PASSIVE AND CONSERVATIVE INFINITE-DIMENSIONAL IMPEDANCE AND SCATTERING SYSTEMS (FROM A PERSONAL POINT OF VIEW)

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Abstract. Let U be a Hilbert space. By a $\mathcal{L}(U)$*-valued positive analytic function* on the open right half-plane we mean an analytic function which satisfies the condition $\widehat{\mathfrak{D}} + \widehat{\mathfrak{D}}^* \geq 0$. This function need not be *proper*, i.e., it need not be bounded on any right half-plane. We give a complete answer to the question under what conditions such a function can be realized as the transfer function of a *impedance passive system*. By this we mean a continuous time state space system whose control and observation operators are not more unbounded than the (main) semigroup generator of the system, and in addition, there is a certain energy inequality relating the absorbed energy and the internal energy. The system is (impedance) *energy preserving* if this energy inequality is an equality, and it is *conservative* if both the system and its dual are energy preserving. A typical example of an impedance conservative system is a system of hyperbolic type with collocated sensors and actuators. We prove that a passive realization exists if and only if a conservative realization exists, and that this is true if and only if $\lim_{s\to+\infty} \frac{1}{s}\widehat{\mathfrak{D}}(s)u = 0$ for every $u \in U$. The physical interpretation of this condition is that the input-output response is not allowed to contain a pure derivative action. We furthermore show that the so called *diagonal transform* (which is a particular rescaled feedback/feedforward transform) maps an *impedance* passive (or energy preserving or conservative) system into a (well-posed) *scattering* passive (or energy preserving or conservative) system. This implies that if we apply negative output feedback to a impedance passive system, then the resulting system is both well-posed and energy stable. Finally, we study *lossless* scattering systems, i.e., scattering conservative systems whose transfer functions are inner.

Keywords

Dissipative, energy preserving, lossless, proper, collocated sensors and actuators, positive real, Caratheodory-Nevanlinna function, Titchmarsh-Weyl function, bounded real lemma, Kalman-Yakubovich-Popov lemma, feedback, Cayley transform, diagonal transform.

1. Introduction. Let U be a Hilbert space. By a $\mathcal{L}(U)$*-valued positive analytic function* on $\mathbb{C}^+$ (= the open right half-plane) we mean an analytic function which satisfies the condition $\widehat{\mathfrak{D}} + \widehat{\mathfrak{D}}^* \geq 0$ (many other alternative names are also used for this class of functions, such as (impedance) *passive functions*, *Caratheodory-Nevanlinna functions*, *Weyl functions*, or *Titchmarsh-Weyl functions*; see, e.g., [1] and [3] for more detailed discussions of the history of this class of functions). This function need not be *proper*, i.e., it need not be bounded on any right half-plane. For example, the scalar functions $\widehat{\mathfrak{D}}(s) = 1/s$ and $\widehat{\mathfrak{D}}(s) = 1$ are proper (the former is even *strictly* proper since $\widehat{\mathfrak{D}}(\infty) = 0$), whereas $\widehat{\mathfrak{D}}(s) = s$ is not proper (all of these are positive analytic). In this article we introduce a class of continuous time impedance passive systems whose transfer functions are (not necessarily proper) positive analytic. Our class of systems contains all earlier state space realizations of positive analytic functions that we know of, and it is almost complete in the sense that we within this class can realize all positive analytic functions that do not contain a part which corresponds to a pure differentiating input/output action (one of the exceptions being the function $\widehat{\mathfrak{D}}(s) = s$ mentioned above). For example, systems with collocated sensors and actuators belong to the class studied here.

As is well-known, every $\mathcal{L}(U)$-valued function $\widehat{\mathfrak{D}}$ which is analytic and bounded on some right half-plane (i.e., every proper transfer function) has a *well-posed realization*. By this we mean a well-posed linear system Σ whose transfer function is equal to the given function $\widehat{\mathfrak{D}}$. This system Σ has a state space (a Hilbert space) X, an input signal $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+;U)$, a state trajectory $x \in C(\mathbb{R}^+;X)$, and an output signal $y \in L^2_{\mathrm{loc}}(\mathbb{R}^+;U)$ (here $\mathbb{R}^+ = [0,\infty)$). In the absence of an input signal (i.e., for $u = 0$), the evolution of the state x is described by a strongly continuous semigroup. That the transfer function of Σ is $\widehat{\mathfrak{D}}$ means that if the initial state is zero and if the input u is Laplace transformable, then the output y is also Laplace transformable and, on some right half-plane, the Laplace transform $\hat{y}$ of y is given by $\hat{y} = \widehat{\mathfrak{D}}\hat{u}$; here $\hat{u}$ is the Laplace transform of u. In Section 2 we give the formal definition of a well-posed linear system, and there we also describe the basic properties of such systems.

Not every positive analytic function is proper, so to develop a more general theory we need a class of systems which are not necessarily well-posed. The class of systems that we introduce in Section 2 is maybe not the most general one, but it has some nice properties which makes it possible to develop a meaningful theory for this class. We allow both the control and the observation operator to be as unbounded as the generator of the semigroup describing the autonomous behavior of the system. This is roughly twice as much unboundedness as may be present in a well-posed system.

The physical interpretation of a positive analytic function is that it is *energy absorbing* (in an impedance setting). This class of transfer functions appears in certain situations where the input u and the output y are related to each other in a specific way. For example, we could have a pair of wires connected to an electrical circuit, and let u be the voltage between the wires and y the current carried by the wires (or the other way around). In this and many other similar situations, the energy absorbed by the system in the time period $[0,t]$ is proportional to the integral $2\int_0^t \Re\langle u(s), y(s)\rangle\, ds$. It is well-known that if the initial state is zero (so that the Laplace transforms of the input and output satisfy $\hat{y} = \widehat{\mathfrak{D}}\hat{u}$ in some right-half-plane), then this energy is nonnegative for all possible input signals u if and only if $\widehat{\mathfrak{D}}$ is a positive analytic function.

Let us next explain what we mean by an *impedance passive system*.

For simplicity we here stick to the well-posed case. The transfer function of an impedance passive system must be a positive analytic function, but this is not enough. A well-posed system Σ is an impedance passive system if for all initial states $x_0 \in X$, all input signals $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+;U)$, and all $t \geq 0$, the state $x(t)$ at time t and the output signal y satisfy

$$|x(t)|^2 \leq |x_0|^2 + 2\int_0^t \Re\langle u(s), y(s)\rangle\, ds. \tag{1.1}$$

Here $|x(t)|^2$ represents the *energy stored in the state* at time $t \geq 0$. An impedance passive system has the property that if at some time the state $x(t)$ is zero, then at this time moment the system can only absorb energy and not emit any energy (the time derivative of the absorbed energy function is positive). If a system Σ is impedance passive, then so is the dual system Σ^d (this system is defined in Section 2; its transfer function is $\widehat{\mathfrak{D}}^d(z) = \widehat{\mathfrak{D}}(\overline{z})^*$). A system Σ is *impedance energy preserving* if the preceding inequality holds in the form of an equality:

$$|x(t)|^2 = |x_0|^2 + 2\int_0^t \Re\langle u(s), y(s)\rangle\, ds, \tag{1.2}$$

and it is *impedance conservative* if both the original system Σ and the dual system Σ^d are impedance energy preserving. In some sense an impedance conservative realization describes a given positive analytic function in an 'optimal' way: all the energy absorbed or emitted by the system is stored in the state or withdrawn from the state, and the same statement is true also for the dual system. There is no guarantee that all of the state energy can ever be withdrawn, as some of it may be trapped in the state forever. A conservative system is *lossless* if all the energy transferred into the system can eventually be withdrawn.

We begin in Section 2 with a presentation of the class of systems that we use to realize positive analytic functions. In the same section we define what we mean by a *well-posed system.* We continue in Section 3 by recalling the notions of *scattering* passive, energy preserving, and conservative systems, as presented in, e.g., [15], [29], and [40]. (The same classes of systems appear in [2] in a different notation.) These classes of systems are closely related to the corresponding classes of impedance systems introduced above. The only difference is that the expression for the absorbed energy is replaced by $\int_0^t |u(s)|^2\, ds - \int_0^t |y(s)|^2\, ds$, so that (1.1) becomes

$$|x(t)|^2 + \int_0^t |y(s)|^2\, ds \leq |x_0|^2 + \int_0^t |u(s)|^2\, ds, \tag{1.3}$$

and (1.2) becomes

$$|x(t)|^2 + \int_0^t |y(s)|^2\, ds = |x_0|^2 + \int_0^t |u(s)|^2\, ds. \tag{1.4}$$

These systems are always well-posed, and they play an important role in our study of impedance passive, energy preserving, and conservative systems.

In Section 4 we are finally ready to give formal definitions of impedance passive, energy preserving, and conservative systems. We also give a number of equivalent conditions for a system to have one of these properties. For example, if the system is described by a (possibly infinite-dimensional) system of differential equations

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{1.5}$$

where $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U;X)$, $C \in \mathcal{L}(X;U)$, and $D \in \mathcal{L}(U)$, then one of our conditions (see formula (4.4)) says that this system is impedance passive if and only if

$$\begin{bmatrix} A + A^* & B \\ B^* & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & C^* \\ C & D + D^* \end{bmatrix}. \tag{1.6}$$

It is impedance energy preserving if and only if this inequality holds as an equality, and it is impedance conservative if furthermore the corresponding dual identity holds. In this section we also point out that an impedance passive system is well-posed if and only if it is proper.

We then continue to discuss discrete time systems in Sections 5–7. The dynamics of a discrete time system on the three Hilbert spaces U (the input space), X (the state space), and Y (the output space) is described by

$$\begin{aligned} x_{k+1} &= \mathbf{A}x_k + \mathbf{B}u_k, \\ y_k &= \mathbf{C}x_k + \mathbf{D}u_k, \qquad k = 0, 1, 2, \ldots, \\ x_0 &= \text{given}, \end{aligned} \tag{1.7}$$

where $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right] \in \mathcal{L}(\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right])$. We are, in particular, interested in discrete time systems that are passive, energy preserving or conservative in a scattering or impedance setting, meaning that they satisfy identities like (1.1)–(1.4) with the integrals replaced by finite sums. These systems are interesting in their own right, and they play an important role in or study of continuous time systems, too, since many statements about continuous time systems can be reduced to the corresponding statements about discrete time systems by means of the Cayley transform.

There is a simple transform, sometimes called the *diagonal transform*, which maps an impedance passive (or energy preserving or conservative) system into a scattering passive (or energy preserving or conservative) system. This transform is well-known in the finite-dimensional state space case, and also in a very general input/output setting (see [45, Section

8.15]) (it maps a positive analytic function into a contractive analytic function). In Section 6 we show that the same transform works in the discrete time infinite-dimensional state space setting as well if we apply it to an impedance passive system. The range of this transform is *not* the class all scattering passive systems. Instead it consists of those scattering passive systems which satisfy an additional algebraic condition, namely that -1 should not be in the spectrum of the transfer function, at any point in the outside the unit disc.

Section 7 describes the Cayley transform in more detail. In this section we also prove what we consider to be our main result: a continuous time positive analytic function $\widehat{\mathfrak{D}}$ has a simple impedance conservative realization if and only if $\lim_{s\to+\infty} \frac{1}{s}\widehat{\mathfrak{D}}(s)u = 0$ for every $u \in U$. The physical interpretation of this condition is that the input-output response is not allowed to contain a pure derivative action. The proof of this result uses [31, Theorem 3.1, p. 255], which gives the existence of a simple scattering conservative realization of a discrete time contractive analytic function, together with the inverse diagonal and Cayley transforms. From this we may further conclude that the same condition is necessary and sufficient for the existence of a minimal impedance passive realization, which is usually not unique. In the exponentially stable finite-dimensional case the last statement is a consequence of the impedance version of the *Kalman-Yakubovich-Popov lemma*, also known as the *positive (real) lemma*. According to that lemma, a matrix-valued proper rational transfer function $\widehat{\mathfrak{D}}$ with an exponentially stable minimal realization of the type (1.5) (with finite-dimensional X and U) is positive if and only if there exist matrices $P > 0$, Q, and W such that

$$\begin{bmatrix} PA + A^*P & PB \\ B^*P & 0 \end{bmatrix} = \begin{bmatrix} 0 & C^* \\ C & D + D^* \end{bmatrix} - \begin{bmatrix} Q^* \\ W^* \end{bmatrix} \begin{bmatrix} Q & W \end{bmatrix}; \qquad (1.8)$$

see, e.g., [46, Theorems 13.25 and 13.26]. This identity has a simple energy interpretation: if we add another output $z(t) = Qx(t) + Wu(t)$ to the system in (1.5), then the solution x of (1.5) satisfies the energy balance equation

$$\langle x(t), Px(t)\rangle + \int_0^t |z(s)|^2\, ds = \langle x_0, Px_0\rangle + 2\int_0^t \Re\langle u(s), y(s)\rangle\, ds. \qquad (1.9)$$

If we replace the norm in the state space by the new norm $|x|_P = \sqrt{\langle x, Px\rangle}$, then the above identity becomes

$$|x(t)|_P^2 + \int_0^t |z(s)|^2\, ds = |x_0|_P^2 + 2\int_0^t \Re\langle u(s), y(s)\rangle\, ds, \qquad (1.10)$$

and this shows that, with this norm and with the added output z, the system (1.5) can be regarded as an mixed impedance/scattering energy preserving system. (The operator P disappears from (1.8) when we compute the adjoints with respect to the inner product $[x_1, x_2] = \langle x_1, Px_2\rangle$ induced by the new norm.) Dropping the extra output z we get a minimal impedance passive realization of $\widehat{\mathfrak{D}}$. See [44, Sections 5–7] for more details.

As a consequence of our proof of the main result we derive the following interesting conclusion: the diagonal transform can always be applied to a continuous time impedance passive system, and it produces a scattering passive system. This result has a very simple feedback interpretation: if we apply negative feedback to an impedance passive system, then the resulting closed-loop system is both well-posed and stable. This is a generalization of a recent result due to Guo and Luo [10] and Weiss and Tucsnak [41] (independently of each other).

In the last section we briefly discuss *lossless* scattering systems, i.e., scattering conservative systems with the property that all the energy transferred into the system can eventually be withdrawn.

The title of this paper ends with the words "from a personal point of view". This is intended to be a disclaimer: I *make no claims whatsoever on the completeness* of the theory, and I also *make no claims on the historical correctness* of the presentation. I have gradually become aware of the existence of a well-developed theory in this field, and I have tried to fit it into my personal view of how the universe is built. In this process I have (re)discovered many beautiful results, and I do not always know to whom these results should be contributed. I am also aware of the fact that there are many more results still waiting to be (re)discovered. In particular, I have not found Theorem 7.4 in the literature, but it must be related to the existing results on realizations based on non-uniform transmission lines. I was hoping to be able to clarify this relationship before the deadline of this article, but run out of time.

2. Infinite-Dimensional Linear Systems. Many infinite-dimensional linear time-invariant continuous-time systems can be described by the equations (1.5) on a triple of Hilbert spaces, namely, the input space U, the state space X, and the output space Y. We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operator A is supposed to be the generator of a strongly continuous semigroup $t \mapsto \mathfrak{A}^t$. The generating operators A, B and C are usually unbounded, but D is bounded.

By modifying this set of equations slightly we get the class of systems which will be used in this work. In the sequel, we think about the block matrix $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as *one single (unbounded) operator* from $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, and write (1.5) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0. \tag{2.1}$$

The operator S completely determines the system. Thus, we may identify the system with such an operator, which we call the *node* of the system.

There are certain conditions that we need to impose on S in order to get a meaningful theory. First of all, S must be closed and densely defined as an operator from $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$. Let us denote the domain of S by $\mathcal{D}(S)$. Then S can be split into $S = \left[\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix}\right]$, where S_1 maps $\mathcal{D}(S)$ into X and S_2 maps $\mathcal{D}(S)$ into Y. By analogy to the finite-dimensional case, let us denote $A\&B := S_1$ and $C\&D := S_2$, so that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ (the reader who finds this notation confusing may throughout replace $A\&B$ by S_1 and $C\&D$ by S_2). It is not true, in general, that $A\&B$ and $C\&D$ (defined on $\mathcal{D}(S)$) can be decomposed into $A\&B = \begin{bmatrix} A & B \end{bmatrix}$ and $C\&D = \begin{bmatrix} C & D \end{bmatrix}$; this is possible only in the case where $\mathcal{D}(S)$ can be written as the product of one subspace of X times another subspace of U. However, we shall require that an extended version of $A\&B$ can be decomposed as indicated above, so that *$A\&B$ is the restriction to $\mathcal{D}(S)$ of $\begin{bmatrix} A & B \end{bmatrix}$* for suitably defined operators A and B.

The decomposition of $A\&B$ is based on the familiar 'rigged Hilbert space structure' (sometimes referred to as a 'Gelfand triple').[1] Let A be a closed (unbounded) densely defined operator on the Hilbert space X with a nonempty resolvent set. We denote its domain $\mathcal{D}(A)$ by X_1. This is a Hilbert space with the norm $|x|_{X_1} := |(\alpha - A)x|_X$, where α is an arbitrary number in $\alpha \in \rho(A)$ (different numbers α give different but equivalent norms). We also construct a larger Hilbert space X_{-1}, which is the completion of X under the norm $|x|_{X_{-1}} := |(\alpha - A)^{-1}x|_X$. Then $X_1 \subset X \subset X_{-1}$ with continuous and dense injections. The operator A has a unique extension to an operator in $\mathcal{L}(X; X_{-1})$ which we denote by $A_{|X}$ (thereby indicating that the domain of this operator is all of X). The operators A and $A_{|X}$ are similar to each other and they have the same spectrum. Thus, for all $\alpha \in \rho(A)$, the operator $\alpha - A_{|X}$ maps X one-to-one onto X_{-1}. Its inverse $(\alpha - A_{|X})^{-1}$ is the unique extension to X_{-1} of the operator $(\alpha - A)^{-1}$.

We shall also need the dual versions of the spaces X_1 and X_{-1}. If we repeat the construction described above with A replaced by the (unbounded) adjoint A^* of A, then we get two more spaces, that we denote by X_1^d (the analogue of X_1) and X_{-1}^d (the analogue of X_{-1}). Then $X_1^d \subset X \subset X_{-1}^d$ with continuous and dense injections. If we identify the dual of X with X itself, then X_1^d becomes the dual of X_{-1} and X_{-1}^d becomes the dual of X_1.[2] We denote the extension of A^* to an operator in $\mathcal{L}(X; X_{-1}^d)$ by $A^*_{|X}$. This operator can be interpreted as the (bounded) adjoint of the operator A, regarded as an operator in $\mathcal{L}(X_1; X)$.

Definition 2.1. *We call S a* system node *on the three Hilbert spaces (U, X, Y) if it satisfies condition (S) below:*[3]

[1] See, e.g., [15] or [28] or almost any other of the papers listed in the reference list for details.

[2] Often X_{-1} is *defined* to be the dual of X_1^d when we identify the dual of X with X itself.

[3] This definition is equivalent to the corresponding definition used by Smuljan in [21]

(S) $S := \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \supset \mathcal{D}(S) \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ *is a closed linear operator. Here* $A\&B$ *is the restriction to* $\mathcal{D}(S)$ *of* $\begin{bmatrix} A_{|X} & B \end{bmatrix}$*, where* A *is the* generator of a C_0 semigroup *on* X *(the notations* $A_{|X} \in \mathcal{L}(X; X_{-1})$ *and* X_{-1} *were introduced in the text above). The operator* B *is an arbitrary operator in* $\mathcal{L}(U; X_{-1})$*, and* $C\&D$ *is an arbitrary linear operator from* $\mathcal{D}(S)$ *to* Y*. In addition, we require that*

$$\mathcal{D}(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \;\middle|\; A_{|X}x + Bu \in X \right\}.$$

It follows from the above definition that $A\&B\colon \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \supset \mathcal{D}(A\&B) \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, with $\mathcal{D}(A\&B) = \mathcal{D}(S)$, is a closed operator. Thus, $\mathcal{D}(S)$ becomes a Hilbert space with the graph norm of the operator $A\&B$. Furthermore, it is not difficult to show that the assumption that S is closed is equivalent to the assumption that $C\&D$ *is continuous from* $\mathcal{D}(S)$ *(with the graph norm of* $A\&B$*) to* Y.

We shall use the following names of the different parts of the system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. The operator A is the *main operator* or the *semigroup generator*, B is the *control operator*, $C\&D$ is the *combined observation/feedthrough operator*, and the operator C defined by

$$Cx := C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad x \in X_1,$$

is the *observation operator* of S.

An easy algebraic computation (see, e.g., [28, Section 4.7] for details) shows that for each $\alpha \in \rho(A) = \rho(A_{|X})$, the operator $\left[\begin{smallmatrix} 1 & (\alpha - A_{|X})^{-1}B \\ 0 & 1 \end{smallmatrix}\right]$ is an boundedly invertible mapping between $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} X_1 \\ U \end{smallmatrix}\right] \to \mathcal{D}(S)$. Since $\left[\begin{smallmatrix} X_1 \\ U \end{smallmatrix}\right]$ is dense in $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$, this implies that $\mathcal{D}(S)$ is dense in $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$. Furthermore, since the second column $\left[\begin{smallmatrix} (\alpha - A_{|X})^{-1}B \\ 1 \end{smallmatrix}\right]$ of this operator maps U into $\mathcal{D}(S)$, we can define the *transfer function* of S by

$$\widehat{\mathfrak{D}}(s) := C\&D \begin{bmatrix} (s - A_{|X})^{-1}B \\ 1 \end{bmatrix}, \qquad s \in \rho(A), \tag{2.2}$$

which is a $\mathcal{L}(U; Y)$-valued analytic function on $\rho(A)$. By the resolvent formula, for any two α, $\beta \in \rho(A)$,

$$\begin{aligned} \widehat{\mathfrak{D}}(\alpha) - \widehat{\mathfrak{D}}(\beta) &= C\big[(\alpha - A_{|X})^{-1} - (\beta - A_{|X})^{-1}\big]B \\ &= (\beta - \alpha)C(\alpha - A)^{-1}(\beta - A_{|X})^{-1}B. \end{aligned} \tag{2.3}$$

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in 1986. Unfortunately, that paper (written in Russian) has not been properly known and recognized in the English literature, and many of its results have been (independently) rediscovered, among others by this author. The main part of [21] is devoted to system nodes which are well-posed (see our Definition 2.3). System nodes appear also in the work by Salamon [19, 20] in a less implicit way, again primarily in the well-posed case. Our notation $C\&D\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ corresponds to Smuljan's notation $N\langle x, u\rangle$ and Salamon's notation $\big(x - (\alpha - A)^{-1}Bu\big) + \widehat{\mathfrak{D}}(\alpha)u$. Compare this to formula (2.4) below.

It is possible to alternatively define a system node by specifying the main operator A, the control operator B, the observation operator C, and the transfer function $\widehat{\mathfrak{D}}$ evaluated at some point $\alpha \in \rho(A)$.

LEMMA 2.1. *Let A be the generator of a C_0 semigroup on a Hilbert space X, and let X_1, X_{-1} and $A_{|X}$ be the spaces and the operator induced by A, as explained in the text preceding Definition 2.1. Let $B \in \mathcal{L}(U;X_{-1})$, let $C \in \mathcal{L}(X_1;Y)$, and let $D \in \mathcal{L}(U;Y)$, where U and Y are two more Hilbert spaces. Let $A\&B$ be the restriction of $\begin{bmatrix} A_{|X} & B \end{bmatrix}$ to $\mathcal{D}(A\&B) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \;\middle|\; A_{|X}x + Bu \in X \right\}$. Finally, let $\alpha \in \rho(A)$, and define*

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C\big(x - (\alpha - A_{|X})^{-1}Bu\big) + Du, \qquad \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(A\&B).$$

Then $S := \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \mathcal{D}(S) := \mathcal{D}(A\&B) \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ is a system node on (U,X,Y). The control operator of this system node is B, the observation operator is C, and the transfer function satisfies $\widehat{\mathfrak{D}}(\alpha) = D$.

See [26] for the (easy) proof.

Thus, if we replace D by $\widehat{\mathfrak{D}}(\alpha)$ above, then we have written $C\&D$ in terms of A, B, C, and $\widehat{\mathfrak{D}}(\alpha)$:

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = \big(x - (\alpha - A_{|X})^{-1}Bu\big) + \widehat{\mathfrak{D}}(\alpha)u. \tag{2.4}$$

In particular, the right-hand side does not depend on how we choose $\alpha \in \rho(A)$.

As shown in [21, Theorem 1.2] (and also in [2] and [15]), if S is a system node on (U,X,Y), then the (unbounded) adjoint S^* of S is a system node on (Y,X,U). We shall refer to this system node as the *dual system node*, and we sometimes denote it by S^d. If we let A be the main operator of S, and let $B \in \mathcal{L}(U;X_{-1})$ and $C \in \mathcal{L}(X_1;Y)$ be the control and observation operators of S, then the main operator of S^d is $A^d = A^*$ (by this we mean the unbounded adjoint of A; see the paragraph before Definition 2.1), the control operator of S^* is $B^d = C^* \in \mathcal{L}(Y;X^d_{-1})$, and the observation operator is $C^d = B^* \in \mathcal{L}(X^d_1;U)$. Furthermore, if $\widehat{\mathfrak{D}}$ is the transfer function of S, then the transfer function $\widehat{\mathfrak{D}}^d$ of S^d is given by $\widehat{\mathfrak{D}}^d(s) = \widehat{\mathfrak{D}}(\overline{s})^*$ for $s \in \rho(A^*)$.

Every system node induces a 'dynamical system' of a certain type:

LEMMA 2.2. *Let S be a system node on (U,X,Y). Then, for each $x_0 \in X$ and $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+;U)$ with $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$, the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \ge 0, \quad x(0) = x_0, \tag{2.5}$$

has a unique solution (x,y) satisfying $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{D}(S)$ for all $t \ge 0$, $x \in C^1(\mathbb{R}^+;X)$, and $y \in C(\mathbb{R}^+;Y)$.

This lemma is proved in [15] (and also in [28]).[4]

By taking *Laplace transforms* in (2.5) we find that if u is Laplace transformable with transform $\hat{u}$, then the output y is also Laplace transformable with transform

$$\begin{aligned} \hat{x}(s) &= (s-A)^{-1}x_0 + (s-A_{|X})^{-1}B\hat{u}(s), \\ \hat{y}(s) &= C(s-A)^{-1}x_0 + \widehat{\mathfrak{D}}(s)\hat{u}(s), \end{aligned} \tag{2.6}$$

for $\Re s$ large enough. Thus, our definition of the transfer function is equivalent to the standard definition in the classical case.

Definition 2.2. *By the* linear system Σ *generated by a system node* S *we understand the family* Σ *of maps defined by*

$$\Sigma_0^t \begin{bmatrix} x_0 \\ \pi_{[0,t]}u \end{bmatrix} := \begin{bmatrix} x(t) \\ \pi_{[0,t]}y \end{bmatrix},$$

parametrized by $t \geq 0$, *where* x_0, $x(t)$, u, *and* y *are as in Lemma 2.2 and* $\pi_{[0,t]}u$ *and* $\pi_{[0,t]}y$ *are the restrictions of* u *and* y *to* $[0,t]$. *We call* x *the* state trajectory *and* y *the* output function *of* Σ *with initial state* x_0 *and input function* u.

By the *reachable* subspace of S we mean the closure in X of the set of all possible values of $x(t)$ in Lemma 2.2 if we take $x_0 = 0$ (and let u and t vary). Its orthogonal complement is the *unreachable subspace*. By the *unobservable* subspace of S we mean the closure of the set of all $x_0 \in X_1$ for which the output y in Lemma 2.2 with initial state x_0 and zero input function u is identically zero. Its orthogonal complement is the *observable subspace*. It is well-known that the orthogonal complement of the reachable subspace of S is the unobservable subspace of the dual system node S^* (and the same statement is true if we interchange S and S^*). A system is *simple* if the intersection of the unreachable and unobservable subspaces is $\{0\}$.

We call a system node S (and the corresponding system Σ) on (U,X,Y) *(approximately) controllable* if the reachable subspace is all of X and *(approximately) observable* if the observable subspace is all of X. A system which is both controllable and observable is *minimal*.

So far we have *defined* Σ_0^t *only for the class of smooth data* given in Lemma 2.2. It is possible to extend this definition by allowing the state to take values in the larger space X_{-1} instead of in X, and by allowing y to be a distribution.

Let us first take a look at the state, which is supposed to be a solution of the equation $\dot{x}(t) = A_{|X}x(t) + Bu(t)$ for $t \geq 0$, with initial value $x(0) = x_0$. However, since $B \in \mathcal{L}(U;X_{-1})$, if $x_0 \in X$ and if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+;U)$, then this equation has a unique strong solution $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+;X_{-1})$ (see, e.g.,

[4] Well-posed versions of this lemma (see Definition 2.3) are (implicitly) found in [19] and [21] (and also in [29]). In the well-posed case we need less smoothness of u: it suffices to take $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^+;U)$. In addition y will be smoother: $y \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^+;Y)$.

[28, Section 3.8]; the operator $A_{|X}$ is the generator of the C_0 semigroup that we get by extending the semigroup generated by A to X_{-1}). Thus, the notion of the state trajectory causes no problem if we are willing to accept a trajectory with values in X_{-1}.

To get a generalized definition of the output y under the same premises we can do as follows (see [28, Section 4.7] for details). Let $x_0 \in X$, $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+;U)$, and let $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+;X_{-1})$ be the corresponding state trajectory. Define $\left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right]$ by

$$\begin{bmatrix} x_2(t) \\ u_2(t) \end{bmatrix} = \int_0^t (t-s) \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} ds, \qquad t \ge 0$$

(this is the second order integral of $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$). Then $\begin{bmatrix} x_2(t) \\ u_2(t) \end{bmatrix} \in \mathcal{D}(S)$ for all $t \ge 0$, and we may define the output y by

$$y(t) = \left(C\&D \begin{bmatrix} x_2(s) \\ u_2(s) \end{bmatrix} \right)'', \qquad t \ge 0, \tag{2.7}$$

where we interpret the second order derivative in the distribution sense.[5]

The following lemma gives a sufficient set of conditions under which the output y is a a function (as opposed to a distribution):

LEMMA 2.3. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, Y). Let $x_0 \in X$, and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+;U)$, and let x and y be the state trajectory and output of S with initial state x_0, and input function u. If $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+;X)$, then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{D}(S))$, $y \in L^1_{\mathrm{loc}}(\mathbb{R}^+;Y)$, and $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is the unique solution with the above properties of the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \text{ for almost all } t \ge 0, \quad x(0) = x_0. \tag{2.8}$$

If $u \in C(\mathbb{R}^+;U)$ and $x \in C^1(\mathbb{R}^+;X)$, then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(\mathbb{R}^+;\mathcal{D}(S))$, $y \in C(\mathbb{R}^+;Y)$, and the equation (2.8) holds for all $t \ge 0$.

See [28, Section 4.7] for the proof.

Another possibility to extend Σ_0^t to a larger class of data is based on an additional *well-posedness assumption.*

DEFINITION 2.3. *A system node S is* well-posed *if, for some $t > 0$, there is a finite constant $K(t)$ such that the solution (x, y) in Lemma 2.2 satisfies*

$$|x(t)|^2 + \|y\|^2_{L^2(0,t)} \le K(t)\big(|x_0|^2 + \|u\|^2_{L^2(0,t)}\big). \tag{\textbf{WP}}$$

It is energy stable *if there is some $K < \infty$ so that, for all $t \in \mathbb{R}^+$, the solution (x, y) in Lemma 2.2 satisfies*

$$|x(t)|^2 + \|y\|^2_{L^2(0,t)} \le K\big(|x_0|^2 + \|u\|^2_{L^2(0,t)}\big). \tag{\textbf{ES}}$$

[5] In the well-posed case, if $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+;U)$, then it suffices to integrate $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ once, then apply $C\&D$, and finally differentiate once in the distribution sense.

It is not difficult to show that if (WP) holds for *one* $t>0$, then it holds for *all* $t\ge 0$.

If a system node S is well-posed, then the corresponding system Σ can be *extended by continuity* to a family of operators

$$\Sigma_0^t := \left[\begin{array}{c|c} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \hline \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{array}\right]$$

from $\left[\begin{smallmatrix} X \\ L^2([0,t];U)\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} X \\ L^2([0,t];Y)\end{smallmatrix}\right]$. (We still denote the extended family by Σ.)

For more details, explanations and examples we refer the reader to [1], [2], [6], [8, 9] [11], [17], [19, 20], [21], [22, 23, 24, 25, 28], [29, 30], [33], [35, 36, 37, 38, 39], [40], [41], and [42] (and the references therein).

3. Scattering Passive and Conservative Systems. The following definition is a slightly modified version of the definitions in the two classical papers [43, 44] by Willems (although we use a slightly different terminology: our *passive* is the *same as Willems' dissipative*).[6]

DEFINITION 3.1. *Let J be a bounded self-adjoint operator on $\left[\begin{smallmatrix} Y \\ U\end{smallmatrix}\right]$. A system node S on (U,X,Y) is J-*passive *if, for all $t>0$, the solution (x,y) in Lemma 2.2 satisfies*

$$|x(t)|^2 - |x_0|^2 \le \int_0^t \left\langle \left[\begin{smallmatrix} y(s) \\ u(s)\end{smallmatrix}\right], J\left[\begin{smallmatrix} y(s) \\ u(s)\end{smallmatrix}\right]\right\rangle ds. \qquad \textbf{(JP)}$$

*It is J-*energy preserving *if the above inequality holds in the form of an equality: for all $t>0$, the solution (x,y) in Lemma 2.2 satisfies*

$$|x(t)|^2 - |x_0|^2 = \int_0^t \left\langle \left[\begin{smallmatrix} y(s) \\ u(s)\end{smallmatrix}\right], J\left[\begin{smallmatrix} y(s) \\ u(s)\end{smallmatrix}\right]\right\rangle ds. \qquad \textbf{(JE)}$$

Physically, passivity means that *there are no internal energy sources.* An energy preserving system has neither any internal energy sources nor any sinks.

Different choices of J give different passivity notions. The case $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1\end{smallmatrix}\right]$ is known as *scattering*. The case where $U=Y=\left[\begin{smallmatrix} V \\ V\end{smallmatrix}\right]$ and $J=\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right]$ is known as *impedance* (*admittance*, *resistance*, *conductance*). The case where $U=Y=\left[\begin{smallmatrix} V \\ W\end{smallmatrix}\right]$, and $J=\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1\end{array}\right]$ is known as *transmission*

[6] Another difference is that we have replaced Willems' more general *storage function* $S(x)$ by the quadratic function $|x|_X^2$. Our setting becomes the scattering version of the setting which Willems uses in the second part [44] if we simply take the norm in the state space to be $|x|^2 = \sqrt{S(x)}$ (this is possible whenever the storage function is quadratic and strictly positive). See also [32] and [34].

(*chain scattering*). In this article we focus on the *scattering* ($J = \left[\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right]$) and *impedance* ($J = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$) settings.

DEFINITION 3.2. *A system node S is* scattering passive[7] *if, for all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies*

$$|x(t)|^2 - |x_0|^2 \le \|u\|^2_{L^2(0,t)} - \|y\|^2_{L^2(0,t)}. \tag{SP}$$

It is scattering energy preserving *if the above inequality holds in the form of an equality: for all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies*

$$|x(t)|^2 - |x_0|^2 = \|u\|^2_{L^2(0,t)} - \|y\|^2_{L^2(0,t)}. \tag{SE}$$

Finally, it is scattering conservative *if both S and S^* are scattering energy preserving.*

Thus, *every scattering passive system is well-posed*: the passivity inequality **(SP)** implies the well-posedness inequality **(WP)**.

A scattering passive system can be characterized in several different ways:

THEOREM 3.1. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, Y). Then the following conditions are equivalent:*

(i) *S is scattering passive.*

(ii) *For all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2_X \le |u(t)|^2_U - |y(t)|^2_Y. \tag{3.1}$$

(iii) *For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$2\Re\left\langle A\&B \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], x_0\right\rangle_X \le |u_0|^2_U - \left|C\&D \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\right|^2_Y. \tag{3.2}$$

(iv) *For some $\alpha \in \rho(A) \cap \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$), the operator*

$$\begin{bmatrix} \mathbf{A}(\alpha) & \mathbf{B}(\alpha) \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} = \begin{bmatrix} (\overline{\alpha} + A)(\alpha - A)^{-1} & \sqrt{2\Re\alpha}\,(\alpha - A)^{-1}B \\ \sqrt{2\Re\alpha}\,C(\alpha - A)^{-1} & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \tag{3.3}$$

is a contraction. (Here $\mathbb{C}^+$ is the open right half-plane.)

This is (a part of) [29, Theorem 7.4], and it is also found in [2] (see, in particular, Definition 4.1, Proposition 4.1, Subsection 4.5, and Theorem 5.2 of [2]).

A similar result is valid for scattering energy preserving systems:

THEOREM 3.2. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, Y). Then the following conditions are equivalent:*

(i) *S is scattering energy preserving.*

---

[7] In [15, 16], [29, 30], [33], [40], [41], etc., these systems are called *dissipative*.

(ii) *For all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 = |u(t)|_U^2 - |y(t)|_Y^2. \tag{3.4}$$

(iii) *For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$2\Re\big\langle A\&B \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], x_0 \big\rangle_X = |u_0|_U^2 - \left|C\&D \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\right|_Y^2. \tag{3.5}$$

(iv) *For some $\alpha \in \rho(A) \cap \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$), the operator $\left[\begin{smallmatrix} \mathbf{A}(\alpha) & \mathbf{B}(\alpha) \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) \end{smallmatrix}\right]$ defined in (3.3) is isometric.*

This theorem is proved in [15]. Most of this theorem is also found in [2].

By applying Theorem 3.2 both to the original system node S and to the dual system node S^* we get a set of systems which characterize *scattering conservative system nodes.* Some equivalent but *simpler conditions* are given in [15].

A finite-dimensional system is *scattering conservative if and only if it is energy preserving and the input and output spaces have the same dimension.* Some related (but more complicated) results are true also in infinite-dimensions. See [2], [15], and [29, 30] for details.

4. Impedance Passive and Conservative Systems. As we mentioned above, we get into the impedance setting by taking $J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ in Definition 3.1.

DEFINITION 4.1. *A system node S on (U, X, U) (note that $Y = U$) is* impedance passive *if, for all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies*

$$|x(t)|_X^2 - |x_0|_X^2 \le 2\int_0^t \Re\langle y(t), u(t)\rangle_U dt. \tag{\textbf{IP}}$$

It is impedance energy preserving *if the above inequality holds in the form of an equality: for all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies*

$$|x(t)|_X^2 - |x_0|_X^2 = 2\int_0^t \Re\langle y(t), u(t)\rangle_U dt. \tag{\textbf{IE}}$$

Finally, S is impedance conservative *if both S and the dual system node S^* are impedance energy preserving.*

Note that in this case *well-posedness is neither guaranteed, nor relevant.*

The property of being passive is conserved under the passage from a system node S to its dual:

LEMMA 4.1. *A system node S is impedance passive if and only if the dual system node S^* is impedance passive.*

This is proved in [26, Corollary 4.5]. That proof is based on property (v) in the following theorem, which lists a number of equivalent conditions for a system node to be impedance passive.

THEOREM 4.1. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, U). Then the following conditions are equivalent:*

(i) *S is impedance passive.*

(ii) *For all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 \le 2\Re\langle y(t), u(t)\rangle_U. \tag{4.1}$$

(iii) *For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$\Re\big\langle A\&B \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], x_0\big\rangle_X \le \Re\langle C\&D \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], u_0\rangle_U. \tag{4.2}$$

(iv) *For some $\alpha \in \rho(A) \cap \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$), the operator $\left[\begin{smallmatrix} \mathbf{A}(\alpha) & \mathbf{B}(\alpha) \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) \end{smallmatrix}\right]$ defined in* (3.3) *satisfies*

$$\begin{bmatrix} \mathbf{A}(\alpha)^*\mathbf{A}(\alpha) & \mathbf{A}(\alpha)^*\mathbf{B}(\alpha) \\ \mathbf{B}(\alpha)^*\mathbf{A}(\alpha) & \mathbf{B}(\alpha)^*\mathbf{B}(\alpha) \end{bmatrix} \le \begin{bmatrix} 1 & \mathbf{C}(\alpha)^* \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) + \widehat{\mathfrak{D}}(\alpha)^* \end{bmatrix}. \tag{4.3}$$

(v) *The system node $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is a maximal dissipative operator in $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$, i.e., $\alpha - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is invertible for every $\alpha \in \mathbb{C}^+$, and for all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$\Re\left\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \le 0. \tag{4.4}$$

(vi) *For some $\alpha \in \rho(A) \cap \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$), the operator $\alpha - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is invertible, and*

$$\begin{bmatrix} \mathbf{A}^\times(\alpha) & \mathbf{B}^\times(\alpha) \\ \mathbf{C}^\times(\alpha) & \mathbf{D}^\times(\alpha) \end{bmatrix} = \left(\overline{\alpha} + \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}\right)\left(\alpha - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}\right)^{-1} \tag{4.5}$$

is a contraction.

This is a slightly abbreviated version of [26, Theorem 4.2 and Corollary 4.4]. The energy preserving case is similar.

THEOREM 4.2. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, U). Then the following conditions are equivalent:*

(i) *S is impedance energy preserving.*

(ii) *For all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 = 2\Re\langle y(t), u(t)\rangle_U. \tag{4.6}$$

(iii) *For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$\Re\left\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} = 0. \tag{4.7}$$

(iv) *For some* $\alpha \in \rho(A) \cap \mathbb{C}^+$ *(or equivalently, for all* $\alpha \in \mathbb{C}^+$*), the operator* $\left[\begin{smallmatrix} \mathbf{A}(\alpha) & \mathbf{B}(\alpha) \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) \end{smallmatrix}\right]$ *defined in* (3.3) *satisfies*

$$\begin{bmatrix} \mathbf{A}(\alpha)^*\mathbf{A}(\alpha) & \mathbf{A}(\alpha)^*\mathbf{B}(\alpha) \\ \mathbf{B}(\alpha)^*\mathbf{A}(\alpha) & \mathbf{B}(\alpha)^*\mathbf{B}(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{C}(\alpha)^* \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) + \widehat{\mathfrak{D}}(\alpha)^* \end{bmatrix}. \tag{4.8}$$

(v) *The system node* $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is skew-symmetric, i.e.,* $\mathcal{D}(S) = \mathcal{D}(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]) \subset \mathcal{D}(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]^*)$*, and*

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}^* \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathcal{D}(S). \tag{4.9}$$

(vi) *For some* $\alpha \in \rho(A) \cap \mathbb{C}^+$ *(or equivalently, for all* $\alpha \in \mathbb{C}^+$*), the operator* $\alpha - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is invertible, and the operator* $\left[\begin{smallmatrix} \mathbf{A}^\times(\alpha) & \mathbf{B}^\times(\alpha) \\ \mathbf{C}^\times(\alpha) & \mathbf{D}^\times(\alpha) \end{smallmatrix}\right]$ *defined in* (4.5) *is an isometry.*

This is a slightly abbreviated version of [26, Theorem 4.6]. An analogous but even simpler result is true for impedance conservative systems:

Theorem 4.3. *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a system node on* (U, X, U)*. Then the following conditions are equivalent:*

(i) S *is impedance conservative.*

(ii) *For all* $t > 0$*, the solution* (x, y) *in Lemma 2.2 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 = 2\Re\langle y(t), u(t)\rangle_U, \tag{4.10}$$

and the same identity is true for the adjoint system.

(iii) *The system node* $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is skew-adjoint, i.e.,*

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}^* = - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}. \tag{4.11}$$

(iv) $A^* = -A$, $B^* = C$, *and* $\widehat{\mathfrak{D}}(\alpha) + \widehat{\mathfrak{D}}(-\overline{\alpha})^* = 0$ *for some (or equivalently, for all)* $\alpha \in \rho(A)$ *(in particular, this identity is true for all* α *with* $\Re\alpha \neq 0$*).*

(v) *For some* $\alpha \in \rho(A) \cap \mathbb{C}^+$ *(or equivalently, for all* $\alpha \in \mathbb{C}^+$*), the operator* $\alpha - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is invertible, and the operator* $\left[\begin{smallmatrix} \mathbf{A}^\times(\alpha) & \mathbf{B}^\times(\alpha) \\ \mathbf{C}^\times(\alpha) & \mathbf{D}^\times(\alpha) \end{smallmatrix}\right]$ *defined in* (4.5) *is unitary.*

This is [26, Theorem 4.7].

Many impedance passive systems are well-posed. There is a simple way of characterizing such systems:

Theorem 4.4. *An impedance passive system node is well-posed if and only if its transfer function* $\widehat{\mathfrak{D}}$ *is bounded on some (or equivalently, on every) vertical line in* $\mathbb{C}^+$*. When this is the case, the growth bound of the system is zero, and, in particular,* $\widehat{\mathfrak{D}}$ *is bounded on every right half-plane* $\mathbb{C}^+_\epsilon = \{s \in \mathbb{C} \mid \Re s > \epsilon\}$ *with* $\epsilon > 0$*.*

This is [26, Theorem 5.1]. It can be used to show that many systems with *collocated actuators and sensors* are well-posed.

EXAMPLE 1. *Let A be the generator of a contraction semigroup on X. Define $S = \begin{bmatrix} A_{|X} & A_{|X} \\ -A_{|X} & -A_{|X} \end{bmatrix}$ with $\mathcal{D}(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right] \;\middle|\; x + u \in \mathcal{D}(A) \right\}$. Then S is an impedance passive system node on (X, X, X) (use part (v) of Theorem 4.1 and note that $\left[\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] A \begin{bmatrix} 1 & 1 \end{bmatrix}$ can be interpreted as the dissipative operator A surrounded by another operator and its adjoint). The transfer function of this node is easily computed, and it turns out to be $\widehat{\mathfrak{D}}(s) = -sA(s-A)^{-1}$, $s \in \mathbb{C}^+$. This example is impedance energy preserving if and only if A generates an isometric semigroup (i.e, A is skew-symmetric), and it is impedance conservative if and only if A generates a unitary group (i.e, A is skew-adjoint).*

EXAMPLE 2. *Let A be a normal operator on X which generates a contraction semigroup (i.e., A is dissipative, or equivalently, the spectrum of A lies in the closed left half-plane). We can then use the polar decomposition of A to write A in the form $A = V|A| = |A|V$, where $|A|$ is the positive square root of $A^*A = AA^*$ and V is unitary. Let $\frac{1}{2} \le p \le 1$, and let us consider the system node*[8]

$$S = \begin{bmatrix} A & |A|^p V \\ -|A|^p V & -|A|^{2p-1} V \end{bmatrix}$$

on (X, X, X), with $\mathcal{D}(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right] \;\middle|\; Ax + |A|^p V u \in X \right\}$. Formulated in this way it is not obvious that the observation/feedthrough operator is continuous. However, we may (formally) compute the corresponding transfer function, which turns out to be given by

$$\widehat{\mathfrak{D}}(s) = -s|A|^{2p-1}(s-A)^{-1}V, \qquad s \in \mathbb{C}^+.$$

We know from Lemma 2.1 that there is a system node S whose semigroup generator is A, whose control operator is $|A|^p V$, whose observation operator is $-|A|^p V$, and whose transfer function satisfies $\widehat{\mathfrak{D}}(1) = |A|^{2p-1}(1-A)^{-1}V$. If we compute the observation/feedthrough operator of this node as described in Lemma 2.1, then we get for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}(S)$, (recall that V, A, and all positive powers of $|A|$ commute)

$$\begin{aligned} C\&D \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] &= -|A|^p V\big(x - (1-A)^{-1}|A|^p V u\big) - |A|^{2p-1}(1-A)^{-1} V u \\ &= -|A|^p V x - |A|^{2p-1} V(1 - |A|V)(1-A)^{-1} u \\ &= -|A|^p V x - |A|^{2p-1} V u. \end{aligned}$$

This shows that the original formula that we gave for the system node S indeed defines a system node S on (X, X, X), with $\mathcal{D}(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right] \;\middle|\; |A|x + |A|^p u \in X \right\}$.

[8] The operators A and the powers of $|A|$ that appear in this example can be extended so that they map all of X into an appropriate larger space. We still denote the extended operators by the same letters.

We claim that this system node is impedance passive. To prove this it suffices to show that the operator

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} = \begin{bmatrix} A & |A|^p V \\ |A|^p V & |A|^{2p-1} V \end{bmatrix}$$

is dissipative. However, this follows from the fact that we can factor this operator in the form

$$\begin{bmatrix} A & |A|^p V \\ |A|^p V & |A|^{2p-1} V \end{bmatrix} = \begin{bmatrix} |A|^{1-p} \\ 1 \end{bmatrix} |A|^{p-1/2} V |A|^{p-1/2} \begin{bmatrix} |A|^{1-p} & 1 \end{bmatrix},$$

and this operator is dissipative since V is dissipative (here V is surrounded by another operator and its adjoint). From the same computation we can conclude that S is impedance conservative whenever A generates a unitary group, i.e., whenever A is skew-adjoint. (The impedance energy preserving version of this example is irrelevant, because if A is normal and generates an isometric semigroup, then this semigroup is unitary.)

5. Discrete Time Systems. There is a close connection between the passive continuous time systems that we have considered so far and the corresponding *discrete time systems*. In these systems the input $u = \{u_k\}_{k=0}^\infty$, the state $x = \{x_k\}_{k=0}^\infty$, and the output $y = \{y_k\}_{k=0}^\infty$ are sequences with values in the Hilbert spaces U, X, respectively Y, and the dynamics is described by

$$\begin{aligned} x_{k+1} &= \mathbf{A} x_k + \mathbf{B} u_k, \\ y_k &= \mathbf{C} x_k + \mathbf{D} u_k, \qquad k = 0, 1, 2, \ldots, \\ x_0 &= \text{given}, \end{aligned} \tag{5.1}$$

where $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right] \in \mathcal{L}(\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right])$. We still call $\mathbf{A}$ the main operator, $\mathbf{B}$ the control operator, $\mathbf{C}$ the observation operator, and $\mathbf{D}$ the feedthrough operator. We define the *transfer function* $\widehat{\mathbf{D}}$ of $\mathbf{\Sigma}$ in the same way as in the continuous time (with $C\&D$ replaced by $\begin{bmatrix} \mathbf{C} & \mathbf{D} \end{bmatrix}$), namely

$$\widehat{\mathbf{D}}(z) = \mathbf{C}(z - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}, \qquad z \in \rho(\mathbf{A}).$$

Obviously, the transfer function $\widehat{\mathbf{D}}^d$ of the dual system $\mathbf{\Sigma}^d = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]^* = \left[\begin{smallmatrix} \mathbf{A}^* & \mathbf{C}^* \\ \mathbf{B}^* & \mathbf{D}^* \end{smallmatrix}\right]$ is

$$\widehat{\mathbf{D}}^d(z) = \mathbf{B}^*(z - \mathbf{A}^*)^{-1}\mathbf{C}^* + \mathbf{D}^* = \widehat{\mathbf{D}}(\overline{z})^*, \qquad z \in \rho(\mathbf{A}^*).$$

Observability, controllability, simplicity, and minimality of a discrete time system is defined in exactly the same way as in continuous time, with continuous time trajectories replaced by discrete time trajectories.

The system (5.1) is *scattering passive* if it is true for all $x_0 \in X$, all input sequences $u_k \in U$, and all $m = 0, 1, 2, \ldots$ that

$$|x_{m+1}|_X^2 - |x_0|_X^2 \le \sum_{k=0}^{m} |u_k|_U^2 - \sum_{k=0}^{m} |y_k|_Y^2.$$

It can easily be seen that this condition holds for all $m = 0, 1, 2, \ldots$ if and only if it holds for $m = 0$, and this is true if and only if $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ is a contraction from $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ with the natural norms:

$$\left|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right|^2_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} = |x|_X^2 + |u|_U^2, \qquad \left|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right|^2_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} = |x|_X^2 + |u|_U^2.$$

The system is *scattering energy preserving* if we have equality above, and it is *scattering conservative* if both the original system and the dual system are energy preserving. The main operator $\mathbf{A}$ of every discrete time scattering passive system is a contraction, so $\widehat{\mathbf{D}}$ is defined and analytic (at least) on $\mathbb{D}^+ = \{z \in \mathbf{C} \mid |z| > 1\} \cup \{\infty\}$. It is well-known that $\widehat{\mathbf{D}}(z)$ is a *contractive analytic function on* $\mathbb{D}^+$ (a *Schur function*), i.e., $\widehat{\mathbf{D}}(z)$ is a contraction for every $z \in \mathbb{D}^+$.

Discrete time impedance passive systems are defined in an analogous way. The system (5.1) is *impedance passive* if it is true for all $x_0 \in X$, all input sequences $u_k \in U$, and all $m = 0, 1, 2, \ldots$ that

$$|x_{m+1}|_X^2 - |x_0|_X^2 \le \sum_{k=0}^{m} 2\Re\langle u_k, y_k\rangle_U.$$

Again, this condition holds for all $m = 0, 1, 2, \ldots$ if an only if it holds for $m = 0$, and this is true if and only if $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ satisfies the operator inequality

$$\begin{bmatrix} \mathbf{A}^* \\ \mathbf{B}^* \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} \le \begin{bmatrix} 1 & \mathbf{C}^* \\ \mathbf{C} & \mathbf{D} + \mathbf{D}^* \end{bmatrix}.$$

The system is *impedance energy preserving* if we have equality above, and it is *impedance conservative* if both the original system and the dual system are energy preserving. The main operator $\mathbf{A}$ of every impedance passive system is a contraction, so $\widehat{\mathbf{D}}$ is defined (at least) on $\mathbb{D}^+$. It is well-known that (5.1) is passive if and only if the dual system is passive, and that $\widehat{\mathbf{D}}(z)$ is a *positive analytic function on* $\mathbb{D}^+$.

6. The Discrete Time Diagonal Transform. There is a simple transform which maps a (discrete time) impedance passive system into a scattering passive system. Following [14], we shall refer to this transform as the *diagonal transform*. Usually the parameter β below is taken to be one, or possibly to be a positive number, but we shall allow β to be any number in $\mathbb{C}^+$.

Let $\beta \in \mathbb{C}^+$, and suppose that $\beta + \mathbf{D}$ is invertible. Then we can replace the original input u_k in (5.1) by a new independent input $u_k^\times = \frac{1}{\sqrt{2\Re\beta}}(\beta u_k + y_k)$, and at the same time replace the original output y_k in (5.1) by a new output $y_k^\times = \frac{1}{\sqrt{2\Re\beta}}(\overline{\beta} u_k - y_k)$ to get the new discrete time system

$$\begin{aligned} x_{k+1}^\times &= \mathbf{A}^\times x_k^\times + \mathbf{B}^\times u_k^\times, \\ y_k^\times &= \mathbf{C}^\times x_k^\times + \mathbf{D}^\times u_k^\times, \qquad k = 0,1,2,\dots, \\ x_0^\times &= \text{given}, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \mathbf{\Sigma}^\times &= \begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}(\beta+\mathbf{D})^{-1}\mathbf{C} & \sqrt{2\Re\beta}\,\mathbf{B}(\beta+\mathbf{D})^{-1} \\ -\sqrt{2\Re\beta}\,(\beta+\mathbf{D})^{-1}\mathbf{C} & (\overline{\beta}-\mathbf{D})(\beta+\mathbf{D})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2\Re\beta} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & \overline{\beta}-\mathbf{D} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{C} & \beta+\mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix}. \end{aligned} \tag{6.2}$$

Equivalently,

$$\begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & 1+\mathbf{D}^\times \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{C} & \beta+\mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix}. \tag{6.3}$$

It is not difficult to show that the transfer function of the new system $\mathbf{\Sigma}^\times$ is $\widehat{\mathbf{D}}^\times(z) = (\overline{\beta} - \widehat{\mathbf{D}}(z))(\beta + \widehat{\mathbf{D}}(z))^{-1}$. We shall refer to the transform from $\mathbf{\Sigma}$ to $\mathbf{\Sigma}^\times$ as the *(discrete time) diagonal transform* with parameter $\beta \in \mathbb{C}^+$. The *inverse diagonal transform* with the same parameter β is the mapping from $\mathbf{\Sigma}^\times$ to $\mathbf{\Sigma}$, and it is explicitly given by

$$\begin{aligned} \mathbf{\Sigma} &= \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\times - \mathbf{B}^\times(1+\mathbf{D}^\times)^{-1}\mathbf{C}^\times & \sqrt{2\Re\beta}\,\mathbf{B}^\times(1+\mathbf{D}^\times)^{-1} \\ -\sqrt{2\Re\beta}\,(1+\mathbf{D}^\times)^{-1}\mathbf{C}^\times & (\overline{\beta}-\beta\mathbf{D}^\times)(1+\mathbf{D}^\times)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2\Re\beta} \end{bmatrix} \begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ -\beta\mathbf{C}^\times & \overline{\beta}-\beta\mathbf{D}^\times \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{C}^\times & 1+\mathbf{D}^\times \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix}. \end{aligned} \tag{6.4}$$

Equivalently,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \beta+\mathbf{D} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix} \begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{C}^\times & 1+\mathbf{D}^\times \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix}. \tag{6.5}$$

This corresponds to a change of input and output variables from $\left[\begin{smallmatrix} u^\times \\ y^\times \end{smallmatrix}\right]$ in (6.1) to $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in (5.1) according to the rule $u_k = \frac{1}{\sqrt{2\Re\beta}}(u_k^\times + y_k^\times)$ and $y_k = \frac{\sqrt{2}}{\sqrt{\Re\beta}}(u_k^\times - y_k^\times)$. Observe that the inverse diagonal transform coincides with the direct diagonal transform if (and only if) $\beta = 1$ (and this explains why the choice $\beta = 1$ is the most common one).

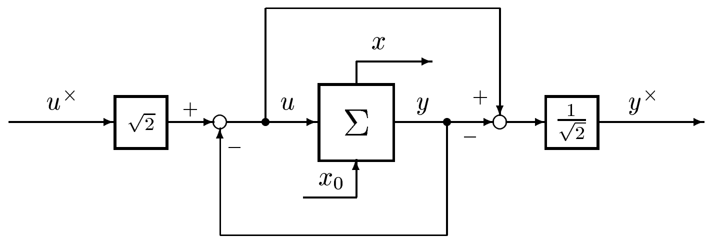

FIG. 1. *The diagonal transform with* $\beta = 1$

Clearly, the diagonal transform described above is well-defined (i.e., it produces a new discrete time system with bounded operators $\mathbf{A}^\times$, $\mathbf{B}^\times$, $\mathbf{C}^\times$, and $\mathbf{D}^\times$) if and only if $-\beta \notin \sigma(\mathbf{D})$, i.e., if and only if $\beta + \mathbf{D}$ is invertible. Furthermore, it is also clear (by construction) that the state trajectory $\{x_k^\times\}_{k=0}^\infty$ of (6.1) coincides with the state trajectory $\{x_k\}_{k=0}^\infty$ of (5.1) if $x_0^\times = x_0$ and $u_k^\times = \frac{1}{\sqrt{2\Re\beta}}(\beta u_k + y_k)$ for $k \ge 0$. Moreover, in this case the two outputs are related by $y_k^\times = \frac{1}{\sqrt{2\Re\beta}}(\beta u_k - y_k)$ for $k \ge 0$. These relationships have specifically been chosen in such a way that

$$|u_k^\times|_U^2 - |u_k^\times|_U^2 = 2\Re\langle y_k, u_k\rangle.$$

This immediately implies the following lemma.

LEMMA 6.1.

(i) *A discrete time system* $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ *is impedance passive (or energy preserving or conservative) if and only if the diagonal transform is well-defined for some* $\beta \in \mathbb{C}^+$ *(or equivalently, for all* $\beta \in \mathbb{C}^+$*) and the diagonally transformed system* $\mathbf{\Sigma}^\times = \begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{bmatrix}$ *is scattering passive (or energy preserving or conservative).*

(ii) *A discrete time scattering passive system* $\mathbf{\Sigma}^\times = \begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{bmatrix}$ *is the diagonal transform for some* $\beta \in \mathbb{C}^+$ *(or equivalently, for all* $\beta \in \mathbb{C}^+$*) of a impedance passive system* $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ *if and only if* $-1 \notin \sigma(\mathbf{D}^\times)$.

(iii) *The two systems have identical controllability properties: if one of the two systems* $\mathbf{\Sigma}$ *and* $\mathbf{\Sigma}^\times$ *is controllable (or observable, or simple, or minimal) then so is the other, and their reachable, unreachable, observable, and unobservable subspaces coincide.*

Thus, a discrete time scattering passive system can be regarded as a slightly more general object than a discrete time impedance passive system, since the diagonal transform maps the latter class into a subclass of the former.

Let us now look at the *realization problem*: which contractive or positive analytic functions on $\mathbb{D}^+$ can be interpreted as the transfer functions of a scattering or impedance conservative system? The answer to the first question is given in the following theorem.

THEOREM 6.1 ([31, Section VI.3, pp. 248–259]). *Every contractive analytic function on $\mathbb{D}^+$ has a simple discrete time scattering conservative realization, which is unique modulo a unitary similarity transform in the state space.*[9]

This is a slightly reformulated version of the results presented in [31, Section VI.3, pp. 248–259]. From this result, combined with Lemma 6.1 we can derive the following analogous realization result for positive analytic functions.

THEOREM 6.2. *Every positive analytic function on $\mathbb{D}^+$ has a simple discrete time impedance conservative realization, which is unique modulo a unitary similarity transform in the state space.*[10]

Proof. Let $\widehat{\mathbf{D}}$ be a discrete time positive analytic function. Then $z \mapsto \widehat{\mathbf{D}}^\times(z) = (1-\widehat{\mathbf{D}}(z))(1+\widehat{\mathbf{D}}(z))^{-1}$ is a contractive analytic function, so, by Theorem 6.1 it has a (essentially unique) simple scattering conservative realization, which we choose to denote by $\mathbf{\Sigma}^\times$. The feedthrough operator $\mathbf{D}^\times$ of this realization is given by

$$\mathbf{D}^\times = \widehat{\mathbf{D}}(\infty)^\times = (1-\widehat{\mathbf{D}}(\infty))(1+\widehat{\mathbf{D}}(\infty))^{-1} = (1-\mathbf{D})(1+\mathbf{D})^{-1},$$

so $1+\mathbf{D}^\times = 2(1+\mathbf{D})^{-1}$ is invertible. We can therefore apply the (inverse) diagonal transform to the system $\Sigma^\times$ to get a simple impedance conservative realization of $\widehat{\mathbf{D}}$, which is still unique modulo a unitary similarity transform in the state space. □

7. The Cayley Transform. Above we have shown how to pass from an impedance passive (or energy preserving or conservative) system to a scattering passive (or energy preserving or conservative) system by using the diagonal transform. There is another very similar transform that is in widespread use to pass from a continuous time system to a discrete time system, namely the Cayley transform. If $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a system node on (U, X, Y) with main operator A, control operator B, observation operator C, and transfer function $\widehat{\mathfrak{D}}$, and if, in addition, $\alpha \notin \sigma(A)$, $\alpha \in \mathbb{C}^+$, then we define the *Cayley transform* with parameter α of S to be the discrete time

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[9] To get this result from [31] it suffices to combine [31, Theorem 3.1, p. 255] with the decomposition of a contractive function into a unitary part and a strictly contractive part described in [31, Proposition 2.1, p. 188]. The same theorem is proved by a different method in [4, Theorem 5.1], and there it is credited to [5].

[10] We do not know whom this theorem should be credited.

system

$$\begin{aligned}\mathbf{\Sigma} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} &= \begin{bmatrix}(\overline{\alpha}+A)(\alpha-A)^{-1} & \sqrt{2\Re\alpha}\,(\alpha-A_{|X})^{-1}B\\ \sqrt{2\Re\alpha}\,C(\alpha-A)^{-1} & \widehat{\mathfrak{D}}(\alpha)\end{bmatrix}\\ &= \begin{bmatrix}1/\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}\overline{\alpha} & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}A\&B\\ C\&D\end{bmatrix}\right)\\ &\qquad\times\left(\begin{bmatrix}\alpha & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}A\&B\\ 0 \quad 0\end{bmatrix}\right)^{-1}\begin{bmatrix}\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix}.\end{aligned} \tag{7.1}$$

Equivalently,

$$\begin{aligned}\begin{bmatrix}\mathbf{A}+1 & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} &= \begin{bmatrix}\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 \quad 0\\ C\&D\end{bmatrix}\\ &\qquad\times\left(\begin{bmatrix}\alpha & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}A\&B\\ 0 \quad 0\end{bmatrix}\right)^{-1}\begin{bmatrix}\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix}.\end{aligned} \tag{7.2}$$

The (discrete time) transfer function of this system is given by $\widehat{\mathbf{D}}(z) = \widehat{\mathfrak{D}}(\frac{\alpha z-\overline{\alpha}}{z+1})$. Note the similarity to (6.2), which becomes even more striking if we replace $A\&B$ by $\begin{bmatrix}A & B\end{bmatrix}$ and $C\&D$ by $\begin{bmatrix}C & D\end{bmatrix}$ (which is permitted when, e.g., A is bounded). If, for example, S is continuous time scattering or impedance passive, then A is a maximal dissipative operator, so in this case the Cayley transform is well-defined for all parameters $\alpha \in \mathbb{C}^+$. Observe that

$$\mathbf{A}+1 = 2\Re\alpha(\alpha-A)^{-1},$$

$$\begin{bmatrix}1/\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{A}+1 & \mathbf{B}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1/\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix} = \left(\begin{bmatrix}\alpha & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}A\&B\\ 0 \quad 0\end{bmatrix}\right)^{-1}$$

so both the operators above are one-to-one, the former maps X onto X_1, and the latter maps $\left[\begin{smallmatrix}X\\ U\end{smallmatrix}\right]$ onto $\mathcal{D}(S)$. This makes it possible to invert the Cayley transform in the form

$$\begin{aligned}S = \begin{bmatrix}A\&B\\ C\&D\end{bmatrix} &= \begin{bmatrix}1/\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha\mathbf{A}-\overline{\alpha} & \alpha\mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\\ &\qquad\times\begin{bmatrix}\mathbf{A}+1 & \mathbf{B}\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix}.\end{aligned} \tag{7.3}$$

Equivalently,

$$\begin{bmatrix}A\&B\\ C\&D\end{bmatrix} - \begin{bmatrix}\alpha & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{A}+1 & \mathbf{B}\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}\sqrt{2\Re\alpha} & 0\\ 0 & 1\end{bmatrix}. \tag{7.4}$$

More specifically, the different operators needed in Lemma 2.1 to construct the node S are given by

$$\begin{aligned} A &= (\alpha\mathbf{A} - \overline{\alpha})(\mathbf{A}+1)^{-1}, & B &= \frac{1}{\sqrt{2\Re\alpha}}(\alpha - A_{|X})\mathbf{B}, \\ C &= \sqrt{2\Re\alpha}\,\mathbf{C}(\mathbf{A}+1)^{-1}, & \widehat{\mathfrak{D}}(s) &= \widehat{\mathbf{D}}\Big(\frac{\overline{\alpha}+s}{\alpha-s}\Big). \end{aligned} \tag{7.5}$$

These considerations lead us to the following conclusion:

LEMMA 7.1. *A discrete time system $\mathbf{\Sigma} = \left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ on (U,X,Y) is the Cayley transform with parameter $\alpha \in \mathbb{C}^+$ of a continuous time system node S on (U,X,Y) if and only if -1 is not an eigenvalue of $\mathbf{A}$ and the operator $A = (\alpha\mathbf{A} - \overline{\alpha})(\mathbf{A}+1)^{-1}$ is the generator of a C_0-semigroup on X.*

The same result (with $\alpha = 1$) is found in [2, Proposition 5.2], where it is used as a central tool which permits a number of results that have earlier been proved for discrete time systems to be transferred to continuous time.

Specializing this result to the case where the original node is scattering or impedance passive we get the following result.

THEOREM 7.1.

(i) *A continuous time system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D\end{smallmatrix}\right]$ is scattering or impedance passive (or energy preserving or conservative) if and only if the Cayley transform of this system is well-defined for some parameter $\alpha \in \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$) and the Cayley transformed system $\mathbf{\Sigma} = \left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ is a discrete time scattering or impedance passive (or energy preserving or conservative) system.*

(ii) *A discrete time scattering or impedance passive system $\mathbf{\Sigma} = \left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ is the Cayley transform for some parameter $\alpha \in \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$) of a continuous time scattering or impedance passive system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D\end{smallmatrix}\right]$ if and only if -1 is not an eigenvalue of $\mathbf{A}$.*

(iii) *The two systems have identical controllability properties: if one of S and $\mathbf{\Sigma}$ is controllable (or observable, or simple, or minimal) then so is the other, and their reachable, unreachable, observable, and unobservable subspaces coincide.*

The scattering version of this theorem (with $\alpha = 1$) is found in [2, Theorem 5.2].

Proof. (i) The necessity of the given conditions on $\mathbf{\Sigma}$ follows from Theorems 3.1 and 4.1 and Lemma 7.1.

(ii) The sufficiency of the same conditions follows from the same results if we are able to show that in all cases $A = (\alpha\mathbf{A} - \overline{\alpha})(\mathbf{A}+1)^{-1}$ is the generator of a C_0-semigroup. In all cases $\mathbf{A}$ is a contraction, and this implies that the operator A given above is dissipative (for all $\alpha \in \mathbb{C}^+$). Since $\alpha - A = 2\Re\alpha(1+\mathbf{A})^{-1}$, we find that $\alpha - A$ has the bounded inverse $1/\sqrt{2\Re\alpha}\,(1+\mathbf{A})^{-1}$. Thus, $\alpha \in \rho(A)$, so A is maximal dissipative, and A generates a C_0 (contraction) semigroup.

(iii) This follows from the fact that there is a one-to-one correspondence between the trajectories and outputs of the discrete time system (6.1) and the continuous time system (2.1): we take the discrete time values of input, state, and output to be the Fourier coefficients of the continuous times input, state, and output with respect to a Laguerre basis (scaled by the parameter α). See [28, Section 11.4] for details. □

In [2] the above result was used to prove the following realization result:

THEOREM 7.2 ([2, Theorem 6.4]). *Every contractive analytic function on $\mathbb{C}^+$ can be realized as the transfer function of a simple continuous time scattering conservative system node, which is unique modulo a unitary similarity transform in the state space.*

Let us now combine the two transforms. Starting from a continuous time impedance passive system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, we may first use the Cayley transform with parameter $\alpha \in \mathbb{C}^+$ to get a discrete time impedance passive system $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$, and then we may use the diagonal transform with parameter $\beta \in \mathbb{C}^+$ to get the discrete time scattering passive system $\mathbf{\Sigma}^\times = \left[\begin{smallmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{smallmatrix}\right]$. The class of all discrete time scattering passive system that can be obtained in this way is described in the following lemma.

THEOREM 7.3. *A discrete time system $\mathbf{\Sigma}^\times = \left[\begin{smallmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{smallmatrix}\right]$ on (U, X, U) is the diagonal transform with parameter $\beta \in \mathbb{C}^+$ of the Cayley transform with the parameter $\alpha \in \mathbb{C}^+$ of a continuous time impedance passive system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ if and only $\mathbf{\Sigma}^\times$ is a (discrete time) scattering passive system, -1 is not an eigenvalue of $\left[\begin{smallmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{smallmatrix}\right]$, and $-1 \notin \sigma(\mathbf{D}^\times)$. In this case*

$$\begin{aligned} &\begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2\Re\alpha} & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix} \left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \right)^{-1} \begin{bmatrix} \sqrt{2\Re\alpha} & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix}, \end{aligned} \tag{7.6}$$

and $\mathbf{\Sigma}^\times$ is also the Cayley transform (with the same parameter α) of a continuous time scattering passive system node $S^\times$.

Proof. We fix the two parameters α, $\beta \in \mathbb{C}^+$ once and for all. Suppose that $\mathbf{\Sigma}^\times = \left[\begin{smallmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{smallmatrix}\right]$ is the diagonal transform (with parameter β) of the Cayley transform $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ (with parameter α) of a continuous time impedance passive system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. Then $\mathbf{\Sigma}^\times$ is a discrete time scattering passive system, and $-1 \notin \sigma(\mathbf{D}^\times)$ since $1 + \mathbf{D}^\times = 2\Re\alpha(\alpha + \mathbf{D})^{-1}$ has a bounded inverse. A short algebraic computation based on (6.3) and (7.2) then gives (7.6). Clearly (7.6) implies that $\mathbf{\Sigma}^\times + 1$ is one-to-one, i.e., -1 is not an eigenvalue of $\mathbf{\Sigma}^\times$. Since $\mathbf{\Sigma}^\times = \left[\begin{smallmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{smallmatrix}\right]$ is a contraction, this implies that -1 is not an eigenvalue of $\mathbf{A}^\times$ either, and by Theorem 7.1, this implies that $\mathbf{\Sigma}^\times$ is the Cayley transform of a continuous time scattering passive system node $S^\times$.

Conversely, suppose that $\mathbf{\Sigma}^\times$ is a discrete time scattering passive system such that -1 is not an eigenvalue of $\mathbf{\Sigma}^\times$ and $-1 \notin \sigma(\mathbf{D}^\times)$. Then the inverse diagonal transform (with parameter β) can be applied to $\mathbf{\Sigma}^\times$, and this transform gives us a discrete time impedance passive system $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$. Clearly, formula (6.3) can be rewritten in the form

$$\begin{aligned}
&\begin{bmatrix} \mathbf{A}^\times + 1 & \mathbf{B}^\times \\ \mathbf{C}^\times & 1 + \mathbf{D}^\times \end{bmatrix} \\
&\quad = \begin{bmatrix} 1 & \frac{1}{\sqrt{2\Re\beta}}\mathbf{B} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A} + 1 & 0 \\ 0 & 2\Re\beta\,(\beta + \mathbf{D})^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{\sqrt{2\Re\beta}}\mathbf{C} & 1 \end{bmatrix}
\end{aligned}$$

and this shows that $\mathbf{A}+1$ must be one-to-one since $\mathbf{\Sigma}^\times + 1$ is one to one. By Theorem 7.1, $\mathbf{\Sigma}$ is the Cayley transform (with parameter α) of a continuous time impedance passive system node. ◻

THEOREM 7.4. *A necessary and sufficient condition for a $\mathcal{L}(U)$-valued positive analytic function $\widehat{\mathfrak{D}}$ on $\mathbb{C}^+$ to have a simple impedance conservative realization is that*

$$\lim_{s\to+\infty} s^{-1}\widehat{\mathfrak{D}}(s)u = 0 \tag{7.7}$$

for all $u \in U$. This realization is unique modulo a unitary similarity transform in the state space.

Thus, the only positive analytic functions that cannot be realized in this way are those that contain a pure derivative action.

Proof. Let us begin by proving the necessity of condition (7.7). Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node whose transfer function is $\widehat{\mathfrak{D}}$. Then

$$\frac{\widehat{\mathfrak{D}}(s) - \widehat{\mathfrak{D}}(1)}{s-1} = -C(s-A)^{-1}(1-A_{|X})^{-1}B.$$

Here the right-hand side tends strongly to zero as $s \to +\infty$ (since $B \in \mathcal{L}(U; X_{-1})$ and $C \in \mathcal{L}(X_1; U)$), and this implies (7.7).

Conversely, suppose that $\widehat{\mathfrak{D}}$ is a positive analytic function satisfying (7.7). By applying the same formula which is used in the Cayley transform (with $\alpha = 1$) to $\widehat{\mathfrak{D}}$ we get the discrete time positive analytic function

$$\widehat{\mathbf{D}}(z) = \widehat{\mathfrak{D}}\Big(\frac{z-1}{z+1}\Big).$$

By Theorem 6.2, this function has a simple impedance conservative realization $\mathbf{\Sigma} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ which is unique modulo a unitary similarity transform in the state space. If we can show that -1 is not an eigenvalue of $\mathbf{A}$, then we can use the inverse Cayley transform to get a simple impedance conservative system node S whose transfer function is the given function $\widehat{\mathfrak{D}}$. Thus, to complete the proof we still need to show that -1 is not an eigenvalue of $\mathbf{A}$.

Recall the conservativity (in the discrete time impedance setting) of $\mathbf{\Sigma}$ means that

$$\begin{bmatrix}\mathbf{A}^*\mathbf{A} & \mathbf{A}^*\mathbf{B}\\ \mathbf{B}^*\mathbf{A} & \mathbf{B}^*\mathbf{B}\end{bmatrix} = \begin{bmatrix}1 & \mathbf{C}^*\\ \mathbf{C} & \mathbf{D}+\mathbf{D}^*\end{bmatrix}, \qquad \begin{bmatrix}\mathbf{A}\mathbf{A}^* & \mathbf{A}\mathbf{C}^*\\ \mathbf{C}\mathbf{A}^* & \mathbf{C}\mathbf{C}^*\end{bmatrix} = \begin{bmatrix}1 & \mathbf{B}\\ \mathbf{B}^* & \mathbf{D}+\mathbf{D}^*\end{bmatrix}.$$

In particular, $\mathbf{A}$ is unitary, $\mathbf{B} = \mathbf{A}\mathbf{C}^*$, and $\mathbf{C} = \mathbf{B}^*\mathbf{A}$. The transfer function of this node is given by $\widehat{\mathbf{D}}(z) = \mathbf{C}(z-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \widehat{\mathfrak{D}}((z-1)/(z+1))$. After a short algebraic computation we get the alternative formulas

$$\begin{aligned}\widehat{\mathbf{D}}(z) &= \mathbf{B}^*\mathbf{A}(z-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \mathbf{C}(z-\mathbf{A})^{-1}\mathbf{A}\mathbf{C}^* + \mathbf{D}\\ &= z\mathbf{B}^*(z-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}^* = z\mathbf{C}(z-\mathbf{A})^{-1}\mathbf{C}^* + \mathbf{D}^*.\end{aligned} \tag{7.8}$$

Let X_0 be the eigenspace of $\mathbf{A}$ corresponding to the eigenvalue -1 (which we want to prove to be $\{0\}$). Let $X_0^\perp$ be its orthogonal complement. Then both X_0 and $X_0^\perp$ are invariant under A (since $\mathbf{A}$ is unitary), and we can decompose the discrete time system $\mathbf{\Sigma}$ into

$$\mathbf{\Sigma} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \left[\begin{array}{cc|c}-1 & 0 & \mathbf{B}_0\\ 0 & \mathbf{A}_1 & \mathbf{B}_1\\ \hline \mathbf{C}_0 & \mathbf{C}_1 & \mathbf{D}\end{array}\right],$$

in accordance with the decomposition of X into $X = \left[\begin{smallmatrix}X_0\\ X_0^\perp\end{smallmatrix}\right]$. The corresponding decomposition of the transfer function $\widehat{\mathbf{D}}$ is

$$\begin{aligned}\widehat{\mathbf{D}}(z) &= \mathbf{D}^* + z\begin{bmatrix}\mathbf{B}_0^* & \mathbf{B}_1^*\end{bmatrix}\begin{bmatrix}z+1 & 0\\ 0 & z-\mathbf{A}_1\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{B}_0\\ \mathbf{B}_1\end{bmatrix}\\ &= \mathbf{D}^* + \frac{z}{z+1}\mathbf{B}_0^*\mathbf{B}_0 + z\mathbf{B}_1^*(z-\mathbf{A}_1)^{-1}\mathbf{B}_1.\end{aligned}$$

Since $\mathbf{A}_1$ is unitary (hence normal), it has a spectral representation: there is a (self-adjoint projection-valued) resolution of the identity E such that for all $x \in X$ and all $z \in \rho(\mathbf{A}_1)$,

$$\begin{aligned}\langle \mathbf{A}x, x\rangle &= \int_{|\zeta|=1} \zeta\langle x, E(\,d\zeta)x\rangle,\\ \langle (z-\mathbf{A})^{-1}x, x\rangle &= \int_{|\zeta|=1} \frac{1}{z-\zeta}\langle x, E(\,d\zeta)x\rangle.\end{aligned}$$

Thus, the transfer function $\widehat{\mathbf{D}}$ has the representation (for all $u \in U$ and all $z \in \rho(\mathbf{A}_1)$)

$$\langle \widehat{\mathbf{D}}(z)u, u\rangle = \langle \mathbf{D}^*u, u\rangle + \frac{z}{z+1}\|\mathbf{B}_0 u\|^2 + \int_{|\zeta|=1} \frac{z}{z-\zeta}\langle \mathbf{B}_1 u, E(\,d\zeta)\mathbf{B}_1 u\rangle. \tag{7.9}$$

The spectral representation of $\mathbf{A}_1$ does not have a point mass at -1 (such a point mass would be the orthogonal projection onto X_0), and therefore, the finite positive measure $\zeta \mapsto \langle \mathbf{B}_1 u, E(\,d\zeta)\mathbf{B}_1 u\rangle$ does not have a point mass at -1 either. By the Lebesgue dominated convergence theorem, for all $u \in U$,

$$\begin{aligned}
&\lim_{z\uparrow -1}(z+1)\langle u, \widehat{\mathbf{D}}(z)u\rangle \\
&= -\|\mathbf{B}_0 u\|^2 + \lim_{z\uparrow -1}\int_{|\zeta|=1}\frac{z(z-1)}{z-\zeta}\langle \mathbf{B}_1 u, E(\,d\zeta)\mathbf{B}_1 u\rangle \\
&= -\|\mathbf{B}_0 u\|^2.
\end{aligned}$$

After a change of variable $z = (1+s)/(1-s)$ this becomes

$$\lim_{s\to+\infty}\frac{1}{s-1}\langle \widehat{\mathfrak{D}}(s)u, u\rangle = \frac{1}{2}\|\mathbf{B}_0 u\|^2.$$

If (7.7) holds, then the limit on the left-hand side is zero for all $u \in U$. Therefore $\mathbf{B}_0 = 0$. Decomposing the identity $\mathbf{B} = \mathbf{A}\mathbf{C}^*$ into its components we find that

$$\mathbf{B} = \begin{bmatrix}\mathbf{B}_0\\ \mathbf{B}_1\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & \mathbf{A}_1\end{bmatrix}\begin{bmatrix}\mathbf{C}_0^*\\ \mathbf{C}_1^*\end{bmatrix} = \begin{bmatrix}-\mathbf{C}_0^*\\ \mathbf{A}_1\mathbf{C}_1^*\end{bmatrix};$$

hence also $\mathbf{C}_0^* = 0$. This implies that the realization $\boldsymbol{\Sigma} = \left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ cannot be simple unless $X_0 = \{0\}$, since X_0 is both unreachable and unobservable. But by construction, the system $\boldsymbol{\Sigma}$ is simple, and therefore $X_0 = \{0\}$. □

As a byproduct of the proof of this theorem we get the *Herglotz-Nevanlinna integral representation formula* for a positive analytic function on $\mathbb{C}^+$ (see, e.g., [45]).[11]

The realizations described in Theorems 6.1, 6.2, 7.2 and 7.4 will not be minimal in general. However, from these realization we can derive minimal realizations, e.g., as follows (see [2, Section 7] or [28, Section 9.1] for details). We proceed in two steps. Let $\mathcal{R}$ be the reachable subspace of Σ. By 'restricting Σ to $\mathcal{R}$' we get a controllable system Σ_1 on $(U, \mathcal{R}, Y)$ whose main operator is $A_1 = A_{|\mathcal{R}}$, control operator is $B_1 = B$, observation operator is $C_1 = C_{|\mathcal{R}}$, and transfer function $\widehat{\mathfrak{D}}$ is the same as the original transfer function. It is not difficult to show that if the original system is conservative (scattering or impedance), then the new system is a energy preserving (scattering or impedance), and it is unique among all controllable energy preserving (scattering or impedance) realizations of $\widehat{\mathfrak{D}}$ modulo a unitary similarity transform in the state space. This implies the following result:

Corollary 7.1. *Theorems 6.1, 6.2, 7.2, and 7.4 remain true in the controllable energy preserving setting: every discrete or continuous time*

[11] Use the Cayley transform to map (7.9) into continuous time.

impedance or scattering passive transfer function has a controllable energy preserving realization, which is unique modulo a unitary similarity transform in the state space (in the continuous time impedance setting we still have to impose the extra necessary and sufficient additional condition (7.7)*).*

Here, by an 'impedance passive transfer function' we mean a positive analytic function, and by a 'scattering passive transfer function' we mean a contractive analytic function.

If the system Σ_1 that we constructed above is observable, then we have obtained a minimal passive realization. If not, then we let $\mathcal{O}_1$ be the observable subspace of Σ_1, denote the orthogonal projection of $\mathcal{R}$ onto $\mathcal{O}_1$ by π, and 'project Σ_1 onto $\mathcal{O}_1$' to get the minimal system Σ_2 on $(U, \mathcal{O}_1, Y)$ whose main operator is $A_2 = \pi A_1 = \pi A_{|\mathcal{R}}$, control operator is $B_1 = \pi B_1 = \pi B$, observation operator is $C_2 = C_{1|\mathcal{R}} = C_{|\mathcal{R}}$, and transfer function $\widehat{\mathfrak{D}}$ is still the same as the original transfer function. This system is passive (scattering or impedance) whenever Σ_1 is passive. Thus, we arrive at the following result:

COROLLARY 7.2. *Every discrete or continuous time impedance or scattering passive function has a minimal passive realization (in the continuous time impedance setting we still have to impose the extra necessary and sufficient additional condition* (7.7)*).*

The above realizations are not unique in general.[12] For example, we could, instead first have projected the system onto the observable subspace to get a system whose adjoint is energy preserving, and then restricted the new system to the reachable subspace. It is possible to make them unique by requiring them to be 'optimal' in a certain sense.[13] See [2, Section 7] and [44, Section 4] for details.

8. The Continuous Time Diagonal Transform. In Section 6 we defined the discrete time diagonal transform. The same transform can also be carried out in continuous time. One way to do this is to first perform a Cayley transform, then a discrete time diagonal transform, and finally an inverse Cayley transform to get a new continuous time system, which we call the diagonal transform of the original continuous time system.

DEFINITION 8.1. *The system node* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *on* (U, X, Y) *is* diagonally transformable *with parameter* $\beta \in \mathbb{C}^+$ *if it is true for some* $\alpha \in \mathbb{C}^+$ *that the operator* $\left(\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right] - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]\right)$ *maps* $\mathcal{D}(S)$ *one-to-one onto* $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$*, and if, in addition, the operator* $\mathbf{A}^\times$ *defined in* (7.6) *is the Cayley transform*

[12] One exceptional case is described in Corollary 10.2. A complete discussion of the uniqueness of a minimal scattering passive realization modulo a unitary similarity transform in the state space is given in [1, Theorem 12].

[13] The above construction produces Arov's optimal realization. This is the minimal realization which uses the norm in the state space induced by Willems' *available storage*. If we instead first project onto the observable subspace and then restrict to the reachable subspace, then we get Arov's $*$-optimal realization. This is the realization which uses the norm induced by Willems' *required supply* function.

with parameter α *of the generator* $A^\times$ *of a* C_0 *semigroup. In this case we refer to the system node* $S^\times = \left[\begin{smallmatrix}[A\&B]^\times \\ [C\&D]^\times\end{smallmatrix}\right]$ *that we get by applying the inverse Cayley transform with parameter* α *to the discrete time system* $\mathbf{\Sigma}^\times = \left[\begin{smallmatrix}\mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times\end{smallmatrix}\right]$ *defined in* (7.6) *by the name* diagonal transform of S *(with parameter* β*).*

An important fact is that *the diagonal transform* $S^\times$ *of* S *that we get in this way does not depend on the parameter* α (although it depends on β). To see this it suffices to write out the explicit formulas for the two successive transforms to verify the following relationship between $S^\times$ and S (see [26] for the well-posed case of this identity with $\beta = 1$), *valid for all* $\alpha \in \rho(A) \cap \rho(A^\times)$:

$$\begin{aligned}
&\begin{bmatrix} (\alpha - A^\times)^{-1} & (\alpha - A^\times_{|X^\times_{-1}})^{-1} B^\times \\ C^\times(\alpha - A^\times)^{-1} & 1 + \widehat{\mathfrak{D}}^\times(\alpha) \end{bmatrix} \\
&\quad = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix} \left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2\Re\beta} \end{bmatrix} \\
&\quad = \begin{bmatrix} (\alpha - A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\
&\qquad + \begin{bmatrix} (\alpha - A_{|X})^{-1} B \\ \sqrt{2\Re\beta} \end{bmatrix} (\beta + \widehat{\mathfrak{D}}(\alpha))^{-1} \begin{bmatrix} -C(\alpha - A)^{-1} & \sqrt{2\Re\beta} \end{bmatrix}.
\end{aligned} \tag{8.1}$$

It is possible to alternatively introduce the continuous time diagonal transform without use of the Cayley transform in the same way as was done in [26, Section 5] in the well-posed case. This alternative approach uses a generalized version of flow-inversion for non-well-posed systems described in [28, Section 6.3]. Arguing in this way we find that S is diagonally transferable (with parameter $\beta \in \mathbb{C}$) if and only if $-\beta \notin \sigma(\widehat{\mathfrak{D}}(\alpha))$ for some $\alpha \in \rho(A)$, the operator

$$R(\alpha, A^\times) = \big(1 - (\alpha - A_{|X})^{-1} B(\beta + \widehat{\mathfrak{D}}(\alpha))^{-1} C\big)(\alpha - A)^{-1}$$

is one-to-one, and the operator

$$A^\times = \alpha - (R(\alpha, A^\times))^{-1}$$

is the generator of a C_0 semigroup.

The following result is a direct consequence of Theorem 7.3 combined with Definition 8.1.

Theorem 8.1.

(i) *A system node* $S = \left[\begin{smallmatrix}A\&B \\ C\&D\end{smallmatrix}\right]$ *on* (U, X, U) *is impedance passive (or energy preserving or conservative) if and only if it is diagonally transformable for some parameter* $\beta \in \mathbb{C}^+$ *(or equivalently, for all* $\beta \in \mathbb{C}^+$*), and the diagonally transformed system node*

$S^\times = \begin{bmatrix} [A\&B]^\times \\ [C\&D]^\times \end{bmatrix}$ is scattering passive (or energy preserving, or conservative) (in particular, it is well-posed).

(ii) *A scattering passive system node $S^\times = \begin{bmatrix} [A\&B]^\times \\ [C\&D]^\times \end{bmatrix}$ with transfer function $\widehat{\mathfrak{D}}^\times$ on (U, X, U) is the diagonal transform for some $\beta \in \mathbb{C}^+$ (or equivalently, for all $\beta \in \mathbb{C}^+$) of an impedance passive system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ on (U, X, U) if and only if it is true for some $\alpha \in \mathbb{C}^+$ (or equivalently, for all $\alpha \in \mathbb{C}^+$) that $-1 \notin \sigma(\widehat{\mathfrak{D}}^\times(\alpha))$ and that -1 is not an eigenvalue of the Cayley transformed discrete time system $\begin{bmatrix} \mathbf{A}^\times & \mathbf{B}^\times \\ \mathbf{C}^\times & \mathbf{D}^\times \end{bmatrix}$ which is defined as in (7.1) with S replaced by $S^\times$.*

(iii) *The two nodes have identical controllability properties: if one of the two nodes S and $S^\times$ is controllable (or observable, or simple, or minimal) then so is the other, and their reachable, unreachable, observable, and unobservable subspaces coincide.*

Proof. Parts (i) and (ii) follow directly from Theorem 7.3 and Definition 8.1. To get part (iii) we need, in addition, Lemma 6.1 and Theorem 7.1. ∎

Theorem 8.1 can be regarded as a generalization of the main result of [41] by Weiss and Tucsnak (the same result was proved independently by Guo and Luo in [10]). In these papers a specific compatible system node is considered. It represents a second order partial differential equation of hyperbolic type with collocated sensors and actuators. Mathematically, the node S in that paper is of the following type (here we ignore a number of additional properties S that are irrelevant for the present discussion; see [27] for details). The operator A is the generator of a unitary group on a Hilbert space X. We let $\Re\alpha > 0$, define the fractional power $(\alpha - A)^{1/2}$ in the standard way (see, e.g., [28, Section 3.9]), and let $X_{1/2}$ be the domain of $(\alpha - A)^{1/2}$. The node S is supposed to be compatible over $X_{1/2}$, with a representation of the type $\left[\begin{smallmatrix} A & C^* \\ C & 0 \end{smallmatrix}\right]$, where $C \in \mathcal{L}(X_{1/2}; U)$. By Theorem 4.3, this is an impedance conservative system node, and by Theorem 8.1, the transformed node $S^\times$ is scattering conservative (hence well-posed). The latter statement implies the main result of [41]. The node S itself does not appear explicitly in [41], but it does appear in the paper [39] by Weiss. There the diagonal transform is used to give a simple state feedback solution of a particular optimal control problem. See [27] for details.

EXAMPLE 3. *We return to Example 1. The diagonal transform with $\beta = 1$ gives the following transformed system:*

$$\begin{bmatrix} A^\times & B^\times \\ C^\times & D^\times \end{bmatrix} = \begin{bmatrix} A & \sqrt{2}A \\ \sqrt{2}A & 1 + A \end{bmatrix} (1 - A)^{-1}.$$

Note that all the operators above are bounded *on X. The diagonally transformed system is always scattering passive, it is scattering energy preserving if and only if the original system is impedance energy preserving, and* *it is scattering conservative if and only if the original system is impedance conservative.*

EXAMPLE 4. *We return to Example 2. The corresponding diagonally transformed system $S^\times$ can be computed from* (8.1), *and it turns out to be (in compatibility form)*

$$S^\times = \begin{bmatrix} A^\times & B^\times \\ C^\times & D^\times \end{bmatrix} = \begin{bmatrix} A & \sqrt{2}\,|A|^p V \\ \sqrt{2}\,|A|^p V & 1 + |A|^{2p-1} V \end{bmatrix} (1 - |A|^{2p-1} V)^{-1}.$$

This agrees with Example 3 in the case where $p = 1$.

The preceding example has some interesting properties. Comparing the unboundedness of the different operators appearing in S and $S^\times$ to the unboundedness of the original operator A we find the following. The control and observation operators of the original node S are both as unbounded as the operator $|A|^p$, where $p \in [\frac{1}{2}, 1]$. The semigroup generator $A^\times$ of the scattering passive system is only as unbounded as the operator $|A|^{2(1-p)}$ where $2(1-p) < 1$ as soon as $p > \frac{1}{2}$. The control and observation operators of $S^\times$ are as unbounded as $|A|^{1-p}$, so they have only half as much unboundedness as $A^\times$. In particular, for $p = \frac{1}{2}$ the systems S and $S^\times$ have 'the same amount of unboundedness', but for $p > \frac{1}{2}$ 'the more unbounded the system S is, the less unbounded is $S^\times$'. Intuitively, the more unbounded B and C are, the stronger is the feedback, hence more regularizing.

9. A Feedback Interpretation. The diagonal transform in Definition 8.1 has a natural output feedback interpretation. Let us, for simplicity, take $\beta = 1$ as in Figure 1. Then the input-output interpretation of the diagonal transform is that we introduce a new input signal $u^\times$, choose the input of the original system Σ to be $u = \sqrt{2}\,u^\times - y$, and regard the new output signal to be $y^\times = \frac{1}{\sqrt{2}}(u - y)$ (this interpretation is valid both in continuous and in discrete time). If we ignore the trivial scaling factors $\sqrt{2}$ and $1/\sqrt{2}$, then the replacement of u by the new input $u^\times$ is a typical negative identity state feedback, whereas the replacement of y by $y^\times$ just amounts to the addition of an extra feedthrough term to the resulting closed loop system (see Figure 1). Recall that if S is a system node on (U, X, Y), then $K \in \mathcal{L}(Y; U)$ is called an admissible feedback operator for S if the replacement of the input signal u by $u = u^K + Ky$ results in a new system node whose input signal is u^K. In the special case where $U = Y$ considered above we may use *negative identity output feedback*, i.e., we let $K = -1$. Thus, Theorem 8.1 implies the following result:

COROLLARY 9.1. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be an impedance passive system node on (U, X, U). Then -1 is an admissible feedback operator for S, and the closed loop system corresponding to this feedback operator is (well-posed and) energy stable (in the sense of Definition 2.3).*

10. Lossless Scattering Systems. In an energy preserving system no energy is lost, but it may be first transferred from the input to the state, and then 'trapped' in the state space forever, so that it can no longer be retrieved from the outside. Thus, from the point of view of an external observer, a conservative system may be 'lossy'. To specifically exclude this case we need another notion, that we shall refer to as *losslessness.* This notion can be studied both in the scattering and in the impedance setting, but for simplicity we here restrict ourselves to the (simpler) scattering case (we shall return to the slightly more complicated impedance setting elsewhere). The theory is essentially the same in discrete and continuous time, so let us here restrict ourselves to the continuous time setting.

DEFINITION 10.1. *A system node* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *on* (U, X, Y) *is* scattering semi-lossless *if the solution* (x, y) *in Lemma 2.2 satisfies*

$$\int_0^\infty |y(s)|_Y^2 \, ds = \int_0^\infty |u(s)|_U^2 \, ds$$

whenever $x_0 = 0$ *and* $u \in L^2(\mathbb{R}^+; U)$ *(in particular, this implies that* $y \in L^2(\mathbb{R}^+; Y)$*). It is* scattering lossless *if both* S *and the dual system node* S^* *are semi-lossless.*

Thus, semi-losslessness is the input/output version of energy preservation, and losslessness is the input/output version of conservativity.

As is well known, semi-losslessness can be interpreted as a property of the transfer function:

PROPOSITION 10.1. *A system node* S *is scattering semi-lossless if and only if its transfer function* $\widehat{\mathfrak{D}}$ *is left-inner in the following sense:* $\widehat{\mathfrak{D}}$ *has an extension to a contractive analytic function in* $\mathbb{C}^+$*, the restriction of* $\widehat{\mathfrak{D}}$ *to every separable subspace of* U *has a strong limit from the right a.e. at the imaginary axis, and this limit is isometric a.e.*

This proposition follows, e.g., from [28, Theorem 10.4.5] (the original reference is [7]) and [31, Proposition 2.2, p. 190]. (We may always, without loss of generality, assume that U is separable, since the values of u lie in a separable subspace of U.)

If S is passive, or more generally, if the growth bound of S is zero, then $\widehat{\mathfrak{D}}$ is defined (at least) on $\mathbb{C}^+$, so in this case no extension is needed. Furthermore, at every point $\alpha \in \rho(A) \cap i\mathbb{R}$, $\widehat{\mathfrak{D}}(\alpha)$ will be isometric if S is semi-lossless and unitary if S is lossless.

THEOREM 10.1. *A controllable semi-lossless scattering passive system node* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *on* (U, X, Y) *is necessarily scattering energy preserving and observable (hence minimal). Furthermore, in this case the system node* S *is uniquely determined by its transfer function* $\widehat{\mathfrak{D}}$ *within the class of all controllable scattering passive realizations of* $\widehat{\mathfrak{D}}$*, modulo a unitary similarity transform in the state space.*

Proof. We begin by showing that if $x_0 = 0$, then the solution (x, y) in Lemma 2.2 satisfies

$$|x(t)|_X^2 + \int_0^t |y(s)|_Y^2 \, ds = \int_0^t |u(s)|_U^2 \, ds, \qquad t \geq 0. \tag{10.1}$$

Because of the passivity, we know that

$$|x(t)|_X^2 + \int_0^t |y(s)|_Y^2\, ds \le \int_0^t |u(s)|_U^2\, ds, \qquad t \ge 0. \tag{10.2}$$

By the well-posedness, we may further let u be an arbitrary function in $L^2(\mathbb{R}^+;U)$. Fix $v > 0$, and let $u(s) = 0$ for $s \ge v$. Then

$$|x(t)|_X^2 + \int_0^t |y(s)|_Y^2\, ds \le \int_0^v |u(s)|_U^2\, ds, \qquad t \ge v.$$

By the lossless property, $\int_0^\infty |y(s)|_Y^2\, ds = \int_0^v |u(s)|_U^2\, ds$, so by letting $t \to \infty$ we conclude that $x(t) \to 0$ as $t \to \infty$. Furthermore, by passivity and the fact that (x, y) is the state and output of S on the time interval $[v, \infty)$ with initial state $x(v)$ and input function 0, and by (10.2),

$$\begin{aligned} 0 &\le |x(v)|_X^2 - \int_v^\infty |y(s)|_Y^2\, ds \\ &\le \int_0^v |u(s)|_U^2\, ds - \int_0^v |y(s)|_Y^2\, ds - \int_v^\infty |y(s)|_Y^2\, ds = 0. \end{aligned}$$

Thus, both the inequalities in this chain must be equalities, and this implies (10.1).

Let us next suppose that $x_0 = x(v)$ where (x, y) is a solution of the type described in Lemma 2.2 with $x(0) = 0$ for some input function u and some $v \ge 0$. Then, by (10.1), $|x(t)|_X^2 = \int_0^t |u(s)|_U^2\, ds - \int_0^t |y(s)|_Y^2\, ds$ for all $t \ge 0$, and by subtracting two copies of this identity from each other, with t replaced by v in one of them, we get (recall that $x_0 = x(v)$)

$$|x(t)|_X^2 + \int_v^t |y(s)|_Y^2\, ds = |x_0|_X^2 + \int_v^t |u(s)|_U^2\, ds, \quad t \ge v.$$

Denote $u_1(t) = u(t - v)$, $x_1(t) = x(t - v)$, and $y_1(t) = y(t - v)$ for $t \ge 0$. Then the above identity becomes the energy balance equation **(SE)** with u, x, and y replaced by u_1, x_1 and y_1. Since S is assumed to be controllable, the set of data (x_0, u_1) considered above is a dense subset of the full set of initial states and input functions allowed in Definition 3.2, so the system must be energy preserving.

The proof of the uniqueness of the S (modulo a unitary similarity transform in the state space) was a part of the argument leading to Corollary 7.2.

As we observed above, if $u = 0$ (or more generally, if u vanishes on some interval $[t, \infty)$), then $x(t) \to 0$ as $t \to \infty$. This, combined with the fact that S is energy-preserving implies that $|x_0|_X^2 = \int_0^\infty |y(s)|_Y^2\, ds$. Thus, S is observable. □

There are a number of fairly transparent conditions which can be used to determine if a given controllable scattering energy preserving system is semi-lossless. One of them is exact observability in infinite time, which means the following. A system node S is *exactly observable in time* $T > 0$ if there is a constant $\epsilon > 0$ such that the output y in Lemma 2.2 with zero input function u and initial state $x_0 \in X_1$ satisfies 'the reverse inequality'

$$\int_0^T |y(t)|_Y^2 \geq \epsilon |x_0|_X^2.$$

It is *exactly observable in infinite time* if this condition is true with $T = \infty$ (the left-hand side of the above inequality may be $+\infty$ if the system is not energy stable). The system node S is *exactly controllable in time* $T > 0$ if, for every $x_0 \in X$, there is an input function $u \in L^2(0, T; U)$ such that the generalized solution x of the equation $\dot{x}(t) = A_{|X} x(t) + Bu(t)$ (as defined in Section 2) satisfies $x(T) = x_0$. Equivalently, S is exactly controllable in time $T > 0$ if and only if S^* is exactly observable in time T. A direct definition of exact controllability in infinite time is more difficult to state, so we define a system node S to be *exactly controllable in infinite time* if S^* is exactly observable in infinite time.

THEOREM 10.2. *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a scattering energy preserving system node* (Y, X, U)*. Then the following conditions are equivalent:*

(i) *the system semigroup of* S *is strongly stable, i.e., the state* $x(t)$ *in Lemma 2.2 tends to zero as* $t \to \infty$ *whenever* $u = 0$*;*

(ii) *the observability gramian of* S *is the identity operator, i.e., the output* y *in Lemma 2.2 with zero input function* u *and initial state* $x_0 \in X_1$ *satisfies* $\int_0^\infty |y(t)|_Y^2 = |x_0|_Y^2$*;*

(iii) S *is exactly observable in infinite time.*

If these conditions hold, then

(iv) S *is semi-lossless.*

If S *is controllable, then (iv) is equivalent to (i)–(iii).*[14]

Proof. The implication (i) $\Rightarrow$ (ii) follows directly from **(SE)** with $u = 0$. The implication (ii) $\Rightarrow$ (iii) is trivial. Thus, to prove the equivalence of (i)–(iii) only the implication (iii) $\Rightarrow$ (i) remains to be established.

Suppose that S is exactly observable in infinite time. Then it follows from **(SE)** that there exists a constant $\eta \in (0, 1)$ such that, for all $x_0 \in X_1$, there exists some $t_1 > 0$ that the solution x in Lemma 2.2 with $x(0) = x_0$ and $u = 0$ satisfies $|x(t_1)|_X \leq \eta |x_0|_X$. We repeat the same argument with x_0 replaced by $x(t_1)$: there is some $t_2 > t_1$ such that the solution x_1 in Lemma 2.2 with $x_1(0) = x(t_1)$ and $u = 0$ satisfies $|x_1(t_2 - t_1)|_X \leq \eta |x(t_1)|_X \leq \eta^2 |x_0|_X$. However, by the uniqueness of the solution x in Lemma 2.2, we must have $x_1(t) = x(t + t_1)$ for all $t \geq 0$, so $|x(t_2)|_X = |x_1(t_2 - t_1)|_X \leq \eta^2 |x_0|$. Continuing in the same way, for each integer $k > 0$, we can find some t_k such that $|x(t_k)|_X \leq \eta^k |x_0|_X$. In particular, $x(t_k) \to 0$ as $k \to \infty$. This combined with the fact that $|x(t)|_X$ is a non-decreasing

[14] The discrete time version of this result (without condition (iii)) is essentially contained in [31, Theorem 2.3, p. 248].

function of t implies that $x(t) \to 0$ as $t \to \infty$, and we have shown that (iii) $\Rightarrow$ (i).

In order to prove that (i) $\Rightarrow$ (iv), suppose that (i) holds. If u in Lemma 2.2 vanishes on some interval $[t_1, \infty)$, then $x(t) \to 0$ as $t \to \infty$ (because $x(t - t_1)$ is the solution obtained from Lemma 2.2 with initial state $x(t_1)$ and zero input function). It the follows from **(SE)** that $\int_0^\infty |y(s)|_Y^2\, ds = \int_0^\infty |u(s)|_U^2\, ds$ in this case. The set of functions u with bounded support is dense in $L^2(\mathbb{R}^+; U)$, so the same identity must be true for all $u \in L^2(\mathbb{R}^+; U)$. Thus S is lossless, and we have proved that (i) $\Rightarrow$ (iv).

The converse direction (iv) $\Rightarrow$ (i) in the controllable case was established as a part of the proof of Theorem 10.1. ☐

There is also an exponentially stable version of Theorem 10.2.

THEOREM 10.3. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a scattering energy preserving system node (Y, X, U). Then the following conditions are equivalent:*

(i) *$i\mathbb{R} \in \rho(A)$ and $\sup_{\omega \in \mathbb{R}} \|(i\omega - A)^{-1}\|$ is finite;*

(ii) *the system semigroup $\mathfrak{A}$ of S is exponentially stable, i.e., $\|\mathfrak{A}^t\| \le Me^{-\epsilon t}$ for some $\epsilon > 0$;*

(iii) *S is exactly observable in some finite time T.*

If these conditions hold, then S is semi-lossless and $\widehat{\mathfrak{D}}(\alpha)$ is isometric for all $\alpha \in i\mathbb{R}$.

Proof. That (i) and (ii) are equivalent follows from Prüss theorem [18, Proposition 2]. As is well known, $\mathfrak{A}$ is exponentially stable if and only if $\|\mathfrak{A}^T\| < 1$ for some $T > 0$. Equivalently, there is some $\epsilon > 0$ such that the solution x in Lemma 2.2 with $u = 0$ satisfies

$$|x(T)|_X^2 \le (1 - \epsilon)|x_0|_X^2.$$

Since S is energy preserving, this identity is equivalent to the identity

$$\int_0^T |y(t)|_Y^2 \ge \epsilon |x_0|_X^2,$$

i.e., S is exactly observable in time T.

The final claim follows from Proposition 10.1 and Theorem 10.2. ☐

By applying the preceding results both to the original node S and the dual node S^* we can derive some further conclusions.

COROLLARY 10.1. *A minimal lossless scattering passive system node S is conservative, and it has the following additional properties:*

(i) *Both the system semigroup $\mathfrak{A}$ and its adjoint $\mathfrak{A}^*$ are strongly stable;*

(ii) *S is exactly observable in infinite time;*

(iii) *S is exactly controllable in infinite time;*

(iv) *The observability gramian of S is the identity operator (see condition (ii) in Theorem 10.2);*

(v) *The controllability gramian of S is the identity operator (this is equivalent to the statement that the observability gramian of S^* is the identity operator);*

(vi) *The transfer function* $\widehat{\mathfrak{D}}$ *of* S *is bi-inner (i.e., both* $\widehat{\mathfrak{D}}$ *and the dual transfer function* $\widehat{\mathfrak{D}}^d$ *are left-inner).*

(vii) *The system node* S *is uniquely determined by its transfer function* $\widehat{\mathfrak{D}}$*, modulo a unitary similarity transform in the state space.*

This follows immediately from Proposition 10.1 and Theorems 10.1 and 10.2. The systems studied in [13] are of this type.

There also exist other conservative versions of Theorem 10.2, for example, those found in [15, 16], [40, Proposition 6.1], and [33, Proposition 3.4]. Instead of stating those results here, let us observe that the following result is true as well.

THEOREM 10.4 ([33, Proposition 3.3]). *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a scattering conservative system node* (Y, X, U)*. Then the following conditions are equivalent:*

(i) $i\mathbb{R} \in \rho(A)$ *and* $\sup_{\omega\in\mathbb{R}}\|(i\omega - A)^{-1}\|$ *is finite;*

(ii) *the system semigroup* $\mathfrak{A}$ *of* S *is exponentially stable, i.e.,* $\|\mathfrak{A}^t\| \le Me^{-\epsilon t}$ *for some* $\epsilon > 0$*;*

(iii) S *is exactly observable in some finite time* T*;*

(iv) S *is exactly controllable in some finite time* T*;*

If these conditions hold, then S *is lossless and* $\widehat{\mathfrak{D}}(\alpha)$ *is unitary for all* $\alpha \in i\mathbb{R}$.

This follows directly from Theorem 10.3. Some further equivalent conditions are given in [33, Proposition 3.3]. See also [12].

Let us finish this section by pointing out that systems of the type described in Theorem 10.1 and Corollary 10.1 do exist.

COROLLARY 10.2.

(i) *Every left-inner analytic function on* $\mathbb{C}^+$ *can be realized as the transfer function of a minimal (semi-lossless) scattering energy preserving system node, which is is unique (within the class of all controllable scattering passive system nodes) modulo a unitary similarity transform in the state space.*

(ii) *Every bi-inner analytic function on* $\mathbb{C}^+$ *can be realized as the transfer function of a minimal (lossless) scattering conservative system node, which is is unique (within the class of all scattering passive system nodes which are controllable or observable, and also within the class of all simple scattering conservative system nodes) modulo a unitary similarity transform in the state space.*

This follows from Corollaries 7.2 and 10.1, and Theorem 10.1. For further related results, see [1, Section 4].

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Weiss and Marius Tucsnak that we discussed in Section 8. Much of my present knowledge of scattering conservative system comes out of numerous discussions with Jarmo Malinen and George Weiss, and Sections 3 and 10 are partially based on unpublished joint work with them (this applies, in particular, to Theorems 3.1, 3.2, and 10.2; see [15] and [16]).

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