

# Compatible Versus Regular Well-Posed Linear Systems

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**Keywords:** Compatible system, regular system, feedback, flow-inversion, time-inversion, time-flow-inversion.

## Abstract

We discuss different regularity notions for  $L^p$ -well-posed linear systems with  $1 \leq p < \infty$ , namely (from the weakest to the strongest) compatibility in the sense of Helton (1976), weak regularity in the sense of Weiss and Weiss (1997) or Staffans and Weiss (2000), strong regularity in the sense of Weiss (1994a), and uniform regularity in the sense of Helton (1976) and Ober and Wu (1996). We further investigate how compatibility and regularity are preserved under various transformations on the system, such as the duality transformation, feedback, flow-inversion, time-inversion, and time-flow inversion. Compatibility is the minimal assumption under which it is possible to describe the input-state-output behavior of a system with initial time zero, initial state  $x_0$ , input  $u$ , state trajectory  $x$  and output  $y$  in the standard way

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= \bar{C}x(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0. \end{aligned} \quad (1)$$

Here  $A$  is the generator of a strongly continuous semigroup  $\mathfrak{A}$ ,  $B$  is the (possibly unbounded) control operator,  $\bar{C}$  is a (possibly unbounded, possibly extended) observation operator, and  $D$  is a bounded feedthrough operator. Also the standard formula for the transfer function,

$$\hat{\mathfrak{D}}(z) = \bar{C}(zI - A)^{-1}B + D, \quad \Re z > \omega_{\mathfrak{A}}, \quad (2)$$

where  $\omega_{\mathfrak{A}}$  is the growth bound of  $\mathfrak{A}$ , is based on compatibility. The operators  $A$  and  $B$  are always unique. The operators  $\bar{C}$  and  $D$  are unique in the different regular cases, but not in the general compatible case. We give formulas for the operators  $A$ ,  $B$ ,  $\bar{C}$  and  $D$  of the transformed systems listed above in many of the cases where compatibility is preserved.

## 1 Well-Posed Linear Systems

Many infinite-dimensional systems can be described by the equations (1) on a triple of Banach spaces, namely, the input space  $U$ , the state space  $X$ , and the output space  $Y$ . We have  $u(t) \in U$ ,  $x(t) \in X$  and  $y(t) \in Y$ . The operators  $A$ ,  $B$ , and  $\bar{C}$  are often unbounded whereas  $D$  is bounded. However, it is not known if all well-posed linear systems can be written in this form, and it is therefore (and also for other reasons) often more convenient to use an “integral” representation of the system, which consists of the four operators from the initial state  $x_0$  and the input function  $u$  to the final state  $x(t)$  and the output function  $y$ :

$$\begin{aligned} x(t) &= \mathfrak{A}^t x_0 + \mathfrak{B}_0^t u, \quad t \geq 0, \\ y &= \mathfrak{C}x_0 + \mathfrak{D}_0 u. \end{aligned} \quad (3)$$

Here,  $\mathfrak{A}^t$  is the semigroup which maps the initial state  $x_0$  into the final state  $x(t)$ ,  $\mathfrak{B}_0^t$  is the map from the input  $u$  (restricted to the interval  $(0, t)$ ) to the final state  $x(t)$ ,  $\mathfrak{C}$  is the map from the initial state  $x_0$  to the output  $y$ , and  $\mathfrak{D}_0$  is the input-output map from  $u$  (restricted to  $\mathbf{R}^+ = [0, \infty)$ ) to  $y$ .

We regard  $L_{\text{loc}}^p(\mathbf{R}^+; U)$  as a Fréchet space, with the sequence of seminorms  $\|u\|_n = \|u\|_{L^p([0, n]; U)}$ . We regard  $L_{\text{loc}}^p(\mathbf{R}^+; U)$  as a subspace of  $L_{\text{loc}}^p(\mathbf{R}; U)$  and on the latter we define the bilateral left shift by  $t$ , denoted  $\tau^t$ , by  $(\tau^t u)(s) = u(s+t)$ ,  $-\infty < s, t < \infty$ .

The *well-posedness* assumption is that (3) behaves well in an  $L^p$ -setting, where  $1 \leq p < \infty$ , i.e.,  $x(t) \in X$  and  $y \in L_{\text{loc}}^p(\mathbf{R}^+; Y)$  depend continuously on  $x_0 \in X$  and on  $u \in L_{\text{loc}}^p(\mathbf{R}^+; U)$ . If this is the case, we call the operators  $\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  a *well-posed linear system*, where

$$\mathfrak{B}u = \lim_{t \rightarrow \infty} \mathfrak{B}_0^t \tau^{-t} u,$$

$$\mathfrak{D}u = \lim_{t \rightarrow \infty} \tau^t \mathfrak{D}_0 \tau^{-t} u,$$

each defined for those  $u \in L_{\text{loc}}^p(\mathbf{R}; U)$  for which the respective limit exists. It is possible to define a well-posed linear system  $\Sigma = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  without any reference to the system of equations (1). See, for example (alphabetically) Arov and Nudelman

(1996), Curtain and Weiss (1989), Salamon (1987, 1989), Staffans (1997, 1998a,c,b, 1999, 2000), Weiss (1989a,b,c, 1991, 1994a,b), Weiss and Weiss (1997) (and the references therein). Most of the available literature deals with Hilbert spaces and  $p = 2$ .

Each well-posed linear system induces operators  $A$ ,  $B$ , and  $C$  corresponding to the operators in (1). Here  $A$  is the generator of the semigroup  $\mathfrak{A}$ . Before introducing the operators  $B$  and  $C$  we need two auxiliary spaces  $X_1$  and  $X_{-1}$ . Choose any  $\gamma$  in the resolvent set of  $A$ . We let  $X_1$  be the domain of  $A$ , with the norm  $\|x\|_{X_1} = \|(\gamma I - A)x\|_X$ , and  $X_{-1}$  is the completion of  $X$  with the norm  $\|x\|_{X_{-1}} = \|(\gamma I - A)^{-1}x\|_X$ . The semigroup  $\mathfrak{A}$  can be extended to a strongly continuous semigroup on  $X_{-1}$ , which we denote by the same symbol. We denote the space of bounded linear operators from  $U$  to  $Y$  by  $L(U;Y)$ , and let  $L_{\omega}^p(\mathbf{R}^+;U)$  represent the space of functions  $u: \mathbf{R}^+ \rightarrow U$  for which  $t \mapsto e^{-\omega t}u(t)$  belongs to  $L^p(\mathbf{R}^+;U)$ .

**Proposition 1.1.** *Let  $\Sigma = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  be a well-posed linear system on  $U$ ,  $X$  and  $Y$ . Denote the growth bound of  $\mathfrak{A}$  by  $\omega_{\mathfrak{A}}$ .*

- (i)  $\Sigma = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  has a unique control operator  $B \in L(U;X_{-1})$ , determined by the fact that the input term  $\mathfrak{B}_0^t u$  in (3) is given by the standard variation of constants formula (the function inside the integral takes its values in  $X_{-1}$ , but the final result belongs to  $X$ )

$$\mathfrak{B}_0^t = \int_0^t \mathfrak{A}^{t-s} B u(s) ds, \\ t \in \mathbf{R}^+, u \in L_{\text{loc}}^p(\mathbf{R}^+;U).$$

- (ii)  $\Sigma$  has a unique observation operator  $C \in L(X_1;Y)$ , determined by the fact that the output term  $\mathfrak{C}x_0$  in (3) is given by (for almost all  $t \in \mathbf{R}^+$ )

$$(\mathfrak{C}x_0)(t) = C \mathfrak{A}^t x_0, \quad x_0 \in X_1.$$

- (iii)  $\Sigma$  has a unique analytic  $L(U;Y)$ -valued transfer function  $\widehat{\mathfrak{D}}$  defined (at least) on  $\Re z > \omega_{\mathfrak{A}}$ , determined by the fact that the Laplace transform  $\widehat{\mathfrak{D}}_0 u$  of the input-output term  $\mathfrak{D}_0 u$  in (3) is given by

$$\widehat{\mathfrak{D}}_0 u = \widehat{\mathfrak{D}}(z) \hat{u}(z), \quad \Re z > \omega_{\mathfrak{A}}, \quad u \in L_{\omega_{\mathfrak{A}}}^p(\mathbf{R}^+;U),$$

where  $\hat{u}$  is the Laplace transform of  $u$ .

The existence of  $B$  and  $C$  is proved in Salamon (1989), Weiss (1989a), Weiss (1989b), and the existence of a transfer function is proved in Curtain and Weiss (1989) and Weiss (1991). (See also Salamon (1989) and (Weiss, 1994a, Remark 2.4).) The control operator  $B$  is said to be *bounded* if the range of  $B$  lies in  $X$ , in which case  $B \in L(U;X)$ . The observation operator  $C$  is said to be *bounded* if it is continuous with respect to the norm of  $X$ , i.e., if it can be extended to an operator in  $L(X;Y)$ . The operator  $\overline{C}$  in (1) is an extension of  $C$ , as will be explained in Section 2.

To get a time-domain representation for the output  $y$  of a well-posed linear system similar to the second equation in (1) we introduce the subspace  $V$  of  $X \times U$  defined by

$$V = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times U \mid Ax + Bu \in X \right\}. \quad (4)$$

Also this space is a Banach space with the norm

$$\| \begin{bmatrix} x \\ u \end{bmatrix} \|_V = \{ |x|_X^2 + |u|_U^2 + |Ax + Bu|_X^2 \}^{1/2}.$$

If  $X$  and  $U$  are Hilbert spaces, then so is  $V$ .

**Proposition 1.2.** *Let  $\Sigma = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  be a well-posed linear system on  $U$ ,  $X$  and  $Y$ . Denote the growth bound of  $\mathfrak{A}$  by  $\omega_{\mathfrak{A}}$ . We define the operator  $C\&D \in L(V;Y)$  by*

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C[x - (\alpha I - A)^{-1}Bu] + \widehat{\mathfrak{D}}(\alpha)u, \quad (5)$$

where  $\alpha \in \mathbf{C}$  with  $\Re \alpha > \omega_{\mathfrak{A}}$  can be chosen in an arbitrary way (i.e., the result is independent of  $\alpha$  as long as  $\Re \alpha > \omega_{\mathfrak{A}}$ ). We call  $C\&D$  the combined observation/feedthrough operator of  $\Sigma$ .

- (i) The output  $y = \mathfrak{C}x_0 + \mathfrak{D}_0 u$  of  $\Sigma$  defined in (3) is given for all  $t \geq 0$  by

$$y(t) = C\&D \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = C\&D \begin{bmatrix} \mathfrak{A}^t x_0 + \mathfrak{B}_0^t u \\ u(t) \end{bmatrix}, \quad (6)$$

for all  $x_0 \in X$  and all  $u \in W_{\text{loc}}^{1,p}(\mathbf{R}^+;U)$  satisfying  $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in V$ . In particular,  $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in V$  for all  $t \geq 0$ .

- (ii) The transfer function  $\widehat{\mathfrak{D}}$  of  $\Sigma$  is given by

$$\widehat{\mathfrak{D}}(z) = C\&D \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}, \quad \Re z > \omega_{\mathfrak{A}}. \quad (7)$$

For the proof, see Arov and Nudelman (1996), Curtain and Weiss (1989), Salamon (1987, 1989), Staffans (2000), or Weiss (1989a,b).

We call  $S = \begin{bmatrix} A & B \\ C\&D \end{bmatrix}: V \rightarrow X \times U$  the *system operator* of the system  $\Sigma$ , and we refer to  $A$ ,  $B$ , and  $C\&D$  as the *generating operators* of  $\Sigma$ . In Arov and Nudelman (1996),  $C\&D$  is denoted by  $N$ .

## 2 Compatible Systems

It is usually *not possible* to split  $C\&D$  directly into

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = Cx + Du, \quad (8)$$

where

$$Cx = C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad x \in X_1, \\ Du = C\&D \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad u \in U, \quad (9)$$

because of the fact that  $\{0\} \times U$  is not contained in the domain  $V$  of  $C&D$  in general (this is true only when  $B$  is bounded). (The relationship between  $C$  and  $C&D$  expressed in (9) is always valid.) Thus, to split  $C&D$  in this way we must first *extend*  $C&D$  to a larger domain containing  $\{0\} \times U$  as a subspace. The smallest possible extended domain is  $Z \times U$ , where  $Z$  is defined as follows. We choose any  $\gamma$  in the resolvent set of  $A$ , and let

$$Z = \{z \in X \mid z = (\gamma I - A)^{-1}(x + Bu) \text{ for some } x \in X \text{ and } u \in U\}. \quad (10)$$

This is a Banach space with the norm

$$\|z\|_Z = \inf_{(\gamma I - A)^{-1}(x + Bu) = z} (|x|_X^2 + |u|_U^2)^{1/2},$$

satisfying  $X_1 \subset Z \subset X$ , and it is a Hilbert space if both  $X$  and  $U$  are Hilbert spaces. It is easy to see that  $V \subset Z \times U$ , but the embedding  $V \subset Z \times U$  need not be dense.

**Definition 2.1.** *The well-posed linear system  $\Sigma = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is compatible if its combined observation/feedthrough operator  $C&D$  can be extended to an operator  $\overline{C&D} \in L(Z \times U; Y)$ . We define the corresponding extended observation operator  $\overline{C} \in L(Z; Y)$  and feedthrough operator  $D \in L(U; Y)$  by*

$$\begin{aligned} \overline{C} &= \overline{C&D} \begin{bmatrix} x \\ 0 \end{bmatrix}, & x \in Z, \\ Du &= \overline{C&D} \begin{bmatrix} 0 \\ u \end{bmatrix}, & u \in U. \end{aligned} \quad (11)$$

The extension of  $C&D$  to  $Z \times U$  need not be unique, since  $V$  may not be dense in  $Z \times U$ . This means that  $\overline{C}$  and  $D$  need not be unique either. However, there is a one-to-one correspondence between  $\overline{C&D}$ ,  $\overline{C}$  and  $D$ , i.e., any one of these three operators determines the other two uniquely. This can be seen as follows. Clearly (11) defines  $\overline{C}$  and  $D$  in terms of  $\overline{C&D}$ , and conversely, if we know both  $\overline{C}$  and  $D$  then we know  $\overline{C&D}$ , since, by (11),

$$\overline{C&D} \begin{bmatrix} x \\ u \end{bmatrix} = \overline{C}x + Du, \quad x \in Z, \quad u \in U. \quad (12)$$

But there is also a one-to-one correspondence between  $\overline{C}$  and  $D$  since, by (7),

$$\widehat{\mathfrak{D}}(z) = \overline{C}(zI - A)^{-1}B + D, \quad \Re z > \omega_{\mathfrak{A}}. \quad (13)$$

This defines  $D$  as a function of  $\widehat{\mathfrak{D}}$  and  $\overline{C}$ , and conversely, it defines the values of  $\overline{C}$  on the range of  $(zI - A)^{-1}B$  as a function of  $\widehat{\mathfrak{D}}$  and  $D$  (the values of  $\overline{C}$  on  $X_1$  are determined by the original control operator  $C$ , so they are the same for all extensions). Thus we have shown the following:

**Proposition 2.2.** *The well-posed linear system  $\Sigma$  is compatible if and only if  $C$  can be extended to an operator  $\overline{C} \in L(Z; Y)$ , and (11) is a one-to-one correspondence between the extended  $\overline{C}$  and the extended  $\overline{C&D}$ .*

Often  $Z$  is a Hilbert space and  $X_1$  is a closed subspace of  $Z$ , and then it is clear from the above that  $\Sigma$  is compatible. Sometimes the extension of  $C$  comes naturally, as in the case where  $C$  is bounded and has a (unique) extension to an operator in  $L(X; Y)$ . We can then let this extension determine the values of  $\overline{C}$  on  $Z$ . On the other hand, in some cases (such as boundary control systems) it is possible to specify  $D$  arbitrarily, i.e., to every bounded operator  $D$  it may be possible to find an extended operator  $\overline{C}$  such that  $D$  is the feedthrough operator corresponding to  $\overline{C}$  (see Staffans (2000)). One possible choice in these cases is to take  $D = 0$ . (If the system is a boundary control system with a bounded observation operator, then the two methods described above produce, in general, two different extensions.) A third method will be described later: if the system is the result of some transformation performed on some other system, then we may want to choose  $\overline{C}$  and  $D$  in such a way that they can be computed by a simple formula from the corresponding operators for the original system.

In spite of the possible non-uniqueness of the extended observation operator  $\overline{C}$  and the corresponding feedthrough operator  $D$ , independently of how these operators are chosen, it is still true that the output equation (6) simplifies into

$$y(t) = \overline{C}x(t) + Du(t), \quad t \geq 0, \quad (14)$$

and we observed above that (7) simplifies into (13).

In the sequel, rather than writing  $\overline{C}$  all the time, we write  $C$  instead of  $\overline{C}$ , but we still think of  $C$  as the (non-unique) extended observation operator in  $L(Z; Y)$ . Whenever both  $C$  and  $D$  appear in the same formula we require  $D$  to be the feedthrough operator induced by  $C$ . For a compatible system, we call a quadruple of operators  $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in L(Z \times U; X_{-1} \times Y)$  constructed as described above a set of *generators* or *generating operators* of this system.

As Helton (1976) comments, most physically motivated systems seem to be compatible. This applies, in particular, to all systems where  $B$  or  $C$  is bounded, to all boundary control systems, and to all systems with finite-dimensional input space  $U$ . All regular linear systems (see the next section) are also compatible. The existence of a non-compatible well-posed linear system in the Hilbert space context is still an open problem. A reasonably complete theory for compatible systems is presented in Staffans (2000).

### 3 Regular Systems

The non-uniqueness of the extended observation operator  $C$  and the corresponding feedthrough operator  $D$  in a compatible system is slightly disturbing. It is possible to get rid of this non-uniqueness by using a stronger *regularity* assumption on the system, and by using a specific extension of  $C$  instead of an arbitrary extension.

**Definition 3.1.** *The well-posed linear system  $\Sigma = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is weakly, strongly, or uniformly regular if its transfer function*

$\widehat{\mathcal{D}}(z)$  has a weak, strong, or uniform limit

$$D = \lim_{z \rightarrow +\infty} \widehat{\mathcal{D}}(z) \quad (15)$$

(where the limit is taken along the positive real axis).

As shown by Weiss (1994a) and Staffans and Weiss (2000), every weakly regular system is compatible and it is possible to extend  $C$  to the space  $Z$  in such a way that the corresponding feedthrough operator is  $D$ . It is even possible to extend  $C$  further so that the output formula  $y(t) = Cx(t) + Du(t)$  becomes valid for almost all  $t \geq 0$  under the minimal assumptions that  $x_0 \in X$  and  $u \in L^p_{\text{loc}}(\mathbf{R}^+; U)$ .

According to Staffans and Weiss (1998), every  $L^1$ -well-posed system is weakly regular, hence compatible.

## 4 Duality

In this section we suppose that  $U$ ,  $X$  and  $Y$  are reflexive, and that  $1 < p < \infty$ . Then the (causal) dual of the well-posed system  $\Sigma = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is given by

$$\Sigma^d = \begin{bmatrix} \mathfrak{A}^* & \mathfrak{C}^* \mathfrak{Y} \\ \mathfrak{Y} \mathfrak{B}^* & \mathfrak{Y} \mathfrak{D}^* \mathfrak{Y} \end{bmatrix}, \quad (16)$$

where  $\mathfrak{Y}$  is the *reflection operator* defined by  $(\mathfrak{Y}u)(t) = u(-t)$  for all  $t \in \mathbf{R}$  and for all  $u \in L^p_{\text{loc}}(\mathbf{R})$ . The system operator of the dual is the adjoint of the system operator of the original system:

$$\begin{bmatrix} A^d & B^d \\ C \& D^d \end{bmatrix} = \begin{bmatrix} A & B \\ C \& D \end{bmatrix}^*. \quad (17)$$

In particular,  $A^d = A^*$ ,  $B^d = C^*$  and  $C^d = B^*$ . The question of whether or not  $D^d = D^*$  in the compatible case is more difficult. By Staffans and Weiss (2000) or Staffans (2000), weak and uniform regularity are preserved under the duality transformation and  $D^d = D^*$  in this case. By Staffans (2000), if  $X_1$  is dense in  $Z$  then compatibility is preserved under the duality transformation and again  $D^d = D^*$ . It is not known to what extent this is true when  $X_1$  is not dense in  $Z$ . According to Staffans and Weiss (2000), *strong regularity is not preserved* under duality in general (but, of course, the dual of a strongly regular system is weakly regular, hence compatible).

## 5 Feedback

To get a standard static output feedback connection for the system  $\Sigma$  we choose some operator  $K \in L(Y; U)$  and replace  $u$  in (3) by  $u = Ky + v$ , where  $v \in L^p_{\text{loc}}(\mathbf{R}^+; U)$  is the new input function. Then it is easy to see that

$$\begin{bmatrix} I & 0 \\ -K\mathfrak{C} & I - K\mathfrak{D}_0 \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} = \begin{bmatrix} x_0 \\ v \end{bmatrix}. \quad (18)$$

This equation defines  $x_0$  and  $u$  uniquely and continuously in terms of  $x_0$  and  $v$  if and only if  $\begin{bmatrix} I & 0 \\ -K\mathfrak{C} & I - K\mathfrak{D}_0 \end{bmatrix}$  is invertible in  $X \times L^p_{\text{loc}}(\mathbf{R}^+; U)$ , or equivalently, if and only if  $I - K\mathfrak{D}_0$  is

invertible in  $L^p_{\text{loc}}(\mathbf{R}^+; U)$ . Under this assumption we get a well-posed *closed-loop system*  $\Sigma^K$  whose input-state-output relationship is given by

$$\begin{aligned} \begin{bmatrix} x(t) \\ y \end{bmatrix} &= \begin{bmatrix} (\mathfrak{A}^K)^t & (\mathfrak{B}^K)_0^t \\ \mathfrak{C}^K & \mathfrak{D}_0^K \end{bmatrix} \begin{bmatrix} x_0 \\ v \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \mathfrak{C} & \mathfrak{D}_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -K\mathfrak{C} & I - K\mathfrak{D}_0 \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ v \end{bmatrix}. \end{aligned} \quad (19)$$

The corresponding *closed-loop system operator* is given by

$$\begin{bmatrix} A^K & B^K \\ (C \& D)^K \end{bmatrix} = \begin{bmatrix} A & B \\ C \& D \end{bmatrix} M^{-1}, \quad (20)$$

where

$$M = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ K(C \& D) \end{bmatrix}. \quad (21)$$

In particular,  $M$  maps the space  $V$  onto the corresponding closed-loop space  $V^K$ .

In the classical finite-dimensional case the above feedback system is well-posed (i.e.,  $I - K\mathfrak{D}_0$  is invertible) if and only if  $I - DK$  is invertible, or equivalently, if and only if  $I - KD$  is invertible. When this is the case we can write the closed-loop generators in the form

$$\begin{bmatrix} A^K & B^K \\ C^K & D^K \end{bmatrix} = \begin{bmatrix} A + BEKC & BE \\ (I - DK)^{-1}C & DE \end{bmatrix}, \quad (22)$$

where  $E = (I - KD)^{-1}$ . In the infinite-dimensional well-posed case the situation is more complicated, and the invertibility of  $(I - DK)$  is neither necessary nor sufficient for the closed-loop system to be well-posed. (For example, in a boundary control system  $D$  can be chosen in an arbitrary way, hence  $I - KD$  will be invertible for some  $D$  but not for all  $D$ , independently of whether the closed-loop system is well-posed or not). Let us assume that the closed-loop system is well-posed. Then left-invertibility of  $I - DK$  is sufficient to imply that the closed-loop system is compatible, and if  $I - DK$  is invertible (from both sides), then (20) holds, as shown by Mikkola (2000) and Staffans (2000). (In particular, in a boundary control system we can make  $I - KD$  invertible by taking  $D = 0$ .) Uniform regularity is always preserved under feedback (and  $I - DK$  is invertible in the uniformly regular case). It is not known if strong regularity is always preserved (it is preserved if  $I - DK$  is invertible; see Weiss (1994b)), but strong regularity implies that  $I - DK$  is left-invertible, hence the closed-loop system is compatible. According to Staffans and Weiss (2000), weak regularity is not preserved in general under feedback, even in the case where  $I - DK$  is invertible.

## 6 Flow-Inversion

The idea behind flow inversion is to keep the relationships between the state  $x$  and the signals  $u$  and  $y$  in (1) intact, but to reinterpret  $y$  as the input and  $u$  as the output. By (3),

$$\begin{bmatrix} I & 0 \\ \mathfrak{C} & \mathfrak{D}_0 \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} = \begin{bmatrix} x_0 \\ y \end{bmatrix}. \quad (23)$$

This equation defines  $x_0$  and  $u$  uniquely and continuously in terms of  $x_0$  and  $y$  if and only if  $\begin{bmatrix} I & 0 \\ \mathfrak{C} & \mathfrak{D}_0 \end{bmatrix}$  is invertible as an operator from  $X \times L_{\text{loc}}^p(\mathbf{R}^+; U)$  to  $X \times L_{\text{loc}}^p(\mathbf{R}^+; Y)$ , or equivalently, if and only if  $\mathfrak{D}_0$  is invertible as an operator from  $L_{\text{loc}}^p(\mathbf{R}^+; U)$  to  $L_{\text{loc}}^p(\mathbf{R}^+; Y)$ . We then get a well-posed *flow-inverted system*  $\Sigma^\times$ , whose input-state-output relationship is given by

$$\begin{aligned} \begin{bmatrix} x(t) \\ u \end{bmatrix} &= \begin{bmatrix} (\mathfrak{A}^\times)^t & (\mathfrak{B}^\times)^t_0 \\ \mathfrak{C}^\times & \mathfrak{D}_0^\times \end{bmatrix} \begin{bmatrix} x_0 \\ y \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{A}^\times & \mathfrak{B}^\times_0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathfrak{C} & \mathfrak{D}_0 \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ y \end{bmatrix}. \end{aligned} \quad (24)$$

The corresponding *flow-inverted system operator* is

$$\begin{bmatrix} A^\times & B^\times \\ C^\times & D^\times \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C \& D \end{bmatrix}^{-1}. \quad (25)$$

In particular,  $\begin{bmatrix} I & 0 \\ C \& D \end{bmatrix}$  maps the space  $V$  onto the corresponding flow-inverted space  $V^\times$ .

In the classical finite-dimensional case flow-inversion is possible if and only if  $D$  is invertible, and in that case

$$\begin{bmatrix} A^\times & B^\times \\ C^\times & D^\times \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{bmatrix}. \quad (26)$$

Actually, flow-inversion can be interpreted as a special case of feedback: if we (for simplicity and without loss of generality) suppose that  $U = Y$ , then we get the flow-inverted system by first replacing  $y$  by  $y - u$  (meaning that we replace  $D$  by  $D - I$ ) and then using negative identity feedback. Thus, everything that we said in the feedback section applies to flow-inversion if we replace  $I - K\mathfrak{D}_0$  by  $\mathfrak{D}_0$ ,  $(I - KD)$  by  $D$ , (20) by (25), and (22) by (26). In particular, if  $D$  is invertible, then flow-inversion preserves compatibility and strong and uniform regularity. However, weak regularity is not preserved under flow-inversion. For more details, see Staffans and Weiss (2000) and Staffans (2000).

## 7 Time-Inversion

The idea behind time-inversion is to solve (1) backward in time, i.e., we let  $t$  in (1) be negative. The set of equations (1) with  $t \leq 0$  cannot directly be interpreted as the equations describing the evolution of a well-posed linear system, since such systems evolve in the forward and not in the backward time direction, but this can easily be taken care of by an extra reflection of the time axis. Thus, we replace  $t \leq 0$  by  $-t \geq 0$  in (1). This changes the sign of the derivative in the first equation in (1), but it has no influence on the second equation. Thus, if we are to obtain a well-posed time-inverted system by arguing in this way, then the system operator of the *time-inverted system*  $\Sigma^\mathfrak{A}$  that we get must be given by

$$\begin{bmatrix} A^\mathfrak{A} & B^\mathfrak{A} \\ (C \& D)^\mathfrak{A} \end{bmatrix} = \begin{bmatrix} -A & -B \\ C \& D \end{bmatrix}. \quad (27)$$

By simply ignoring the input  $u$  and the output  $y$  in (1), we realize that a necessary condition for the time-invertibility

of a well-posed linear system  $\Sigma = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is that  $\mathfrak{A}$  can be extended to a group. As shown in Staffans and Weiss (2000), this condition is also sufficient, and (27) holds. By (27), the combined observation/feedthrough operator of the time-inverted system is *the same* as the original one, hence, in particular, compatibility is preserved under time-inversion. Moreover, since the time-inverted transfer function  $\widehat{\mathfrak{D}}^\mathfrak{A}$  can be computed through (7) from the original observation/feedthrough operator  $C \& D$  (defined in (5)), we get that for all  $-z$  in right half-plane  $\Re(-z) > \omega_{\mathfrak{A}}$ ,

$$\begin{aligned} \widehat{\mathfrak{D}}^\mathfrak{A}(-z) &= C \& D \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix} \\ &= C[(zI - A)^{-1} - (\alpha I - A)^{-1}]B + \widehat{\mathfrak{D}}(\alpha), \end{aligned} \quad (28)$$

where  $\omega_{\mathfrak{A}}$  is the growth bound of the semigroup generated by  $-A$  and  $\alpha > \omega_{\mathfrak{A}}$ . Comparing this expression to (7), we realize that  $\widehat{\mathfrak{D}}^\mathfrak{A}(-z)$  and  $\widehat{\mathfrak{D}}(z)$  are obtained from *the same formula*, which in one case is valid for  $\Re z > \omega_{\mathfrak{A}}$  and in the other case for  $\Re z < -\omega_{\mathfrak{A}}$ . The function produced by this formula is analytic in the resolvent set of  $A$ . Thus, if the spectrum of  $A$  does not separate the right half-plane  $\Re z > \omega_{\mathfrak{A}}$  from the left half-plane  $\Re z < -\omega_{\mathfrak{A}}$ , then  $\widehat{\mathfrak{D}}^\mathfrak{A}(-z)$  and  $\widehat{\mathfrak{D}}(z)$  are analytic continuations of each other.

It is shown in Staffans and Weiss (2000) with a scalar counterexample that regularity need not be preserved under time inversion. (In the single input single output case weak, strong, and uniform regularity are all equivalent, and hence none of them is preserved.) Moreover, even if both the original and the time-inverted system are regular, it is still possible that the forward extended observation and feedthrough operators differ from the backward extended observation and feedthrough operators, as can be seen from the simple example

$$\widehat{\mathfrak{D}}(z) = \frac{e^{-z}}{1 - e^{-z}}.$$

## 8 Time-Flow-Inversion

To get a *time-flow inverted system*, we perform the two operations described in Sections 6 and 7 simultaneously. That is, we interpret  $y$  as the input and  $u$  as the output, and replace  $t < 0$  in (1) by  $-t > 0$ . If  $\Sigma = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is flow-invertible, and if the flow-inverted system is time-invertible, then by combining (25) and (27) we get the expected formula for the system operator of the *time-flow-inverted system*  $\Sigma^\leftarrow$ , namely

$$\begin{bmatrix} A^\leftarrow & B^\leftarrow \\ (C \& D)^\leftarrow \end{bmatrix} = \begin{bmatrix} -A & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C \& D \end{bmatrix}^{-1}, \quad (29)$$

which in the compatible case with an invertible operator  $D$  should become

$$\begin{bmatrix} A^\leftarrow & B^\leftarrow \\ C^\leftarrow & D^\leftarrow \end{bmatrix} = \begin{bmatrix} -A + BD^{-1}C & -BD^{-1} \\ -D^{-1}C & D^{-1} \end{bmatrix}. \quad (30)$$

In the finite-dimensional case a time-flow-invertible system is necessarily both time-invertible and flow-invertible

and (30) holds. In particular, the dimensions of  $U$  and  $Y$  must be the same. This is not true in the infinite-dimensional case, as will be shown below.

It is quite simple to find a necessary and sufficient condition for time-flow-invertibility. Let  $t > 0$ , and define  $(\pi_{(0,t)}u)(s) = u(s)$  if  $s \in (0, t)$  and  $(\pi_{(0,t)}u)(s) = 0$  otherwise. Then (3) can be rewritten in the form

$$\begin{aligned} x(t) &= \mathfrak{A}^t x(0) + \mathfrak{B}_0^t \pi_{(0,t)} u, \\ \pi_{(0,t)} y &= \mathfrak{C}_0^t x(0) + \mathfrak{D}_0^t \pi_{(0,t)} u, \quad t \geq 0, \end{aligned} \quad (31)$$

where  $\mathfrak{C}_0^t = \pi_{(0,t)} \mathfrak{C}$  and  $\mathfrak{D}_0^t = \pi_{(0,t)} \mathfrak{D} \pi_{(0,t)}$ . Defining

$$\Sigma_0^t = \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{bmatrix}, \quad (32)$$

we can write this as

$$\begin{bmatrix} x(t) \\ \pi_{(0,t)} y \end{bmatrix} = \Sigma_0^t \begin{bmatrix} x(0) \\ \pi_{(0,t)} u \end{bmatrix}. \quad (33)$$

If the system is time-flow-invertible, then it must be possible to express  $x(0)$  and  $\pi_{(0,t)} u$  in terms of  $x(t)$  and  $\pi_{(0,t)} y$ , and this implies that  $\Sigma_0^t$  must be invertible from  $X \times L^2((0, t); U)$  to  $X \times L^2((0, t); Y)$ , for all  $t > 0$ . It is shown in Staffans and Weiss (2000) that this necessary condition is also sufficient for the existence of the time-flow-inverted system. Moreover, if  $\Sigma_0^t$  is invertible for one  $t > 0$ , then it is invertible for all  $t > 0$ , and the time-flow-inverted system operator is given by (29).

As mentioned earlier, in the finite-dimensional case, a system cannot be time-flow-invertible unless the dimensions of  $U$  and  $Y$  are the same. This is not true in the infinite-dimensional case as the following counterexample shows: take  $U = \{0\}$ ,  $Y = \mathbf{C}$ , let  $\mathfrak{A}$  be the left-shift on  $X = L^2(\mathbf{R}^+)$ , and let  $\mathfrak{C} = I$ . This system is conservative, meaning that the operator  $\Sigma_0^t = \begin{bmatrix} \mathfrak{A}^t \\ \mathfrak{C}_0^t \end{bmatrix}$  is unitary from  $X$  to  $X \times L^2(0, t)$  for all  $t \geq 0$ , hence invertible. The inverse of a unitary operator coincides with its adjoint, and it can be shown that the time-flow-inverted system coincides with the dual system in this case. Thus,  $\mathfrak{A}^{\leftarrow} = \mathfrak{A}^d = \mathfrak{A}^*$  is the right-shift on  $L^2(\mathbf{R}^+)$ ,  $(\mathfrak{B}_0^{\leftarrow})_0^t = (\mathfrak{C}_0^t)^*$ , i.e.,  $\mathfrak{B}_0^t y = \pi_{(0,t)} y$ , and the generators of the time-flow inverted system are  $A^{\leftarrow} = A^*$  and  $B^{\leftarrow} = C^*$ .

The system described above is time-flow-invertible but neither time-invertible nor flow-invertible. Both the system itself and the time-flow-inverted system are regular. The feedthrough operator  $D$  is not invertible, and the right-hand side of (30) is not even well-defined (but of course, (29) is still true).

The question whether compatibility or regularity are preserved under time-flow-inversion is still wide open.

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