{2}(1-A_{|X})^{-1}B \\ \sqrt{2}C(1-A)^{-1} & \widehat{\mathfrak{D}}(1) \end{bmatrix}. \tag{10}$$

This is the **same formula** (3) which we saw in Theorems 1 and 3 (with $\alpha = 1$). Thus, **these theorems say something about the Cayley transformed systems**.

The **inverse Cayley transform** is given by

$$\begin{aligned} A &= (\mathbf{A}-1)(\mathbf{A}+1)^{-1}, & B &= \frac{1}{\sqrt{2}}(1-A_{|X})\mathbf{B}, \\ C &= \sqrt{2}\,\mathbf{C}(\mathbf{A}+1)^{-1}, & \widehat{\mathfrak{D}}(s) &= \widehat{\mathbf{D}}\Big(\frac{1+s}{1-s}\Big). \end{aligned} \tag{11}$$

For the inverse Cayley transform to be well-defined the operator $\mathbf{A}+1$ must be one-to-one, so

$\mathbf{A}$ cannot have -1 as an eigenvalue.

In the passive case we can say more:

Lemma 4 (i) *A* **continuous time** *system node* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is scattering or impedance passive (energy preserving, conservative) if and only if the* **Cayley transform** *of this system is well-defined and the Cayley transformed system* $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ *is* **discrete time** *scattering or impedance passive (energy preserving, conservative).*

(ii) *A discrete time scattering or impedance passive system* $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ **is the Cayley transform** *of a continuous time scattering or impedance passive node* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *if and only if* -1 **is not an eigenvalue of $\mathbf{A}$**.

(iii) *The two systems have* **identical controllability properties***: if one of S and $\mathbf{\Sigma}$ is controllable (observable, simple, minimal) then so is the other, and their reachable, unreachable, observable, and unobservable subspaces coincide.*

This lemma is the main technical tool used in Arov and Nudelman [1996] to transfer a number of discrete time theorems to continuous time (in the **scattering** setting).

Theorem 7 *Every contractive analytic function on $\mathbb{C}^+$ can be realized as the transfer function of a simple continuous time scattering conservative system node, which is unique modulo a unitary similarity transform in the state space.*

Proof: Apply the inverse Cayley transform to Theorem 5.

This is **[Arov and Nudelman, 1996, Theorem 6.4]**.

We now arrive at the **main result** of this section.

Theorem 8 *A necessary and sufficient condition for a $\mathcal{L}(U)$-valued positive analytic function $\widehat{\mathfrak{D}}$ on $\mathbb{C}^+$ to have a simple impedance conservative realization is that*

$$\lim_{s\to+\infty} s^{-1}\widehat{\mathfrak{D}}(s)u = 0 \tag{12}$$

for all $u \in U$. This realization is unique modulo a unitary similarity transform in the state space.

Thus, the only positive transfer functions that cannot be realized in this way are those that contain a pure derivative action.

To what extent is this known?*

*One version of this theorem with (12) replaced by $\widehat{\mathfrak{D}}(s) = O(s^{-1})$ as $s \to +\infty$ is found in Arov [1979]. There both B and C are required to be **bounded**.

Proof.

- Apply the formal Cayley transform to $\widehat{\mathfrak{D}}$ only, to get the discrete time positive transfer function $\widehat{\mathbf{D}}(z) = \widehat{\mathfrak{D}}\left(\frac{z-1}{z+1}\right)$.
- Use Theorem 6 to get a simple impedance conservative realization of $\widehat{\mathbf{D}}$.
- Finally, apply the inverse Cayley transform to get a simple impedance conservative system node S whose transfer function is $\widehat{\mathfrak{D}}$. (This is where (12) comes in.)

As a byproduct of this theorem we get the **Herglotz-Nevanlinna integral representation formula** for a positive analytic function on $\mathbb{C}^+$, found, e.g., in Zemanian [1972].

Controllable Energy Preserving Realizations

From the conservative realizations it is easy to construct **energy preserving** realizations.

Corollary 1 *Theorems 5, 6, 7, and 8 remain true in the controllable energy preserving setting: every discrete or continuous time impedance or scattering* **passive transfer function** *has a* **controllable energy preserving realization***, which is* **unique** *modulo a unitary similarity transform in the state space (in the continuous time impedance setting we still have the extra necessary and sufficient additional condition* (12)*).*

Proof: Restrict the systems in Theorems 5, 6, 7, and 8 to the reachable subspace.

Recall: for **transfer functions**
impedance passive = positive, and
scattering passive = contractive.

Minimal Passive Realizations

From the energy preserving realizations it is easy to construct **passive** realizations.

Corollary 2 *Every discrete or continuous time impedance or scattering passive function has a* **minimal passive realization** *(in the continuous time impedance setting we still have the extra necessary and sufficient additional condition* (12)*).*

Proof: Project the systems in Corollary 1 to the observable subspace.

Without any further conditions these realizations are **not unique**. For example we could first have projected onto the observable subspace, and then restricted to the controllable subspaces to get another passive realizations.*

The realizations that we get as explained in the proof of Corollary 2 are called **optimal** by **Arov and Nudelman [1996]**. To get the same realizations in **Willems [1972a,b]** setting we use the norm given by the **available storage**.

*This way we get realizations which Arov and Nudelman [1996] call $*$-**optimal**. To get the same realizations in Willems [1972a,b] setting we use the norm given by the **required supply**.

7. A FEEDBACK INTERPRETATION

Corollary 3 *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be an impedance passive system node on* (U,X,U)*. Then* -1 *is an admissible feedback operator for* S*, and the closed loop system corresponding to this feedback operator is (well-posed and) energy stable (in the sense of Definition 2).*

This is a classic result in a new setting. It may be regarded as a generalization of the recent 'thin air' result by **Weiss and Tucsnak [2001]**. (That paper was an important source of inspiration for me.)

8. LOSSLESS SYSTEMS

In a energy preserving system **no energy is lost**, but it may be first transferred from the input to the state, and then **trapped in the state space** forever, so that it can no longer be retrieved from the outside.

Lossless = no trapped energy.

For simplicity, let us here only look at the **continuous time scattering** setting (the discrete time is very similar; the impedance setting is slightly more complicated).

Definition 6 *A system node* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *on* (U,X,Y) *is* **semi-lossless** *if the solution* (x,y) *in Lemma 1 satisfies*

$$\int_0^\infty |y(s)|_Y^2\, ds = \int_0^\infty |u(s)|_U^2\, ds$$

whenever $x_0 = 0$ *and* $u \in L^2(\mathbb{R}^+;Y)$*. It is* **lossless** *if both* S *and the dual node* S^* *are semi-lossless.*

Thus, semi-losslessness is the **input/output version** of **energy preservation**, and losslessness is the input/output version of **conservativity**.

As is well known, semi-losslessness can be interpreted as a property of the transfer function:

Proposition 1 *A system node* S *is* **semi-lossless if and only if its transfer function** $\widehat{\mathfrak{D}}$ **is left-inner** *in the following sense:* $\widehat{\mathfrak{D}}$ *has an extension to a contractive analytic function in* $\mathbb{C}^+$*, the restriction of* $\widehat{\mathfrak{D}}$ *to every separable subspace of* U *has a strong limit from the right a.e. at the imaginary axis, and this limit is isometric a.e.*

This proposition follows from Fourès and Segal [1955] and [Sz.-Nagy and Foiaş, 1970, Proposition 2.2, p. 190].

It turns out the the losslessness property puts some severe restrictions on the system node S:

Theorem 9 *A* **controllable semi-lossless scattering passive** *system node* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *on* (U,X,Y) *is necessarily* **scattering energy preserving and observable***. Furthermore, in this case the system node* S *is* **uniquely** *determined by its transfer function* $\widehat{\mathfrak{D}}$ *within the class of all controllable scattering passive realizations of* $\widehat{\mathfrak{D}}$*, modulo a unitary similarity transform in the state space.*

This is proved in Staffans [2002b].

Theorem 10 *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a scattering energy preserving system node (Y,X,U). Then the following conditions are equivalent:*

(i) *the system semigroup of S is* **strongly stable***, i.e., the state $x(t)$ in Lemma 1 tends to zero as $t \to \infty$ whenever $u = 0$;*

(ii) *the* **observability gramian of S is the identity operator***, i.e., the output y in Lemma 1 with zero input function u and initial state $x_0 \in X_1$ satisfies $\int_0^\infty |y(t)|_Y^2 = |x_0|_Y^2$;*

(iii) *S is* **exactly observable in infinite time***, i.e., the output y in Lemma 1 with zero input function u and initial state $x_0 \in X_1$ satisfies $\int_0^\infty |y(t)|_Y^2 \geq \varepsilon |x_0|_Y^2$ for some $\varepsilon > 0$;*

If these conditions hold, then

(iv) S is **semi-lossless***.*

If S is **controllable***, then (iv) is equivalent to (i)–(iii).*

Also this result is proved in Staffans [2002b]. The discrete time version of this result (without condition (iii)) is essentially contained in [Sz.-Nagy and Foiaş, 1970, Theorem 2.3, p. 248].

By applying this result both to the original node S and to the dual node S^* we get a characterization of the (essentially unique) **simple scattering conservative realizations of lossless scattering functions** (inner from both sides). It is both **exactly controllable and exactly observable in infinite time**. This is basically the class of systems studied in Lax and Phillips [1967].

SUMMARY

- By a $\mathcal{L}(U)$**-valued positive analytic function** on the open right half-plane we mean an analytic function which satisfies the condition $\widehat{\mathfrak{D}} + \widehat{\mathfrak{D}}^* \geq 0$. This function need not be **proper**, i.e., it need not be bounded on any right half-plane.
- We have given a complete answer to the question under what conditions such a function can be realized as the transfer function of an **impedance passive state space system**: The impulse response **must not contain a pure derivative**. Passivity means that a certain energy inequality holds.
- The system is **energy preserving** if this energy inequality is an equality, and it is **conservative** if both the system and its dual are (impedance) energy preserving systems.

SUMMARY (continues)

- A typical example of an impedance conservative system is a system of hyperbolic type with **collocated sensors and actuators**.
- The **diagonal transform** maps an **impedance** passive (energy preserving, conservative) system into a (well-posed) **scattering** passive (energy preserving, conservative) system.
- If we apply **negative output feedback** to an impedance passive system, then the resulting system is both **well-posed and energy stable**.
- Finally, we have studied **lossless scattering** systems, i.e., scattering conservative systems whose transfer functions are inner.

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