1

Boundary control state/signal systems and boundary triplets

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Abstract This chapter is an introduction to the basic theory of state/signal systems via boundary control theory. The $\mathcal{LC}$-transmission line illustrates the new concepts. It is shown that every boundary triplet can be interpreted as an impedance representation of a conservative boundary control state/signal system.

1.1 Introduction

We discuss the connection between some basic notions of boundary control state/signal systems on one hand, and classical boundary triplets on the other hand. Boundary triplets and their generalizations have been extensively utilized in the theory of self-adjoint extensions of symmetrical operators in Hilbert spaces, see e.g. [Gorbachuk and Gorbachuk, 1991; Derkach and Malamud, 1995; Behrndt and Langer, 2007], and the references therein.

The notions related to standard input/state/output boundary control systems are discussed in Section 1.2, where we also introduce the boundary control state/signal system. In Section 1.3 we briefly discuss the concept of conservativity in the state/signal framework and in Section 1.4 we illustrate the abstract concepts we have introduced using the example of a finite-length conservative $\mathcal{LC}$-transmission line with distributed inductance and capacitance.

We conclude this chapter in Section 1.5, where we recall the definition of a boundary triplet for a symmetric operator and compare this object to a boundary control state/signal system. In particular, we show that

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every boundary triplet can be transformed into a conservative boundary control state/signal system in impedance form, but that the converse is not true. We make a few final remarks about common generalizations of boundary triplets, which leads over to Chapter 2, where we treat more general passive state/signal systems, not only conservative systems or systems of boundary-control type. There we show how general conservative state/signal systems are related to *boundary relations*.

1.2 Boundary control systems

In this section we introduce boundary control state/signal systems by first describing their predecessors, namely input/state/output systems of boundary-control type.

In boundary control one often investigates systems that can be abstractly written in the form

$$\Sigma_{i/s/o}: \quad \begin{cases} \dot{x}(t) = Lx(t), \\ u(t) = \Gamma_0 x(t), \quad t \in \mathbb{R}^+, \ x(0) = x_0 \text{ given}, \\ y(t) = \Gamma_1 x(t), \end{cases} \tag{1.1}$$

where $\mathbb{R}^+ = [0, \infty)$ and $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$. Here the *initial state* x_0 and the *current state* $x(t)$ belong to the Hilbert state space $\mathcal{X}$, the *input* $u(t)$ belongs to the Hilbert input space $\mathcal{U}$, and the *output* $y(t)$ belongs to the Hilbert output space $\mathcal{Y}$. The *main operator* L is an unbounded operator in $\mathcal{X}$ with domain $\operatorname{dom}(L)$, and the *boundary control operator* Γ_0 is an unbounded operator $\mathcal{X} \to \mathcal{U}$ with the same domain as L. The *observation operator* $\Gamma_1 \colon \mathcal{X} \to \mathcal{Y}$ may be bounded or unbounded, and it is defined at least on $\operatorname{dom}(L)$. All of these operators are linear. We denote the system (1.1) with these properties by $\Sigma_{i/s/o} = (L, \Gamma_0, \Gamma_1; \mathcal{X}, \mathcal{U}, \mathcal{Y})$.

In order for (1.1) to generate a dynamical system with good properties at least the properties listed in the following definition need to be assumed; see e.g. [Salamon, 1987], [Staffans, 2005], or [Malinen and Staffans, 2006] for details.

Definition 1.1 Assume that $\Sigma_{i/s/o} = (L, \Gamma_0, \Gamma_1; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is as described above. Then $\Sigma_{i/s/o}$ is a *boundary control input/state/output (i/s/o) node* if $\Sigma_{i/s/o}$ satisfies the following conditions:

1. The input operator Γ_0 is surjective and *strictly unbounded* in the sense that $\ker(\Gamma_0)$ is *dense* in $\mathcal{X}$.

2. The restriction $A := L|_{\ker(\Gamma_0)}$ of L to $\ker(\Gamma_0)$ *generates* a C_0-semigroup $t \mapsto \mathfrak{A}^t$, $t \in \mathbb{R}^+$.

A *boundary control state/signal system* is analogous to a boundary control i/s/o system, but we no longer specify which part of the "boundary signal" $w(t) := \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$ is the input, and which part is the output. Instead we combine the input and output spaces into one signal space $\mathcal{W} := \begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix} = \mathcal{U} \times \mathcal{Y}$, and denote $\Gamma := \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}$. Then $\Gamma\colon \mathrm{dom}\,(L) \to \mathcal{W}$, and (1.1) can be rewritten in the form

$$\Sigma: \quad \begin{cases} \dot{x}(t) = Lx(t), \\ w(t) = \Gamma x(t), \end{cases} \quad t \in \mathbb{R}^+,\ x(0) = x_0 \text{ given.} \tag{1.2}$$

As before, the *initial state* x_0 and the *current state* $x(t)$ belong to the Hilbert state space $\mathcal{X}$. The (*interaction*) signal $w(t)$ belongs to the signal space $\mathcal{W}$, which we take to be an arbitrary Kreĭn space (the reason for this will be explained below). We thus no longer assume that $\mathcal{W}$ is of the form $\mathcal{W} = \begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}$, where $\mathcal{U}$ and $\mathcal{Y}$ are the input and output spaces of a boundary control i/s/o node. The *main operator* L is still an unbounded operator $\mathcal{X} \to \mathcal{X}$ with domain $\mathrm{dom}\,(L)$, and the *boundary operator* Γ is an unbounded operator $\mathcal{X} \to \mathcal{W}$ with the same domain as L. We denote this system by $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$.

Note that (1.2) can be written in the *graph form*:

$$\Sigma: \quad \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad t \in \mathbb{R}^+,\ x(0) = x_0,$$

where the *generating subspace* V is the graph of $\begin{bmatrix} L \\ \Gamma \end{bmatrix}$:

$$V := \left\{ \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \,\middle|\, x \in \mathrm{dom}\,(L) \right\}. \tag{1.3}$$

The unbounded operator $\begin{bmatrix} L \\ \Gamma \end{bmatrix}$ is assumed to be closed, and this is equivalent to assuming that V is a closed subspace of the *node space* $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$. The generating subspace is the key to generalizing the state/signal theory beyond boundary control, as we shall see in Chapter 2. We define the *dynamics* of a state/signal system using the generating subspace V.

Definition 1.2 Let V be a closed subspace of $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$.

1. The pair $\begin{bmatrix} x \\ w \end{bmatrix}$ is a *classical trajectory* generated by V on $\mathbb{R}^+$ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$, $w \in C(\mathbb{R}^+; \mathcal{W})$, and $\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ for all $t > 0$.

2. The pair $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a *generalized trajectory* generated by V on $\mathbb{R}^+$ if $x \in C(\mathbb{R}^+;\mathcal{X})$, $w \in L^2_{\rm loc}(\mathbb{R}^+;\mathcal{W})$, and there exists a sequence of classical trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ such that $x_n \to x$ uniformly on all bounded intervals $[0,T]$ and $w_n \to w$ in $L^2_{\rm loc}(\mathbb{R}^+;\mathcal{W})$.

Note that $\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ for all $t > 0$ in item 1 of Definition 1.2 if and only if $\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ for all $t \in \mathbb{R}^+$ when we interpret $\dot{x}(0)$ as the right-sided derivative of x at zero. We are now ready to define a boundary control s/s system.

Definition 1.3 A *boundary control state/signal (s/s) node* is a quadruple $\Sigma = (L,\Gamma;\mathcal{X},\mathcal{W})$ such that:

1. The space $\mathcal{X}$ is a Hilbert space and $\mathcal{W}$ is a Kreĭn space.
2. The operator $\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right] : \mathcal{X} \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ is closed and densely defined.
3. The range of Γ is dense in $\mathcal{W}$.

By the *boundary control state/signal system* induced by a boundary control s/s node $(L,\Gamma;\mathcal{X},\mathcal{W})$ we mean this node together with the sets of classical and generalized trajectories generated by V in (1.3) on $\mathbb{R}^+$. We denote both the node and the system by $\Sigma = (L,\Gamma;\mathcal{X},\mathcal{W})$.

In Definition 1.4 below we will equip the node space $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$ with an indefinite inner product which makes it a Kreĭn space.

1.3 Conservative state/signal systems in boundary control

In this section we shall focus our attention on s/s systems Σ whose classical trajectories on $\mathbb{R}^+$ satisfy the *power equality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} = [w(t),w(t)]_{\mathcal{W}}\,, \quad t \in \mathbb{R}^+. \tag{1.4}$$

Here $\|x(t)\|^2_{\mathcal{X}}$ stands for (two times) the *internal energy* stored in the state x at time t and $[w(t),w(t)]_{\mathcal{W}}$ represents (two times) the power (energy flow per time unit) entering the system through the signal $w(t)$ at time t. This explains why we need to take $\mathcal{W}$ to be a Kreĭn space: we must allow the inner product $[\cdot,\cdot]_{\mathcal{W}}$ in $\mathcal{W}$ to be indefinite. If the inner product in $\mathcal{W}$ is non-negative, then no energy can leave the system via the (interaction) signal, and if the inner product in $\mathcal{W}$ is non-positive, then no energy can enter the system via the signal.

The equality (1.4) says that the system has *no internal energy sources or sinks.* However, the equality is not enough to make the system Σ conservative: we need an additional *hypermaximality condition.* We give the full definition of a conservative boundary control s/s system in Definition 1.5 below.

After integration over the interval $[s,t] \subset \mathbb{R}^+$, one can rewrite (1.4) in the equivalent form

$$\|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2 = \int_s^t [w(v), w(v)]_{\mathcal{W}} \, \mathrm{d}v, \quad s,t \in \mathbb{R}^+, \ s \le t.$$

By the continuity of the inner product this inequality remains valid for generalized trajectories as well.

Carrying out the differentiation in (1.4), we get a third equivalent condition in terms of classical trajectories, namely

$$-\left(\dot{x}(t), x(t)\right)_{\mathcal{X}} - \left(x(t), \dot{x}(t)\right)_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} = 0, \quad t \in \mathbb{R}^+. \tag{1.5}$$

Using item 1 of Definition 1.2, we see that (1.5) always holds if

$$-\left(z, x\right)_{\mathcal{X}} - \left(x, z\right)_{\mathcal{X}} + [w, w]_{\mathcal{W}} = 0, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V. \tag{1.6}$$

It is now natural to make the following definition:

Definition 1.4 Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ a Kreĭn space. The corresponding *node space* is the product space $\mathfrak{K} = \mathcal{X} \times \mathcal{X} \times \mathcal{W}$ equipped with the indefinite inner product induced by the quadratic form in (1.6):

$$\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]_{\mathfrak{K}} = -\left(z_1, x_2\right)_{\mathcal{X}} - \left(x_1, z_2\right)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}. \tag{1.7}$$

Note that the the quadratic form in (1.6) is strictly indefinite, i.e., it takes both positive and negative values whenever $\mathcal{X} \neq \{0\}$. Furthermore, the inner product in (1.7) makes the node space $\mathfrak{K}$ a Kreĭn space.

The equality (1.6) says that V is a *neutral subspace* of $\mathfrak{K}$ with respect to the inner product (1.7), i.e., that $[v,v]_{\mathfrak{K}} = 0$ for all $v \in V$. The condition that a subspace V is a neutral subspace of $\mathfrak{K}$ can equivalently be written $V \subset V^{[\perp]}$, where

$$V^{[\perp]} := \left\{ k \in \mathfrak{K} \mid [k, k']_{\mathfrak{K}} = 0 \text{ for all } k' \in V \right\}.$$

If instead $V^{[\perp]} \subset V$, then V is called co-neutral, and if $V^{[\perp]} = V$, then V is called *Lagrangian* or *hypermaximal neutral.*

Definition 1.5 A boundary control s/s system $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ is *conservative* if its generating subspace V in (1.3) is a Lagrangian subspace of the node space $\mathfrak{K}$, i.e., if $V = V^{[\perp]}$.

Since every orthogonal companion is closed, necessarily every Lagrangian subspace is closed. Moreover, in [Kurula et al., 2010, Thm 4.3] it was proved that if V in (1.3) is Lagrangian then $\ker(\Gamma)$ is dense in $\mathcal{X}$ and $\operatorname{ran}(\Gamma)$ is dense in $\mathcal{W}$. Since $\ker(\Gamma) \subset \operatorname{dom}(\Gamma) = \operatorname{dom}(L)$, the operator $\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]$ is closed and automatically densely defined. Thus the conditions in Definition 1.3 are satisfied for every Lagrangian subspace V of the type (1.3). See also [Derkach et al., 2006, Cor. 2.4].

Remark In the boundary control case the neutrality condition $V \subset V^{[\perp]}$ means that

$$(Lx, x)_{\mathcal{X}} + (x, Lx)_{\mathcal{X}} = [\Gamma x, \Gamma x]_{\mathcal{W}}, \quad x \in \operatorname{dom}(L). \tag{1.8}$$

However, if V is only neutral, then V might for instance be the degenerate trivial system $\{0\}$. This case is excluded by the hypermaximality condition $V \supset V^{[\perp]}$, which in the case of boundary control means that

$$(z_1, x)_{\mathcal{X}} + (x_1, Lx)_{\mathcal{X}} = [w_1, \Gamma x]_{\mathcal{W}}, \, x \in \operatorname{dom}(L) \Longrightarrow \left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right] \in V. \tag{1.9}$$

Letting $\mathcal{X}$ be a Hilbert space, $\mathcal{W}$ be a Kreĭn space, and $\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right] : \mathcal{X} \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, we thus have that $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ is a conservative boundary control s/s system if and only if the conditions (1.8) and (1.9) are satisfied.

1.4 An example: the transmission line

An ideal transmission line of length ℓ can be modeled by the following equations, where $\xi \in [0, \ell]$ and $t \in \mathbb{R}^+$:

$$\begin{gathered} \frac{\partial}{\partial t} \begin{bmatrix} i(\xi,t) \\ v(\xi,t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\mathcal{L}(\xi)} \frac{\partial}{\partial \xi} \\ -\frac{1}{\mathcal{C}(\xi)} \frac{\partial}{\partial \xi} & 0 \end{bmatrix} \begin{bmatrix} i(\xi,t) \\ v(\xi,t) \end{bmatrix}, \\ w(t) = \left[\begin{smallmatrix} i(0,t) \\ v(0,t) \\ -i(\ell,t) \\ v(\ell,t) \end{smallmatrix}\right], \quad \begin{bmatrix} i(\xi,0) \\ v(\xi,0) \end{bmatrix} = \begin{bmatrix} i_0(\xi) \\ v_0(\xi) \end{bmatrix}. \end{gathered} \tag{1.10}$$

Here $i(\xi,t)$ and $v(\xi,t)$ are the current and voltage, respectively, at the point $\xi \in [0,\ell]$ at time $t \in \mathbb{R}^+$. The functions $\mathcal{L}(\cdot) > 0$ and $\mathcal{C}(\cdot) > 0$ represent the *distributed inductance and capacitance*, respectively, of the line. For simplicity we assume that $\mathcal{C}(\cdot)$ and $\mathcal{L}(\cdot)$ are continuous on $[0,\ell]$, which implies that $\mathcal{C}$ and $\mathcal{L}$ are both bounded and bounded away from zero. The transmission line is illustrated in Figure 1.1.

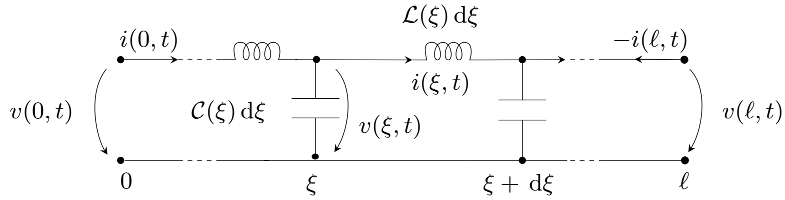

Figure 1.1 An ideal $\mathcal{LC}$-transmission line of length ℓ with *distributed* inductance $\mathcal{L}$ and capacitance $\mathcal{C}$. Here $i(\xi,t)$ and $v(\xi,t)$ denote the current and the voltage, respectively, at the point $\xi \in [0,\ell]$ at time $t \in \mathbb{R}^+$.

The natural state at time t of this transmission line is the current-voltage vector $x(t) = \begin{bmatrix} i(\cdot,t) \\ v(\cdot,t) \end{bmatrix}$, $t \in \mathbb{R}^+$, and the initial state is $x(0) = \begin{bmatrix} i(\cdot,0) \\ v(\cdot,0) \end{bmatrix} = \begin{bmatrix} i_0(\cdot) \\ v_0(\cdot) \end{bmatrix} =: x_0$. We take the state space $\mathcal{X}$ to be $L^2\big([0,\ell];\mathbb{C}^2\big)$ with inner product $(\cdot,\cdot)_{\mathcal{X}}$ defined by

$$\left(\begin{bmatrix} i_1(\cdot) \\ v_1(\cdot) \end{bmatrix}, \begin{bmatrix} i_2(\cdot) \\ v_2(\cdot) \end{bmatrix} \right)_{\mathcal{X}} = \int_0^\ell \big(\mathcal{L}(\xi) i_1(\xi)\overline{i_2(\xi)} + \mathcal{C}(\xi) v_1(\xi)\overline{v_2(\xi)} \big)\,\mathrm{d}\xi. \quad (1.11)$$

In our setting the corresponding quadratic form $(x(t),x(t))_{\mathcal{X}}$ is equivalent to the standard inner product on $L^2([0,\ell];\mathbb{C}^2)$ and its value is twice the energy stored in the state $x(t)$ of the transmission line at time t.

The operator L is given by

$$L := \begin{bmatrix} 0 & -\frac{1}{\mathcal{L}(\xi)}\frac{\partial}{\partial \xi} \\ -\frac{1}{\mathcal{C}(\xi)}\frac{\partial}{\partial \xi} & 0 \end{bmatrix}, \quad \mathrm{dom}\,(L) := W^{1,2}([0,\ell];\mathbb{C}^2),$$

where $W^{1,2}([0,\ell];\mathbb{C}^2)$ is the Sobolev space of absolutely continuous functions in $L^2([0,\ell];\mathbb{C}^2)$ which have a distribution derivative in $L^2([0,\ell];\mathbb{C}^2)$. The signal space $\mathcal{W}$ is $\mathbb{C}^4$ equipped with the indefinite inner product

$$\left[\begin{bmatrix} i_{01} \\ v_{01} \\ i_{\ell 1} \\ v_{\ell 1} \end{bmatrix}, \begin{bmatrix} i_{02} \\ v_{02} \\ i_{\ell 2} \\ v_{\ell 2} \end{bmatrix} \right]_{\mathcal{W}} = \left(\begin{bmatrix} i_{01} \\ v_{01} \\ i_{\ell 1} \\ v_{\ell 1} \end{bmatrix}, J_{\mathcal{W}} \begin{bmatrix} i_{02} \\ v_{02} \\ i_{\ell 2} \\ v_{\ell 2} \end{bmatrix} \right)_{\mathbb{C}^4}, \; J_{\mathcal{W}} = \begin{bmatrix} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] & 0 \\ 0 & \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \end{bmatrix}. \quad (1.12)$$

The boundary operator Γ has the same domain as L, and it is given by

$$\Gamma \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} = \begin{bmatrix} i(0) \\ v(0) \\ -i(\ell) \\ v(\ell) \end{bmatrix}.$$

The operator $\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]$ is closed as an operator from $\mathcal{X}$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ with domain $\mathrm{dom}\left(\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]\right) = \mathrm{dom}\,(L) = W^{1,2}([0,\ell];\mathbb{C}^2)$. With these definitions, the transmission line can be modeled as an example of a boundary control s/s system in the sense of Definition 1.3, as we now show.

We next derive the appropriate Lagrangian identity. Combining $x(t) = \left[\begin{smallmatrix} i(\cdot,t) \\ v(\cdot,t) \end{smallmatrix}\right]$, (1.10), and (1.11), we make the following computations for $t > 0$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 &= 2\mathrm{Re}\ (x(t),\dot{x}(t))_{\mathcal{X}} \\
&= 2\mathrm{Re}\int_0^\ell \left(\mathcal{L}(\xi)i(\xi,t)\overline{\frac{\partial}{\partial t}i(\xi,t)} + \mathcal{C}(\xi)v(\xi,t)\overline{\frac{\partial}{\partial t}v(\xi,t)}\right)\mathrm{d}\xi \\
&= -2\int_0^\ell \mathrm{Re}\left(i(\xi,t)\overline{\frac{\partial}{\partial \xi}v(\xi,t)} + \overline{\frac{\partial}{\partial \xi}i(\xi,t)}v(\xi,t)\right)\mathrm{d}\xi \\
&= -2\int_0^\ell \mathrm{Re}\ \frac{\partial}{\partial \xi}\left(i(\xi,t)\overline{v(\xi,t)}\right)\mathrm{d}\xi \\
&= -2\mathrm{Re}\left[i(\xi,t)\overline{v(\xi,t)}\right]_{\xi=0}^{\ell} \\
&= 2\mathrm{Re}\, i(0,t)\overline{v(0,t)} - 2\mathrm{Re}\, i(\ell,t)\overline{v(\ell,t)} \\
&= \left(\begin{bmatrix} i(0,t) \\ v(0,t) \\ -i(\ell,t) \\ v(\ell,t) \end{bmatrix}, \begin{bmatrix} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] & 0 \\ 0 & \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \end{bmatrix}\begin{bmatrix} i(0,t) \\ v(0,t) \\ -i(\ell,t) \\ v(\ell,t) \end{bmatrix}\right)_{\mathbb{C}^4} \\
&= [\Gamma x(t), \Gamma x(t)]_{\mathcal{W}},
\end{aligned}$$

where we have used that ($'$ denotes spatial derivative)

$$2\mathrm{Re}\,(i\overline{v'} + \overline{i'}v) = i\overline{v'} + \overline{i'}v + \overline{i}v' + i'\overline{v} = 2\mathrm{Re}\,(i\overline{v})'$$

in the fourth equality. Thus, $[w(t),w(t)]_{\mathcal{W}} = [\Gamma x(t), \Gamma x(t)]_{\mathcal{W}}$ is two times the power entering the transmission line through the terminals at the ends $\xi = 0$ and $\xi = \ell$ of the line at time $t \geq 0$.

These computations tell us that the generating subspace V is a neutral subspace of the node space $\mathfrak{K}$, i.e., that (1.8) holds. It is not difficult to show that this subspace is not only neutral, but even Lagrangian, so that (1.9) also holds; see Example 1.9 below for the proof idea. Thus, *the transmission line gives rise to a conservative boundary control s/s system.*

Remark Set $\mathcal{U} := \mathbb{C}^2$, $R := iL|_{\ker\Gamma}$, and

$$\Gamma_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} i(0) \\ -i(\ell) \end{bmatrix} \quad \text{and} \quad \Gamma_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} v(0) \\ v(\ell) \end{bmatrix}. \tag{1.13}$$

Then R is a closed, densely defined and symmetric operator in the

Hilbert space $\mathcal{X}$, and the triple $(\Gamma_0, -\mathrm{i}\Gamma_1; \mathcal{U})$ is a *boundary triplet* for $R^* = \mathrm{i}L$ in the standard sense; see below. The boundary triplet and its connection to boundary-control state/signal systems is the topic of the last section of this chapter.

Recall that $[w(t), w(t)]_{\mathcal{W}}$ is two times the power entering the transmission line through the terminals at the ends $\xi = 0$ and $\xi = \ell$ of the line at time $t \geq 0$. The decomposition in (1.13) of Γ into an input map Γ_0 and an output map Γ_1 corresponds to choosing the current entering the system at $\xi = 0$ and $\xi = \ell$ as input and the voltages at both ends as output, cf. (1.1). We refer to this as an *impedance decomposition* of the external signal w.

Several other choices of input and output would have been possible, such as for example

$$\begin{aligned} \widetilde{\Gamma}_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} &:= \frac{1}{\sqrt{2}}\big(\Gamma_1 + \Gamma_0\big) \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} v(0)+i(0) \\ v(\ell)-i(\ell) \end{bmatrix} \quad \text{and} \\ \widetilde{\Gamma}_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} &:= \frac{1}{\sqrt{2}}\big(\Gamma_1 - \Gamma_0\big) \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} v(0)-i(0) \\ v(\ell)+i(\ell) \end{bmatrix}, \quad \text{or} \end{aligned} \tag{1.14}$$

$$\widehat{\Gamma}_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} i(0) \\ v(0) \end{bmatrix} \quad \text{and} \quad \widehat{\Gamma}_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} -i(\ell) \\ v(\ell) \end{bmatrix}. \tag{1.15}$$

In (1.14) we have

$$\left\| \widetilde{\Gamma}_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} \right\|_{\mathbb{C}^2}^2 - \left\| \widetilde{\Gamma}_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} \right\|_{\mathbb{C}^2}^2 = \left[\Gamma \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix}, \Gamma \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} \right]_{\mathcal{W}},$$

where $[\cdot,\cdot]_{\mathcal{W}}$ still denotes the inner product (1.12). This decomposition is an example of a *scattering decomposition*. In (1.15) we choose voltage and current at $\xi = 0$ as input and the voltage and current at $\xi = \ell$ as output, and this in an example of a *transmission decomposition*.

Remark 1.6 Making a different choice of input and output signals results in a different map from the input to the output, i.e., a different input/state/output representation, with possibly widely different properties. However, the physical system, i.e., the $\mathcal{LC}$-transmission line with length ℓ, is still the same. This “input/output-free” paradigm is inherent in the state/signal philosophy.

1.5 The connection to boundary triplets

Boundary triplets originate from the extension theory of symmetrical operators on Hilbert spaces. The following definition is adapted from

[Gorbachuk and Gorbachuk, 1991, pp. 154–155], using the more recent terminology and notations from [Derkach et al., 2006, Def. 5.1].

Definition 1.7 Let R be a closed densely defined symmetric operator on the Hilbert space $\mathcal{X}$ with equal (finite or infinite) defect numbers $n_\pm := \dim\ker(R \mp i)$. Let $\mathcal{U}$ be another Hilbert space, the "external Hilbert space", and let Γ_j, $j = 0, 1$, be linear operators mapping $\mathrm{dom}\,(R^*)$ into $\mathcal{U}$.

The triplet $(\Gamma_0, \Gamma_1; \mathcal{U})$ is called a *boundary triplet* for the operator R^* if the following two conditions hold:

1. For all $x_1, x_2 \in \mathrm{dom}\,(R^*)$ we have
$$(R^*x_1, x_2)_\mathcal{X} - (x_1, R^*x_2)_\mathcal{X} = (\Gamma_0 x_1, \Gamma_1 x_2)_\mathcal{U} - (\Gamma_1 x_1, \Gamma_0 x_2)_\mathcal{U}.$$
2. The range of the combined operator $\Gamma := \left[\begin{smallmatrix}\Gamma_0\\ \Gamma_1\end{smallmatrix}\right]$ is $\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{U}\end{smallmatrix}\right]$.

Here condition 1 is the *Lagrangian identity* and condition 2 can be interpreted as a *regularity condition* or a *(hyper)maximality condition.*

By a *direct-sum decomposition* $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of a Kreĭn space we mean that $\mathcal{U}$ and $\mathcal{Y}$ are closed subspaces of $\mathcal{W}$, such that $\mathcal{U} + \mathcal{Y} = \mathcal{W}$ and $\mathcal{U} \cap \mathcal{Y} = \{0\}$. This decomposition is *Lagrangian* if $\mathcal{U}$ and $\mathcal{Y}$ are both Lagrangian subspaces: $\mathcal{U} = \mathcal{U}^{[\perp]}$ and $\mathcal{Y} = \mathcal{Y}^{[\perp]}$. For every Hilbert space $\mathcal{U}$, the direct-sum decomposition
$$\mathcal{W} = \widetilde{\mathcal{U}} \dotplus \widetilde{\mathcal{Y}} := \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right] \dotplus \left[\begin{smallmatrix}\{0\}\\ \mathcal{U}\end{smallmatrix}\right] \tag{1.16}$$
of $\mathcal{W} = \mathcal{U}^2$ is *Lagrangian* if $\mathcal{W}$ has the inner product
$$[\left[\begin{smallmatrix}u_1\\ y_1\end{smallmatrix}\right], \left[\begin{smallmatrix}u_2\\ y_2\end{smallmatrix}\right]]_\mathcal{W} = (u_1, y_2)_\mathcal{U} + (y_1, u_2)_\mathcal{U}. \tag{1.17}$$
For instance, the impedance decomposition in the transmission line example, where we take the currents as input and voltages as outputs, is a Lagrangian decomposition.

For a proof of the following result, see [Malinen and Staffans, 2007, Sec. 5]:

Theorem 1.8 *Let R be a closed and densely defined symmetric operator on $\mathcal{X}$ with equal defect numbers, and let* $(\Gamma_0, \Gamma_1; \mathcal{U})$ *be a boundary triplet for R^*. Take $\mathcal{W} := \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{U}\end{smallmatrix}\right]$ with the indefinite inner product* (1.17) *and define $\Gamma := \left[\begin{smallmatrix}\Gamma_0\\ \mathrm{i}\Gamma_1\end{smallmatrix}\right]$ with* $\mathrm{dom}\,(\Gamma) = \mathrm{dom}\,(R^*)$.

Then $\Sigma = (\mathrm{i}R^*, \Gamma; \mathcal{X}, \mathcal{W})$ is a boundary control s/s system *in the sense of Definition 1.3. The system is moreover* conservative*: $V = V^{[\perp]}$, where V is given by* (1.3).

Consider the conservative boundary control s/s system Σ in Theorem 1.8. The *input/state/output representation*

$$\Sigma_{i/s/o} = \left(\mathrm{i}R^*, \begin{bmatrix} \Gamma_0 \\ \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} \\ \mathrm{i}\Gamma_1 \end{bmatrix}; \mathcal{X}, \begin{bmatrix} \mathcal{U} \\ \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} \\ \mathcal{U} \end{bmatrix}\right)$$

corresponding to the Lagrangian decomposition (1.16) is an example of an *impedance representation* of Σ. We investigate these concepts in more detail in Section **??**.

The converse of Theorem 1.8 is not true: *there do exist conservative boundary control s/s systems which are not induced by any boundary triplet of the type in Definition 1.7.* These examples are of two types:

1. The signal space $\mathcal{W}$ *need not have a Lagrangian decomposition.* A necessary and sufficient condition for the existence of a Lagrangian decomposition is that $\mathrm{ind}_+\mathcal{W} = \mathrm{ind}_-\mathcal{W}$ $(\leq \infty)$; see Example 1.9 below. In the case of a boundary triplet we always have at least the Lagrangian decomposition (1.16).
2. Even if the signal space $\mathcal{W}$ has a Lagrangian decomposition *the main operator L need not be closed*, and we can thus not have $L = \mathrm{i}R^*$. Moreover, *the operator* $\Gamma := \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}$ *need not be surjective.* See [Malinen and Staffans, 2007] for an example.

   More precisely, let $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ be a conservative boundary control s/s system. According to [Kurula et al., 2010, Prop. 4.5], *L is closed if and only if the range of Γ is closed.* Combining this with the condition that Γ has dense range, we obtain that L is closed if and only if Γ is surjective. The same conclusion can be made based on [Derkach et al., 2006, Prop. 2.3 and Cor. 2.4].

We now give an example of a conservative boundary control s/s system that is not induced by a boundary triplet. In a scattering setting this system has no input and a one-dimensional output, and the C_0-semigroup describing the system dynamics is the left shift in $L^2(\mathbb{R}^+;\mathbb{C})$.

Example 1.9 Choose $\mathcal{X} := L^2(\mathbb{R}^+;\mathbb{C})$ with its standard Hilbert-space inner product, set $\mathcal{W} := -\mathbb{C}$, and define

$$V := \left\{ \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\xi} \\ x \\ x(0) \end{bmatrix} \middle| \, x \in W^{1,2}(\mathbb{R}^+;\mathbb{C}) \right\} \subset \mathcal{X} \times \mathcal{X} \times \mathcal{W}.$$

It is clear that $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$ implies that $z = 0$, and we will now show that $V = V^{[\perp]}$, i.e., that $(V; \mathcal{X}, \mathcal{W})$ is a conservative boundary control s/s system. Note that the signal space $\mathcal{W}$ has no Lagrangian decompositions.

We first prove that $V^{[\perp]} \subset V$. By definition $\left[\begin{smallmatrix} \widetilde{z} \\ \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in V^{[\perp]}$ if and only if $\left[\begin{smallmatrix} \widetilde{z} \\ \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \mathfrak{K} = L^2(\mathbb{R}^+;\mathbb{C}) \times L^2(\mathbb{R}^+;\mathbb{C}) \times \mathbb{C}$ and for all $x \in W^{1,2}(\mathbb{R}^+;\mathbb{C})$:

$$\begin{aligned} &\left[\begin{bmatrix} \widetilde{z} \\ \widetilde{x} \\ \widetilde{w} \end{bmatrix}, \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\xi} \\ x \\ x(0) \end{bmatrix}\right]_{\mathfrak{K}} \\ &\qquad = -\widetilde{w}\,\overline{x(0)} - \int_0^\infty \left(\widetilde{x}(\xi)\overline{\frac{\mathrm{d}x}{\mathrm{d}\xi}(\xi)} + \widetilde{z}(\xi)\overline{x(\xi)}\right) \mathrm{d}\xi = 0. \end{aligned} \tag{1.18}$$

In particular, if we let x vary over the set of test functions in C^∞ with support contained in $(0,\infty)$, then $x(0) = 0$, and we find that $\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\xi} = \widetilde{z}$ in the distribution sense. Since both $\widetilde{x}$ and $\widetilde{z}$ belong to $L^2(\mathbb{R}^+;\mathbb{C})$, this implies that $\widetilde{x} \in W^{1,2}(\mathbb{R}^+;\mathbb{C})$. This makes it is possible to integrate by parts in (1.18), using that $\widetilde{z}(\xi) = \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\xi}(\xi)$, in order to get that

$$\widetilde{w}\,\overline{x(0)} = \widetilde{x}(0)\,\overline{x(0)}, \quad x \in W^{1,2}(\mathbb{R}^+;\mathbb{C}).$$

Thus $\widetilde{w} = \widetilde{x}(0)$, and this proves that $V^{[\perp]} \subset V$.

In order to show that $V \subset V^{[\perp]}$, we choose $\widetilde{x} \in W^{1,2}(\mathbb{R}^+;\mathbb{C})$ arbitrarily, and we set $\widetilde{z} := \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\xi}$ and $\widetilde{w} := \widetilde{x}(0)$. Then (1.18) holds for all $x, \widetilde{x} \in W^{1,2}(\mathbb{R}^+;\mathbb{C})$, i.e., $V \subset V^{[\perp]}$. We are done proving that $V = V^{[\perp]}$, and therefore, that $(V;\mathcal{X},\mathcal{W})$ is a conservative boundary control s/s system whose signal space $\mathcal{W} = -\mathbb{C}$ has no Lagrangian decompositions.

The i/s/o case where $\Gamma = \left[\begin{smallmatrix} \Gamma_0 \\ \Gamma_1 \end{smallmatrix}\right] : \mathcal{X} \to \mathcal{U}^2$ has dense but non-closed range has been treated using *generalized boundary triplets* in [Derkach and Malamud, 1995] and using *quasi boundary triplets* in [Behrndt and Langer, 2007]. Interconnection of conservative boundary control i/s/o systems with surjective $\left[\begin{smallmatrix} \Gamma_0 \\ \Gamma_1 \end{smallmatrix}\right]$ was worked out in detail in [Kurula et al., 2010].

A considerably more general notion than that of a boundary triplet is that of a *boundary relation* which was extensively studied in e.g. [Derkach et al., 2006]. The topic of Chapter 2, which is more detailed than the present one, is to show how boundary relations are connected to general (non-boundary control) s/s systems. There the main point is to show that the notion of a boundary relation is connected to the notion of a conservative state/signal system in the same way as the boundary triplet is related to the boundary control s/s system: the former arises as a particular i/s/o impedance representation of the latter.

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2

Passive state/signal systems and conservative boundary relations

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Abstract This chapter is a continuation and deepening of Chapter 1. In the present chapter the state/signal theory is extended beyond boundary control and beyond conservative systems. The main aim is to clarify the basic connections between the state/signal theory and that of (conservative) boundary relations. It is described how one can represent a state/signal system using input/state/output systems in different ways by making different choices of input signal and output signal. There is an "almost one-to-one" relationship between conservative state/signal systems and boundary relations, and this connection is used in order to introduce dynamics to a boundary relation. Consquently, a boundary relation is such a general object that it mathematically has rather little to do with boundary control. The Weyl family and γ-field of a boundary relation are connected to the frequency-domain characteristics of a state/signal system.

2.1 Introduction

The theory of boundary relations has been developed by a number of authors in the theory of self-adjoint extensions of symmetrical operators and relations in Hilbert spaces; see e.g. the recent articles [Derkach et al., 2006, 2009; Behrndt et al., 2009].

One way of introducing the notion of a state/signal (s/s) system is to start from an input/state/output (i/s/o) system. By a standard i/s/o system we mean a system of equations of the type

$$\Sigma_{i/s/o}: \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \ x(0) = x_0 \text{ given}, \qquad (2.1)$$

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where $\dot{x}$ stands for the time derivative of x. Here x, u and y take values in the Hilbert spaces $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$, that are called the "state space", the "input space" and the "output space", respectively. For now the linear operators A, B, C, and D are assumed to be bounded, but we soon drop this restrictive assumption.

The system (2.1) can be viewed as an i/s/o representation of a s/s system by setting $\mathcal{W} := \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$ and using the graph V of the operator $\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$:

$$\begin{aligned}
&\Sigma: \quad \begin{bmatrix} \dot{x}(t) \\ x(t) \\ \left[\begin{smallmatrix} u(t) \\ y(t) \end{smallmatrix}\right] \end{bmatrix} \in V, \quad t \in \mathbb{R}^+, \; x(0) = x_0, \quad \text{where} \\
&V := \left\{ \begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{matrix} z = Ax + Bu \\ y = Cx + Du \end{matrix} \right\}.
\end{aligned} \tag{2.2}$$

This reformulation might seem trivial, but many concepts, such as that of a passive or a conservative system, are much simpler to formulate in the s/s framework than in the i/s/o counterpart; see Remark 2.16 below. Moreover, the input/output-free approach of the s/s theory permits the study of a physical system *as such* by looking at the geometric properties of V instead of merely studying *a particular representation* $\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$ of the system, cf. Remark 1.6 in Chapter 1.

We give the general definition of a s/s system, which does not a priori assume a representation (2.2), and we discuss well-posed i/s/o representations in Section 2.2. In Section 2.3 we study passive and conservative systems in more detail. The topic of Section 2.4 is frequency domain theory, and here we introduce the characteristic node bundle of a s/s system, which extends the notions of the γ-field and the Weyl family of a boundary relation. We make the precise connection between s/s systems and boundary relations in Section 2.5, where we also describe exactly how to transform a conservative s/s system into a boundary relation and vice versa.

2.2 Continuous-time state/signal systems

In this section we extend the ideas in Section 1.2 to more general generating subspaces V than those arising from either an i/s/o representation of the type (2.2) or from a boundary control s/s system.

2.2.1 General definitions We first introduce the s/s node and the s/s system that it induces.

Definition 2.1 Let $\mathcal{X}$ be a Hilbert space, let $\mathcal{W}$ be a Kreĭn space, and let V be a *closed* subspace of the node space $\mathfrak{K} = \mathcal{X} \times \mathcal{X} \times \mathcal{W}$ equipped with the indefinite inner product induced by the quadratic form

$$\left[\left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ x_2 \\ w_2 \end{smallmatrix}\right]\right]_{\mathfrak{K}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}.$$

The pair $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a *classical trajectory* generated by V on $\mathbb{R}^+$ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$, $w \in C(\mathbb{R}^+; \mathcal{W})$, and

$$\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V, \quad t \in \mathbb{R}^+, \tag{2.3}$$

where $\dot{x}$ stands for the time derivative of x (at $t = 0$ this is the right-sided derivative of x at zero). The closure of the set of classical trajectories on $\mathbb{R}^+$ in $\left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ is the set of *generalized trajectories* on $\mathbb{R}^+$ generated by V.

Moreover, $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a *state/signal node (s/s node)* if V has the following properties in addition to being closed:

1. The generating subspace V satisfies the condition
$$\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \implies z = 0. \tag{2.4}$$
2. For every $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ there exists a classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+ := [0, \infty)$ that satisfies $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_0 \\ w_0 \\ w_0 \end{smallmatrix}\right]$.

By the *state/signal system (s/s system)* induced by a s/s node $(V; \mathcal{X}, \mathcal{W})$ we mean the s/s node itself together with its sets of classical and generalized trajectories on $\mathbb{R}^+$ generated by V.

It follows immediately from part 2 of Definition 2.1 that a space of classical trajectories determines its generating subspace uniquely through

$$V = \left\{ \left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \text{ is a classical trajectory} \right\}.$$

It is less obvious, but still true, that a space of generalized trajectories determines its generating s/s node uniquely. This is because the space of generalized trajectories of a s/s node determines the space of classical trajectories uniquely. Indeed, a generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is in fact a classical trajectory if and only if $x \in C^1(\mathbb{R}^+; \mathcal{X})$ and $w \in C(\mathbb{R}^+; \mathcal{W})$. For proof, see [Kurula and Staffans, 2011, Cor. 3.2].

Example 2.2 Let $(L, \Gamma; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a boundary control s/s node as given in Definition 1.3. This does in general not imply that $\Sigma := (V; \mathcal{X}, \mathcal{W})$ is a s/s system, where V is given by

$$V = \left\{ \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \,\middle|\, x \in \operatorname{dom}(L) \right\},$$

because V might not have property 2 of Definition 2.1. We prove in Example 2.7 below that Σ is indeed a s/s system when L and Γ arise from a boundary control i/s/o system of the type described in Definition 1.1.

The fact that the generating subspace V is independent of the time variable t means that the state/signal system is *time invariant.* Moreover, condition (2.4) means that V is the graph of some linear operator $G\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \to \mathcal{X}$ with domain $\operatorname{dom}(G) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, i.e., that

$$V = \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \,\middle|\, z = G \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right], \ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \operatorname{dom}(G) \right\}.$$

The assumption that V is closed means that G is a closed operator. Now (2.3) can alternatively be written in the form

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \operatorname{dom}(G) \quad \text{and} \quad \dot{x}(t) = G \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \tag{2.5}$$

and all classical trajectories generated by V satisfy this condition.

Example 2.3 If V is given by (2.2) then the operator G defined above is given by

$$\begin{aligned} G &= \begin{bmatrix} A & \begin{bmatrix} B & 0 \end{bmatrix} \end{bmatrix} \Big|_{\operatorname{dom}(G)} \quad \text{with} \\ \operatorname{dom}(G) &= \left\{ \begin{bmatrix} x \\ \left[\begin{smallmatrix} u \\ Cx+Du \end{smallmatrix}\right] \end{bmatrix} \,\middle|\, \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \right\}. \end{aligned} \tag{2.6}$$

Note, however, that the operators A, B, C, and D in (2.2), and therefore also in (2.6), by construction depend on a particular choice of i/o (input/output) decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, whereas (2.5) does not. In this sense (2.5) is a truly coordinate-free differential-equation representation of a s/s system.

Let us now go back to the general case. Condition 2 in Definition 2.1 means that there for all $\left[\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right] \in \operatorname{dom}(G)$ exists a classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \begin{bmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$ such that $\begin{bmatrix} x(0) \\ w(0) \end{bmatrix} = \left[\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right]$. From the condition $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \in V$ we immediately obtain that this trajectory also satisfies $\dot{x}(0) = G \left[\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right]$.

It is an interesting observation that we can represent an arbitrary s/s system by a closed operator G, and it helps to build intuition, but we shall not make any significant use of this operator in this exposition. For our present purposes it is more convenient to use (2.3).

As is well-known, an arbitrary Kreĭn space $\mathcal{W}$ can be interpreted as a Hilbert space consisting of the same vectors as $\mathcal{W}$. This is done by equipping $\mathcal{W}$ with an *admissible Hilbert-space inner product*; see Remark 2.3 below. An important consequence is that, from a topological point of view, every closed subspace of a Kreĭn space can be regarded as a Hilbert space, and we make frequent use of this.

Definition 2.4 A direct-sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of a Kreĭn space is *i/s/o well-posed* for the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ if the following two conditions hold:

1. For every $x_0 \in \mathcal{X}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ there exists a generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$ with $x(0) = x_0$ and $P^{\mathcal{Y}}_{\mathcal{U}} w = u$.
2. There exists a positive nondecreasing function K on $\mathbb{R}^+$ such that every generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$ satisfies

$$\begin{aligned} &\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|P^{\mathcal{U}}_{\mathcal{Y}} w(s)\|^2_{\mathcal{W}} \,\mathrm{d}s \\ &\qquad \le K(t) \left(\|x(0)\|^2_{\mathcal{X}} + \int_0^t \|P^{\mathcal{Y}}_{\mathcal{U}} w(s)\|^2_{\mathcal{W}} \,\mathrm{d}s \right), \quad t \in \mathbb{R}^+. \end{aligned} \tag{2.7}$$

   Here $\|\cdot\|_{\mathcal{W}}$ stands for an arbitrary admissible norm in $\mathcal{W}$.

The s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is *well-posed* if there exists at least one i/s/o well-posed decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space $\mathcal{W}$.

For more details on well-posed s/s systems, see [Kurula and Staffans, 2009]. In the next section we elaborate on the topic of representing state/signal systems by i/s/o systems.

Input/state/output representations The simplest example of a s/s system may be constructed by starting from a bounded classical linear i/s/o continuous-time system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as we did in the introduction.

However, applications often require that the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is unbounded. In the unbounded case the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ in (2.1) can be replaced by an *i/s/o system node* operator $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. Here the top and bottom rows are denoted by $A\&B$ and $C\&D$ in order to indicate the connection to (2.1), but this notation is purely symbolic. In general it is possible to extend $A\&B$ into an operator $\begin{bmatrix} A_{-1} & B \end{bmatrix}$ which maps $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ continuously into a larger extrapolation space $\mathcal{X}_{-1}$. The operator A_{-1} is the continuous extension to $\mathcal{X}$ of the generator A of a C_0-semigroup on $\mathcal{X}$. Unfortunately, $C\&D$ does not split correspondingly. One can define an operator C, whose domain is a subspace of $\mathcal{X}$ containing the domain of A, but there is no uniquely defined operator corresponding to D in the general unbounded case. See [Staffans, 2005, Chapter 5] for details.

We now give a definition of an abstract system node, which is based on [Staffans, 2005, Lem. 4.7.7].

Definition 2.5 By an *i/s/o-system node* $(S;\mathcal{X},\mathcal{U},\mathcal{Y})$ we mean a triple of Hilbert spaces $\mathcal{X}$ (the state space), $\mathcal{U}$ (the input space), and $\mathcal{Y}$ (the output space), together with a linear operator

$$S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \operatorname{dom}(S) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right], \quad \operatorname{dom}(S) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right],$$

with the following properties:

1. The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is closed as an operator mapping $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with domain $\operatorname{dom}(S)$.
2. The operator $A\&B : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \mathcal{X}$ is closed with domain $\operatorname{dom}(S)$.
3. The *main operator* A of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, defined by
$$Ax = A\&B \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \quad \text{on} \quad \operatorname{dom}(A) = \left\{ x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \operatorname{dom}(S) \right\},$$
generates a strongly continuous semigroup $t \mapsto \mathfrak{A}^t$ on $\mathcal{X}$.
4. For all $u \in \mathcal{U}$ there exists an $x \in \mathcal{X}$ such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)$.

The triple (u, x, y) is said to be a *classical i/s/o trajectory* of the i/s/o system node $(S;\mathcal{X},\mathcal{U},\mathcal{Y})$ if $u \in C(\mathbb{R}^+;\mathcal{U})$, $x \in C^1(\mathbb{R}^+;\mathcal{X})$, $y \in C(\mathbb{R}^+;\mathcal{Y})$, and

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+. \tag{2.8}$$

As we can see from the above definition, in the unbounded case (2.1) is replaced by (2.8). The i/s/o system (2.8) can again be interpreted as an i/s/o representation of a s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ by taking $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ and defining

$$V := \left\{ \begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right],\ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S) \right\}. \tag{2.9}$$

Here $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ stands for the product of $\mathcal{U}$ and $\mathcal{Y}$ which can be turned into a Kreĭn space by equipping it with any of several indefinite inner products. A few important choices of inner product will be described later.

Recall that the component spaces $\mathcal{U}$ and $\mathcal{Y}$ of every direct-sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of a Kreĭn space can be interpreted as Hilbert spaces with inner products inherited from some admissible inner product in $\mathcal{W}$.

Remark In the sequel we call $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are arbitrary closed subspaces of some Kreĭn spaces, an i/s/o system node if $\Sigma_{i/s/o}$ is an i/s/o system node in the sense of Definition 2.5 with $\mathcal{U}$ and $\mathcal{Y}$ equipped with admissible inner products.

We define an i/s/o representation of a general s/s system $(V; \mathcal{X}, \mathcal{W})$.

Definition 2.6 Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system and let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be an arbitrary direct-sum decomposition of the signal space.

Assume that V can be written on the form (2.9), where $\Sigma_{i/s/o} := (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is an i/s/o system node. Then we call $\Sigma_{i/s/o}$ the *i/s/o representation* of Σ corresponding to the i/o (input/output) decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \{0\} \end{smallmatrix}\right] \dotplus \left[\begin{smallmatrix} \{0\} \\ \mathcal{Y} \end{smallmatrix}\right]$, and we call the i/o decomposition *system-node admissible*, or shortly just *admissible*.

The i/s/o representation $\Sigma_{i/s/o}$ is uniquely determined by the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and the decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ (except for the fact that the norms and inner products in $\mathcal{U}$ and $\mathcal{Y}$ are determined only up to equivalence), since V is the graph of S in the sense of (2.9). In general, a s/s system Σ has several i/s/o representations, one induced by every admissible i/o decomposition of $\mathcal{W}$.

Example 2.7 Let $\Sigma_{i/s/o} = (L, \Gamma_0, \Gamma_1; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o boundary control system of the type in Definition 1.1. We let $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, equipped with an arbitrary Kreĭn-space inner product, e.g. the standard Hilbert-space inner product, and we define

$$V := \left\{ \begin{bmatrix} Lx \\ x \\ \left[\begin{smallmatrix} \Gamma_0 x \\ \Gamma_1 x \end{smallmatrix}\right] \end{bmatrix} \middle| \, x \in \operatorname{dom}(L) \right\}. \tag{2.10}$$

We now prove that $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a s/s system with admissible i/o decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \{0\} \end{smallmatrix}\right] \dotplus \left[\begin{smallmatrix} \{0\} \\ \mathcal{Y} \end{smallmatrix}\right]$. We find the corresponding i/s/o representation $(S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ by identifying $\mathcal{U} = \left[\begin{smallmatrix} \mathcal{U} \\ \{0\} \end{smallmatrix}\right]$ and $\mathcal{Y} = \left[\begin{smallmatrix} \{0\} \\ \mathcal{Y} \end{smallmatrix}\right]$, and by noting that the map from $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$, where $\begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in V$, is given by

$$S = \left[\begin{smallmatrix} L \\ \Gamma_1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 \\ \Gamma_0 \end{smallmatrix}\right]^{-1}, \quad \operatorname{dom}(S) = \left\{ \left[\begin{smallmatrix} x \\ \Gamma_0 x \end{smallmatrix}\right] \,\middle|\, x \in \operatorname{dom}(L) \right\}.$$

A detailed investigation of the connections between $\begin{bmatrix} L \\ \Gamma_0 \\ \Gamma_1 \end{bmatrix}$ and S can be found in Section 2 of [Malinen and Staffans, 2006]. In particular, S is a system node by [Malinen and Staffans, 2006, Thm 2.3], and from [Kurula, 2010, Prop. 2.7] it then follows that Σ is a s/s node, as we claimed above. This shows how nicely boundary control can be incorporated into the general s/s framework.

If $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is an i/s/o representation of $\Sigma = (V; \mathcal{X}, \mathcal{W})$, so that V is given by (2.9), then the well-posedness condition (2.7) is equivalent to the condition that every classical trajectory (u, x, y) of $\Sigma_{i/s/o}$ satisfies the following inequality for all $t \in \mathbb{R}^+$:

$$\|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(s)\|_{\mathcal{Y}}^2 \, \mathrm{d}s \leq K(t) \left(\|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(s)\|_{\mathcal{U}}^2 \, \mathrm{d}s \right). \tag{2.11}$$

Definition 2.8 Input/state/output systems whose classical trajectories (u, x, y) satisfy (2.11) for a positive nondecreasing function K, which does not depend on the trajectory, are called *well-posed*.

It follows directly from Definitions 2.4 and 2.6 that if an i/o decomposition is both admissible and well-posed for a s/s system then the corresponding i/s/o representation is i/s/o-well-posed. In fact, every well-posed i/o decomposition is admissible by [Kurula and Staffans, 2009, Thms 4.9 and 6.4]. For more detailed information about i/s/o system nodes and well-posed i/s/o systems we refer the reader to [Staffans, 2005].

2.3 Passive and conservative state/signal systems

In this section we describe the concepts of passivity and conservativity within the state/signal system framework.

We need the notion of an *anti-Hilbert space*. A Kreĭn space $\mathcal{Y}$ is an anti-Hilbert space if $-\mathcal{Y}$, i.e., the space of all vectors in $\mathcal{Y}$ equipped with the inner product $-[\cdot, \cdot]_{\mathcal{Y}}$, is a Hilbert space.

Definition 2.9 A direct-sum decomposition $\mathcal{W} = \mathcal{W}_1 \dot{+} \mathcal{W}_2$ of a Kreĭn space is called:

1. *orthogonal* if every vector $w_1 \in \mathcal{W}_1$ is orthogonal to every vector in $w_2 \in \mathcal{W}_2$: $[w_1, w_2]_{\mathcal{W}} = 0$, and we write this as $\mathcal{W} = \mathcal{W}_1 \boxplus \mathcal{W}_2$.

2. *fundamental* if $\mathcal{W} = \mathcal{W}_1 \boxplus \mathcal{W}_2$, where $\mathcal{W}_1$ is a Hilbert space and $\mathcal{W}_2$ is an anti-Hilbert space in the inner product inherited from $\mathcal{W}$. In this case we denote $\mathcal{W}_+ := \mathcal{W}_1$ and $\mathcal{W}_- := -\mathcal{W}_2$, so that $\mathcal{W}_+$ always is the Hilbert space component and $-\mathcal{W}_-$ is the anti-Hilbert space component, and we write $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$.
3. *Lagrangian* if $\mathcal{W}_1$ and $\mathcal{W}_2$ are both Lagrangian: $\mathcal{W}_j = \mathcal{W}_j^{[\perp]}$. We introduce and explain the special notation $\mathcal{W} = \mathcal{W}_1 \overset{\Psi}{+} \mathcal{W}_2$ for Lagrangian decompositions in Definition 2.19 below.

When $\mathcal{W}$ is the signal space of a s/s system we typically use $\mathcal{W}_1$ as input space and $\mathcal{W}_2$ as output space in i/s/o representations. In this connection we do not always use the inner products inherited from $\mathcal{W}$ in $\mathcal{W}_1$ and $\mathcal{W}_2$. In a Lagrangian decomposition the subspaces do not even inherit a unique inner product from $\mathcal{W}$. In the case of orthogonal (and fundamental) decompositions we throughout take the input space to be $\mathcal{U} := \mathcal{W}_1$ with the inner product inherited from $\mathcal{W}$ and the output space to be $\mathcal{Y} := -\mathcal{W}_2$. Thus in the case of a fundamental decomposition both $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces.

If $\mathcal{W} = \mathcal{U} \boxplus \mathcal{Y}$, then in fact $\mathcal{Y} = \mathcal{U}^{[\perp]}$ and both $\mathcal{U}$ and $\mathcal{Y}$ are themselves Kreĭn spaces. Every Kreĭn space, which is neither a Hilbert space nor an anti-Hilbert space, has an uncountable number of fundamental decompositions $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$. For every fundamental decomposition it holds that

$$\begin{aligned}
[w_+, w_+]_{\mathcal{W}} &= (w_+, w_+)_{\mathcal{W}_+} > 0, && w_+ \in \mathcal{W}_+,\ w_+ \neq 0,\\
[w_-, w_-]_{\mathcal{W}} &= -\,(w_-, w_-)_{\mathcal{W}_-} < 0, && w_- \in -\mathcal{W}_-,\ w_- \neq 0.
\end{aligned}$$

Remark Let $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ be a fundamental decomposition of a Kreĭn space. Then $\mathcal{W}$ can be viewed as a Hilbert space with the inner product

$$\begin{aligned}
(w_{1,+} + w_{1,-}, w_{2,+} + w_{2,-})_{\mathcal{W}} = (w_{1,+}, w_{2,+})_{\mathcal{W}_+} + (w_{1,-}, w_{2,-})_{\mathcal{W}_-},\\
w_{1,+}, w_{2,+} \in \mathcal{W}_+, \quad w_{1,-}, w_{2,-} \in -\mathcal{W}_-.
\end{aligned}$$

This inner product is called an *admissible inner product* and the norm induced by this inner product is called an *admissible norm.*

If $\mathcal{W}$ is either a Hilbert space or an anti-Hilbert space, then $\mathcal{W}$ has one unique fundamental decomposition, but in all other case $\mathcal{W}$ has infinitely many fundamental decompositions, and consequently also infinitely many admissible norms. However, all of these norms are equivalent.

Thus, once a fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ has been fixed, each $w \in \mathcal{W}$ has a unique decomposition $w = w_+ + w_-$ with $w_\pm \in \mathcal{W}_\pm$, and

$$[w, w]_{\mathcal{W}} = (w_+, w_+)_{\mathcal{W}_+} - (w_-, w_-)_{\mathcal{W}_-} = \|w_+\|^2_{\mathcal{W}_+} - \|w_-\|^2_{\mathcal{W}_-}. \quad (2.12)$$

The dimensions of $\mathcal{W}_\pm$ do not depend on the choice of fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$. They are called the positive and negative indices of $\mathcal{W}$ and are denoted by $\operatorname{ind}_\pm \mathcal{W}$. A Lagrangian decomposition of $\mathcal{W}$ exists if and only if $\operatorname{ind}_+ \mathcal{W} = \operatorname{ind}_- \mathcal{W}$.

Passive s/s systems and scattering representations We first recall that a subspace V of a Kreĭn space $\mathfrak{K}$ is called *non-negative, non-positive*, or *neutral* if every vector $v \in V$ satisfies

$$[v, v]_{\mathfrak{K}} \geq 0, \quad [v, v]_{\mathfrak{K}} \leq 0, \quad \text{or} \quad [v, v]_{\mathfrak{K}} = 0,$$

respectively. A non-negative (or non-positive) subspace is called maximal non-negative (or maximal non-positive) if it is not strictly contained in any other non-negative (or non-positive) subspace. Such a subspace is automatically closed. A subspace V is *Lagrangian* if $V = V^{[\perp]}$, where $V^{[\perp]}$ is given by

$$V^{[\perp]} := \left\{ k \in \mathfrak{K} \,\middle|\, [k, k']_{\mathfrak{K}} = 0 \text{ for all } k' \in V \right\}.$$

Since many physical systems lack internal energy sources, it is natural to require the generating subspace V to be non-negative in the node space $\mathfrak{K} := \mathcal{X} \times \mathcal{X} \times \mathcal{W}$ which is equipped with the inner product

$$\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]_{\mathfrak{K}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}; \quad (2.13)$$

cf. Definition 2.1.

The node space $\mathfrak{K}$ is a Kreĭn space with the fundamental decomposition $\mathfrak{K} = \mathfrak{K}_+ \boxplus -\mathfrak{K}_-$, where

$$\mathfrak{K}_\pm = \left\{ \begin{bmatrix} \mp x \\ x \\ w_\pm \end{bmatrix} \,\middle|\, x \in \mathcal{X},\ w_\pm \in \mathcal{W}_\pm \right\}$$

and $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ is an arbitrary fundamental decomposition of $\mathcal{W}$. As an immediate consequence, we have that $\operatorname{ind}_\pm \mathfrak{K} = \dim \mathcal{X} + \operatorname{ind}_\pm \mathcal{W}$.

Just as in the case of boundary control, it is immediate that all classical trajectories on $\mathbb{R}^+$ generated by a non-negative V satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|^2_{\mathcal{X}} \leq [w(t), w(t)]_{\mathcal{W}}, \quad t \in \mathbb{R}^+, \quad \text{and} \quad (2.14)$$

$$\|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2 \le \int_s^t [w(v), w(v)]_{\mathcal{W}} \, \mathrm{d}v, \quad s,t \in \mathbb{R}^+, \; s \le t, \quad (2.15)$$

where the second inequality holds also for the generalized trajectories.

However, non-negativity of V does not yet imply that $(V; \mathcal{X}, \mathcal{W})$ is a s/s node. The situation is analogous to the situation in semigroup theory: the generator of a contraction semigroup is not just dissipative, but even *maximal* dissipative; see the Lumer-Phillips Theorem [Staffans, 2005, Thm 3.4.8].

Definition 2.10 A s/s system $\Sigma = (V; \mathcal{X}; \mathcal{W})$ is said to be *passive* if V is a maximal non-negative subspace of the node space $\mathfrak{K}$, i.e., with respect to the inner product (2.13). The system Σ is *conservative* if $V = V^{[\perp]}$.

I/s/o representations corresponding to fundamental decompositions of the signal space of a passive s/s system are exceptionally well-behaved, and we now investigate these in more detail.

Definition 2.11 Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system and let $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o representation of Σ in the sense of Definition 2.6. Then $\Sigma_{i/s/o}$ is called a *scattering representation* of Σ if $\mathcal{U} = \mathcal{W}_+$ and $\mathcal{Y} = \mathcal{W}_-$, where $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ is a fundamental decomposition.

Let $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ be a fundamental decomposition, and set $\mathcal{U} := \mathcal{W}_+$ and $\mathcal{Y} := \mathcal{W}_-$. Combining (2.15) and (2.12) we obtain that every classical trajectory of a passive s/s system satisfies (with $u(v) \in \mathcal{U}$ and $y(v) \in \mathcal{Y}$):

$$\|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2 \le \int_s^t \|u(v)\|_{\mathcal{U}}^2 - \|y(v)\|_{\mathcal{Y}}^2 \, \mathrm{d}v \qquad (2.16)$$

for every $s, t \in \mathbb{R}^+$ such that $s \le t$. This is the well-known *scattering-passivity inequality.* Note that (2.16) implies (2.7) with $K(t) = 1$, $t \in \mathbb{R}^+$.

The first part of the following further development of the above ideas was proved as Theorem 4.5 and Proposition 5.8 in [Kurula, 2010]. The second part follows from the first part and Definition 2.4.

Theorem 2.12 *Assume that V is a maximal non-negative subspace of $\mathfrak{K}$ satisfying* (2.4)*: $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$ only if $z = 0$. Then $(V; \mathcal{X}, \mathcal{W})$ is a passive well-posed s/s node for which every fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ is (admissible and) well-posed and the corresponding scattering representation with input space $\mathcal{U} = \mathcal{W}_+$ and output space $\mathcal{Y} = \mathcal{W}_-$ is well-posed.*

In particular, for every $x_0 \in \mathcal{X}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ there exists a unique generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$ with $x(0) = x_0$ and $P^{\mathcal{Y}}_{\mathcal{U}} w = u$.

Thus a triple $(V; \mathcal{X}, \mathcal{W})$ is a passive s/s system if and only if V is a maximal non-negative subspace of $\mathfrak{K}$ with the property (2.4).

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system. Each different fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ gives rise to a different scattering representation, so there always exist uncountably many scattering representations of a given *passive* s/s system (except for the degenerate cases where the energy exchange through the external signal is unidirectional).

Now suppose that $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has the property that V is *maximal non-positive.* Then (2.14) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \geq [w(t), w(t)]_{\mathcal{W}}, \quad t \in \mathbb{R}^+,$$

and an analogue of Theorem 2.12 can be formulated for Σ, which says that Σ is *well-posed in the backward time direction*, and that every fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$ yields a well-posed i/s/o representation if we take the *output* space to be $\mathcal{Y} = \mathcal{W}_+$ and the *input* space to be $\mathcal{U} = \mathcal{W}_-$.

Definition 2.13 We call a triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with a maximal non-positive generating subspace V satisfying (2.4) an *anti-passive s/s node* (in the backward time direction), i.e., it has properties 1 and 2 in Definition 2.1 with $\mathbb{R}^+$ replaced by $\mathbb{R}^-$.

It is well-known that $V = V^{[\perp]}$ if and only if V is both maximal non-negative and maximal non-positive. A conservative s/s system is thus one that is at the same time both passive and anti-passive. We conclude that conservative s/s systems are *i/s/o well-posed both in the forward and in the backward time directions.* This does *not* imply that the signal space $\mathcal{W}$ has a direct sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ which is i/s/o well-posed both in the forward and backward time direction. We provided a conservative system for which no decomposition of the signal space is admissible both in the forward and backward time directions in Example 1.9. Indeed, when that system is solved in forward time, $x(0) \in \mathbb{C}$ is the unique output and there is no input, and when the system is solved in backward time, $x(0)$ is the unique input and there is no output. See [Kurula, 2010, Thm 4.11] for more details on conservative s/s systems.

Remark 2.14 The *maximal* non-negativity of V in a passive s/s system

$\Sigma = (V; \mathcal{X}, \mathcal{W})$ intuitively means that it has "enough" trajectories to make sense as a system.

More precisely, the maximal non-negativity of V implies that the *state/signal dual* $(V^{[\perp]}; \mathcal{X}, \mathcal{W})$ of Σ is anti-passive and thus very well-structured. If V is replaced by a smaller space then the dual becomes larger, and in particular, if $V = \{0\}$, then $V^{[\perp]} = \mathfrak{K}$ which has no meaning as a s/s system at all.

We also note that a s/s system is conservative if and only if it coincides with its own s/s dual.

See [Kurula, 2010, Sec. 3] for more details on the dual s/s system and its i/s/o representations.

We now return to i/s/o representations of passive s/s systems. It is well known that the adjoint S^* of an i/s/o system node operator S is also an i/s/o system node operator which represents the adjoint system; see [Staffans, 2005, Lemma 6.2.14]. If $\mathcal{U}$ is the input space and $\mathcal{Y}$ is the output space of S then $\mathcal{Y}$ is the input space and $\mathcal{U}$ is the output space of S^*.

Definition 2.15 An i/s/o system node $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is *scattering passive* if all its classical trajectories (u, x, y) satisfy (2.16) for every $s, t \in \mathbb{R}^+$ such that $s \leq t$.

The i/s/o system node $\Sigma_{i/s/o}$ is *scattering conservative* if all classical trajectories (u, x, y) and (y^d, x^d, u^d) on $\mathbb{R}^+$ of S and S^*, respectively, satisfy

$$\begin{aligned} \|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2 &= \int_s^t \|u(v)\|_{\mathcal{U}}^2 - \|y(v)\|_{\mathcal{Y}}^2 \,\mathrm{d}v \quad \text{and} \\ \|x^d(t)\|_{\mathcal{X}}^2 - \|x^d(s)\|_{\mathcal{X}}^2 &= \int_s^t \|y^d(v)\|_{\mathcal{Y}}^2 - \|u^d(v)\|_{\mathcal{U}}^2 \,\mathrm{d}v \end{aligned}$$

for every $s, t \in \mathbb{R}^+$ such that $s \leq t$. (Compare this to (2.16).)

Remark 2.16 Recall that a triple $(V; \mathcal{X}, \mathcal{W})$ is a passive s/s system if and only if V is a maximal non-negative subspace of $\mathfrak{K}$ with the property (2.4). Comparing this to Definitions 2.5 and 2.15, which are necessary for defining only a special class of passive i/s/o systems, we see that the s/s definition is both more general and considerably simpler. Moreover, the definitions of conservative i/s/o systems are even more complicated, since we need to formulate conditions on the *dual system* but a conservative s/s system is very elegantly characterized by the properties $V = V^{[\perp]}$ and (2.4).

The following proposition was proved in [Kurula, 2010, Prop. 5.6].

Proposition 2.17 *All scattering representations of a* passive (conservative) *s/s system are scattering passive (conservative) i/s/o systems.*

Conversely, let $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *be a scattering passive (conservative) i/s/o system node, so that* $\mathcal{U}$ *and* $\mathcal{Y}$ *are both Hilbert spaces. Define* $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{U} \\ -\mathcal{Y} \end{smallmatrix}\right]$ *with inner product* $[\left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right]] := (u_1, u_2)_{\mathcal{U}} - (y_1, y_2)_{\mathcal{Y}}$. *Then* $\mathcal{W}$ *is a Kreĭn space with fundamental decomposition* $\Sigma = \left[\begin{smallmatrix} \mathcal{U} \\ \{0\} \end{smallmatrix}\right] \boxplus \left[\begin{smallmatrix} \{0\} \\ -\mathcal{Y} \end{smallmatrix}\right]$. *Moreover,* $(V; \mathcal{X}, \mathcal{W})$ *with* V *given by* (2.9)*, is the unique passive (conservative) s/s system whose scattering representation induced by the above fundamental decomposition is* $\Sigma_{i/s/o}$.

Scattering passive i/s/o systems are discussed in, e.g., [Arov and Nudelman, 1996] and [Staffans, 2005, Chapter 11]. The connection between different well-posed i/s/o representations of a s/s system, and thus in particular, between different scattering representations of a passive s/s system, is described in [Kurula and Staffans, 2009, Section 4].

Impedance and transmission representations In the context of boundary relations, another type of i/s/o representation is in fact more important than the scattering representation, namely the *impedance* representation.

Definition 2.18 An *impedance representation* of a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is an i/s/o representation corresponding to a system-node admissible *Lagrangian* decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space $\mathcal{W}$.

Since not all Kreĭn spaces have Lagrangian decompositions, there exist passive s/s systems which have no impedance representations. However, assume that $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ indeed is a Lagrangian decomposition of $\mathcal{W}$, i.e., that $\mathcal{U}$ and $\mathcal{Y}$ are both Lagrangian subspaces of $\mathcal{W}$. By [Arov and Staffans, 2007a, Lemma 2.3] there exist admissible Hilbert-space inner products on $\mathcal{U}$ and $\mathcal{Y}$ and a unitary operator $\Psi : \mathcal{Y} \to \mathcal{U}$, such that the Kreĭn-space inner product on $\mathcal{W}$ is given by the following (where $u_1, u_2 \in \mathcal{U},\ y_1, y_2 \in \mathcal{Y}$):

$$[y_1 + u_1, y_2 + u_2]_{\mathcal{W}} = (\Psi y_1, u_2)_{\mathcal{U}} + (u_1, \Psi y_2)_{\mathcal{U}}. \tag{2.17}$$

Definition 2.19 By writing $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$ we mean that the Kreĭn space $\mathcal{W}$ is decomposed into the direct sum of $\mathcal{U}$ and $\mathcal{Y}$, and that the inner product $[\cdot,\cdot]_{\mathcal{W}}$ in $\mathcal{W}$ may be written in the form (2.17), where Ψ is a unitary operator from $\mathcal{Y}$ to $\mathcal{U}$.

It follows from (2.17) that both $\mathcal{U}$ and $\mathcal{Y}$ are Lagrangian subspaces of $\mathcal{W}$, i.e., that the decomposition in Definition 2.19 is always Lagrangian. See Section 2 of [Arov and Staffans, 2007a] for more details on Lagrangian decompositions of $\mathcal{W}$. If $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$, then the inequality (2.14) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \leq 2\mathrm{Re}\,\left(u(t), \Psi y(t)\right)_{\mathcal{U}}, \quad t \in \mathbb{R}^+.$$

Moreover, the inequality (2.15) takes the form

$$\|x(t)\|^2_{\mathcal{X}} - \|x(s)\|^2_{\mathcal{X}} \leq 2\mathrm{Re}\int_s^t (u(v), \Psi y(v))_{\mathcal{U}}\,\mathrm{d}v,\ s,t \in \mathbb{R}^+,\ t \geq s, \quad (2.18)$$

and this is the *impedance-passivity inequality.*

Definition 2.20 An i/s/o system node $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is *impedance passive* if all its classical trajectories (u,x,y) satisfy (2.18) for some unitary operator $\Psi : \mathcal{Y} \to \mathcal{U}$. (One commonly has $\mathcal{Y} = \mathcal{U}$ and $\Psi = 1_{\mathcal{U}}$.)

The i/s/o system node $\Sigma_{i/s/o}$ is *impedance conservative* if all classical trajectories (u,x,y) and (y,x,u) on $\mathbb{R}^+$ of S and S^*, respectively, satisfy (2.18) with equality instead of inequality.

An analogue of Proposition 2.17 relating impedance representations and impedance passive i/s/o systems can be formulated simply by replacing "scattering" by "impedance" and the fundamental decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \{0\} \end{smallmatrix}\right] \boxplus \left[\begin{smallmatrix} \{0\} \\ -\mathcal{Y} \end{smallmatrix}\right]$ by a Lagrangian decomposition $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$, where $\Psi : \mathcal{Y} \to \mathcal{U}$ is an arbitrary unitary operator.

Theorem 2.21 *The following claims are true for an impedance conservative i/s/o system node* $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$.

1. *The main operator A of S, see item 3 of Definition 2.5, is skew-adjoint and A generates a unitary group $t \mapsto \mathfrak{A}^t$, $t \in \mathbb{R}$, on $\mathcal{X}$.*
2. *For every $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ and initial state $x_0 \in \mathcal{X}$, such that $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathrm{dom}\,(S)$, the system*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad (2.19)$$

has a unique classical *trajectory (u,x,y) with $x(0) = x_0$; $W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ denotes the space of functions that together with their first and second distribution derivatives lie in $L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$.*

3. *For every* $x_0 \in \mathcal{X}$ *there exists a* generalized *trajectory* (u,x,y) *of* $\Sigma_{i/s/o}$*, such that* $x(0) = x_0$*. This trajectory is uniquely determined by the initial state* x_0 *and the input* u*.*
4. *The system* (2.19) *can also be solved in backwards time, i.e., for* $t \in \mathbb{R}^- = (-\infty, 0]$*, with the initial state* $x_0 \in \mathcal{X}$ *given at* $t = 0$*. In particular, every trajectory* (u,x,y) *of* $\Sigma_{i/s/o}$ *with* $x(0) = x_0$ *and* $u = 0$ *satisfies* $x(t) = \mathfrak{A}^t x_0$ *for all* $t \in \mathbb{R}$*, and if* $x_0 \in \operatorname{dom}(A)$*, then this trajectory is classical. This trajectory is the unique trajectory of* $\Sigma_{i/s/o}$ *with the given state* x_0 *at time* 0 *and input* $u(t) = 0$*,* $t \in \mathbb{R}$*.*

Proof One can verify that the i/s/o system node $\Sigma_{i/s/o}$ is impedance conservative with some given Ψ if and only if V defined in (2.9) is a Lagrangian subspace of $\mathfrak{K}$: $V = V^{[\perp]}$, where $\mathcal{W} := \mathcal{U} \overset{\Psi}{+} \mathcal{Y}$.

1. According to [Staffans, 2002a, Thm 4.7(4)] we have $A = -A^*$, and thus A generates a unitary group by Stone's theorem [Pazy, 1983, Thm 10.8].

2. This follows from [Staffans, 2005, Lem. 4.7.8].

3. The s/s system induced by an impedance conservative i/s/o system node is conservative, and therefore in particular passive. By Theorem 2.12, $(V; \mathcal{X}, \mathcal{W})$ is well-posed, and according to condition 1 of Definition 2.4, every $x_0 \in \mathcal{X}$ can be taken as the initial state of some generalized trajectory. Moreover, if $x_0 = 0$ and $u(t) = 0$ for all $t \in \mathbb{R}^+$, then $x(t) = 0$ and $y(t) = 0$ for all ≥ 0 by claim 2.
This is a consequence of Remark 2.14, [Staffans, 2005, Theorem 3.8.2], and the previous claims in this theorem. □

There are several ways to add dynamics to a boundary relation. Using Theorem 2.21 is one way, as we will show at the end of Section 2.5.

Remark 2.22 Note that the input u in Theorem 2.21 corresponds to a system-node admissible Lagrangian decomposition of the signal space of a conservative s/s system, and that this decomposition need not be well-posed in general. Indeed, the corresponding impedance representation $(S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ need not be well-posed, i.e., the i/s/o system node S in (2.9) need not satisfy (2.11).

If the decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ happens to be well-posed then we have from Definition 2.4 that the set

$$\big\{u \mid (u,x,y) \text{ is a generalized trajectory of } \Sigma_{i/s/o} \text{ with } x(0) = x_0\big\}$$

equals all of $L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{U})$ for all $x_0 \in \mathcal{X}$, but in the ill-posed case we can make no such conclusion.

On the contrary, every scattering representation of a passive s/s system is well-posed, cf. Theorem 2.12. This explains why the scattering formalism is sometimes useful for solving technical difficulties in the boundary relations theory, cf. [Behrndt et al., 2009], where this technique is used extensively.

There exist conservative s/s systems for which no Lagrangian decompositions are system-node admissible, see [Arov and Staffans, 2007a, Ex. 5.13], which can also be formulated for continuous time with trivial modifications. It follows from Theorem 2.34 and Proposition 2.35 below that the following two conditions together are sufficient and necessary for a Lagrangian decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ to be admissible for a conservative s/s system $(V; \mathcal{X}, \mathcal{W})$:

1. $\left[\begin{smallmatrix} z \\ 0 \\ \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V \Longrightarrow \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = 0.$
2. for each $u \in \mathcal{U}$ there exist $z, x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $\left[\begin{smallmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V.$

Well-posed impedance passive i/s/o systems were studied in [Staffans, 2002a]; the ill-posed impedance case were considered in [Staffans, 2002b].

Remark The energy inequalities (2.16) and (2.18) correspond to fundamental and Lagrangian decompositions of $\mathcal{W}$, respectively, but the *property of passivity* is characterized by the maximal non-negativity of V. Thus passivity is a *state/signal characteristic*, i.e., passivity *does not depend on any particular decomposition of the signal space into an input space and an output space.*

A third, fairly common, type of representation is the *transmission representation.*

Definition 2.23 An i/s/o representation of a passive s/s system corresponding to an admissible *orthogonal* decomposition $\mathcal{W} = \mathcal{W}_1 \boxplus \mathcal{W}_2$ of the signal space, with input space $\mathcal{U} = \mathcal{W}_1$ and output space $\mathcal{Y} = -\mathcal{W}_2$ is called a *transmission representation.*

Every scattering representation can also be interpreted as a transmission representation.

Example 2.24 We continue the transmission line example in Section 1.4. As we saw there, this is a conservative boundary control system. The choice of input and output maps Γ_0, Γ_1 in (1.13) corresponds to the Lagrangian decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathbb{C} \\ \{0\} \\ \mathbb{C} \\ \{0\} \end{smallmatrix}\right] \overset{\Psi}{\dotplus} \left[\begin{smallmatrix} \{0\} \\ \mathbb{C} \\ \{0\} \\ \mathbb{C} \end{smallmatrix}\right]$, where $\Psi = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and according to Example 2.7, this decomposition is admissible. The choice $\widetilde{\Gamma}_0, \widetilde{\Gamma}_1$ made in (1.14) corresponds to the fundamental decomposition $\mathcal{W} = \left\{ \left[\begin{smallmatrix} a \\ a \\ b \\ b \end{smallmatrix}\right] \,\middle|\, a, b \in \mathbb{C} \right\} \boxplus \left\{ \left[\begin{smallmatrix} -a \\ a \\ -b \\ b \end{smallmatrix}\right] \,\middle|\, a, b \in \mathbb{C} \right\}$, and according to Theorem 2.12, also this decomposition is admissible. The choice $\widehat{\Gamma}_0, \widehat{\Gamma}_1$ in (1.15) corresponds to the orthogonal (but non-fundamental) decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathbb{C} \\ \mathbb{C} \\ \{0\} \\ \{0\} \end{smallmatrix}\right] \boxplus \left[\begin{smallmatrix} \{0\} \\ \{0\} \\ \mathbb{C} \\ \mathbb{C} \end{smallmatrix}\right]$, which is *not* admissible.

The non-admissible orthogonal and Lagrangian decompositions which do not yield i/s/o representations can be treated using continuous-time analogues of the *affine representations* developed in [Arov and Staffans, 2007b].

2.4 The frequency domain characteristics of a state/signal system

The input-state/state-output resolvent matrix Suppose that x, $\dot{x}$, y, and u are all Laplace transformable, with the Laplace transforms converging in $\mathbb{C}^+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$, the right half-plane. Take Laplace transforms in the i/s/o equation

$$\Sigma_{i/s/o}: \quad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0,$$

in order to get

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{C}^+. \tag{2.20}$$

At least in the case where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a bounded operator in a scattering representation of a passive s/s system it is possible to solve $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right]$ from the identity (2.20) for all $\lambda \in \mathbb{C}^+$. The map $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$ turns out to be a bounded linear operator that we denote by $\widehat{\mathfrak{S}}(\lambda) = \left[\begin{smallmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{smallmatrix}\right]$. More explicitly,

$$\begin{gathered} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{C}^+, \quad \text{where} \\ \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} = \begin{bmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1}B \\ C(\lambda - A)^{-1} & C(\lambda - A)^{-1}B + D \end{bmatrix}. \end{gathered} \tag{2.21}$$

Definition 2.25 The operator $\widehat{\mathfrak{S}} := \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ is called the *is/so (input-state/state-output) resolvent matrix of* $\Sigma_{i/s/o}$. The different components of this resolvent matrix are named as follows:

1. $\widehat{\mathfrak{A}}$ is the *state/state* resolvent function,
2. $\widehat{\mathfrak{B}}$ is the *input/state* resolvent function,
3. $\widehat{\mathfrak{C}}$ is the *state/output* resolvent function, and
4. $\widehat{\mathfrak{D}}$ is the *input/output* resolvent function.

Of course, the state/state resolvent function is the familiar *resolvent* of the main operator A. The other components of $\widehat{\mathfrak{S}}$ has different names in different parts of the literature, and we make the connections to the corresponding notions in the theory of boundary relations in Theorem 2.33 below. In the i/s/o tradition the input/output resolvent function is usually called the *transfer function* of $\Sigma_{i/s/o}$.

Remark A significant part of formula (2.21) remains valid with the appropriate interpretation of the operators A, B, and C if we replace $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ by a *system node* operator S of the type described in Definition 2.5; see [Staffans, 2002b, Sec. 2].

In the case of a scattering passive i/s/o system $\Sigma_{i/s/o}$, the function $\widehat{\mathfrak{D}}$ is often called the *scattering matrix* of $\Sigma_{i/s/o}$. If in addition, $\Sigma_{i/s/o}$ is *conservative*, then $\widehat{\mathfrak{D}}$ is also called the *characteristic function* of the corresponding i/s/o system node, or of its main operator A; see Definition 2.5. In this case A is a maximal dissipative operator in $\mathcal{X}$.

In the case where $\Sigma_{i/s/o}$ is transmission passive, $\widehat{\mathfrak{D}}$ is called the *transmission matrix* of $\Sigma_{i/s/o}$. Also here $\widehat{\mathfrak{D}}$ is called the *characteristic function* if $\Sigma_{i/s/o}$ is conservative; see e.g. [Tsekanovskiĭ and Šmuljan, 1977]. In a transmission passive i/s/o system, the main operator A is often not dissipative, and this lack of dissipativity causes many of the technical problems associated with transmission passive systems.

Finally, in the case where $\Sigma_{i/s/o}$ is impedance passive, $\widehat{\mathfrak{D}}$ is called the *impedance matrix* of $\Sigma_{i/s/o}$. If $\Sigma_{i/s/o}$ is conservative then the main operator A is skew-adjoint, cf. Theorem 2.21.

See [Šmuljan, 1986; Salamon, 1987; Curtain and Weiss, 1989; Arov and Nudelman, 1996], or [Staffans, 2005] for more information on transfer functions (input/output resolvent functions).

The characteristic node bundle In order to derive the analogue of an i/s/o resolvent matrix for a s/s system, we rewrite the identity

(2.20) so that it uses the generating subspace V instead of the system node operator S.

Suppose therefore that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory of a s/s node, and that x, $\dot{x}$, and w are all Laplace transformable with the Laplace transforms converging in the whole right half-plane $\mathbb{C}^+$. Taking Laplace transforms in $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V$, $t \in \mathbb{R}^+$, we get

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \widehat{w}(\lambda) \end{bmatrix} \in V, \quad \lambda \in \mathbb{C}^+.$$

Definition 2.26 Let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be a direct sum decomposition of $\mathcal{W}$. The domain of the *generalized i/s/o resolvent matrix* with respect to this decomposition and the generalized i/s/o resolvent matrix itself are defined by

$$\operatorname{dom}(\widehat{\mathfrak{S}}) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} \text{for all } \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \text{ there exists} \\ \text{a unique pair } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] \\ \text{such that } \left[\begin{smallmatrix} \lambda x - x_0 \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V \end{array} \right\},$$

$$\widehat{\mathfrak{S}}(\lambda) \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] := \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] := \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \ \lambda \in \operatorname{dom}(\widehat{\mathfrak{S}}),$$

$$\text{where } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \text{ is the unique pair for which } \left[\begin{smallmatrix} \lambda x - x_0 \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V.$$

Of course, in this definition only those decompositions $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space for which the domain of the generalized resolvent matrix $\widehat{\mathfrak{S}}$ is nonempty are interesting.

Example 2.27 We continue Example 2.7 by computing the i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ of the boundary control i/s/o system $\Sigma_{i/s/o} = (L, \Gamma_0, \Gamma_1; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ in Definition 1.1. Therefore we again let V be given by (2.10) and we carry out the following computations:

$$\left[\begin{smallmatrix} \lambda x - x_0 \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right] \in V = \left\{ \left[\begin{smallmatrix} Lx \\ x \\ \left[\begin{smallmatrix} \Gamma_0 x \\ \Gamma_1 x \end{smallmatrix}\right] \end{smallmatrix}\right] \;\middle|\; x \in \operatorname{dom}(L) \right\} \iff$$

$$x_0 = (\lambda - L)x, \quad y = \Gamma_1 x, \quad \text{and} \quad u = \Gamma_0 x,$$

$$\text{so that} \quad \widehat{\mathfrak{S}}(\lambda): \begin{bmatrix} (\lambda - L)x \\ \Gamma_0 x \end{bmatrix} \mapsto \begin{bmatrix} x \\ \Gamma_1 x \end{bmatrix}, \quad x \in \operatorname{dom}(L).$$

One can show that $\mathbb{C}^+ \subset \operatorname{dom}(\widehat{\mathfrak{S}})$ if the system $\Sigma_{i/s/o}$ is *passive*, i.e., if V is maximal non-negative, and if $\mathcal{U} = \mathcal{W}_+$ for some fundamental decomposition $\mathcal{W} = \mathcal{W}_+ \boxplus -\mathcal{W}_-$.

It is possible to further extend the notion of a generalized i/s/o resolvent matrix by allowing $\widehat{\mathfrak{S}}(\lambda)$ to be a *relation* instead of a function. This extension is implemented by the following notion:

Definition 2.28 The *characteristic node bundle* of the (not necessarily passive) s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is the family $\{\widehat{\mathfrak{E}}(\lambda)\}_{\lambda\in\mathbb{C}}$ of subspaces of the node space $\mathfrak{K}$, where each $\widehat{\mathfrak{E}}(\lambda)$ is given by

$$\widehat{\mathfrak{E}}(\lambda) = \left\{ \begin{bmatrix} x_0 \\ x \\ w \end{bmatrix} \middle| \begin{bmatrix} \lambda x - x_0 \\ x \\ w \end{bmatrix} \in V \right\}.$$

The subspace $\widehat{\mathfrak{E}}(\lambda)$ is called the *fiber of* $\widehat{\mathfrak{E}}$ *at* $\lambda \in \mathbb{C}$.

By using the above state/signal characteristic node bundle we can reformulate the definition of the generalized i/s/o resolvent matrix $\begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}$ as follows.

Remark Let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be a direct sum decomposition of $\mathcal{W}$. The domain of the *generalized i/s/o resolvent matrix* $\widehat{\mathfrak{S}}$ of the passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with respect to this decomposition consists of those points $\lambda \in \mathbb{C}$ for which *the fiber* $\widehat{\mathfrak{E}}(\lambda)$ *of the characteristic node bundle is the graph of a bounded linear operator* $\begin{bmatrix} \mathcal{X} \\ \{0\} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \{0\} \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$, and $\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}$ is defined to be this operator. Note that we require that $\mathrm{dom}\,(\widehat{\mathfrak{S}}(\lambda)) = \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ for all $\lambda \in \mathrm{dom}\,(\widehat{\mathfrak{S}})$.

However, even if $\widehat{\mathfrak{E}}(\lambda)$ is not the graph of an operator, it can always be interpreted *as the graph of a closed relation* $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$. With this interpretation it makes sense to call this relation the *is/so resolvent relation at the point* $\lambda \in \mathbb{C}$. This resolvent relation is *defined for all* $\lambda \in \mathbb{C}$ but now $\mathrm{dom}\,(\widehat{\mathfrak{E}}(\lambda))$ may depend on λ.

Observe that unlike the above mentioned resolvent matrices and resolvent relations, *the fiber* $\widehat{\mathfrak{E}}(\lambda)$ *is a state/signal characteristic*, i.e., it does not depend on any particular decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space. Thus, although the s/s system Σ has many different resolvent relations, each corresponding to a different decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, *all resolvent relations have the same graph.* The different resolvent relations are simply different *representations* of the characteristic node bundle corresponding to different input/output decompositions.

We refer the reader to [Arov and Staffans, 2011] for more information on characteristic node bundles.

2.5 Conservative boundary relations

As we showed in Section 1.5, boundary triplets can be obtained as the i/s/o representations of conservative boundary control systems in case the boundary mapping Γ is surjective and the external signal space $\mathcal{W}$ has equal positive and negative indices. Here we show that a Lagrangian decomposition of the signal space of a conservative s/s system gives rise to a *boundary relation*, even if the decomposition of the signal space does not induce an i/s/o representation. We also prove the converse: every conservative boundary relation can be interpreted as a conservative state/signal system.

2.5.1 Definitions The following definition of a boundary relation has been adapted from [Derkach et al., 2009, Def. 3.1], with some minor change of notation.

Definition 2.29 Let R be a closed symmetric linear relation in a Hilbert space $\mathcal{X}$ (with arbitrary defect numbers), and let $\mathcal{U}$ be an auxiliary Hilbert space. A linear relation $\Gamma\colon \mathcal{X}^2 \to \mathcal{U}^2$ is called a *conservative boundary relation* for R^* if

1. $\mathrm{dom}\,(\Gamma)$ is dense in R^*,
2. the identity
$$(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} = (y_1, u_2)_{\mathcal{U}} - (u_1, y_2)_{\mathcal{U}} \tag{2.22}$$
holds for every $\{\left[\begin{smallmatrix} x_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\}$, $\{\left[\begin{smallmatrix} x_2 \\ z_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right]\} \in \Gamma$, and
3. Γ is maximal in the sense that if $\{\left[\begin{smallmatrix} x_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\} \in \mathcal{X}^2 \times \mathcal{U}^2$ satisfies (2.22) for every $\{\left[\begin{smallmatrix} x_2 \\ z_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right]\} \in \Gamma$, then $\{\left[\begin{smallmatrix} x_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\} \in \Gamma$.

We remark that what we here call "conservative boundary relation" is simply called "boundary relation" in [Derkach et al., 2009]. We have added the word "conservative" because of the close resemblance to conservative s/s systems. As shown in [Derkach et al., 2009, Proposition 3.1], $\ker(\Gamma) = R$ for the relation Γ and the operator R in Definition 2.29.

The following definition is an adaptation of [Derkach et al., 2009, Defs 3.4 and 3.5].

Definition 2.30 Let R be a closed symmetric linear relation in the Hilbert space $\mathcal{X}$ and let $\Gamma : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{U} \end{smallmatrix}\right]$ be a conservative boundary relation for R^*.

The *Weyl family* (of $R = \ker(\Gamma)$) corresponding to Γ is the family

$$M(\lambda) := \left\{ \{u, y\} \mid \{ \left[\begin{smallmatrix} x \\ \lambda x \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \} \in \Gamma \right\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

The γ*-field* (of $R = \ker(\Gamma)$) corresponding to Γ is the relation

$$\gamma(\lambda) := \left\{ \{u, x\} \mid \{ \left[\begin{smallmatrix} x \\ \lambda x \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \} \in \Gamma \right\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

By [Derkach et al., 2006, Sec. 4.2], the γ-field of a boundary relation is in fact single-valued for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Note that

$$\mathrm{dom}\,(M(\lambda)) = \mathrm{dom}\,(\gamma(\lambda)) = \left\{ u \mid \{ \left[\begin{smallmatrix} x \\ \lambda x \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \} \in \Gamma \right\}$$

in general depends on λ. This is analogous to the dependence of the domain of the i/s/o resolvent relation of a s/s system on λ in the general case. In [Derkach et al., 2006, Sect. 4.3] it is studied in which cases $\mathrm{dom}\,(M(\lambda))$ is independent of λ.

Connections to conservative state/signal systems We now proceed essentially in the same way as we did in Section 1.5 in order to explain the connection between a conservative boundary relation and a conservative s/s system.

Let R be a closed symmetric linear relation in $\mathcal{X}$ and let $\Gamma : \mathcal{X}^2 \to \mathcal{U}^2$ be a conservative boundary relation for R^*. We construct a s/s system by taking the signal space $\mathcal{W}$ to be $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{U} \end{smallmatrix}\right]$ with the indefinite inner product

$$[\left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right]]_{\mathcal{W}} := (u_1, y_2)_{\mathcal{U}} + (y_1, u_2)_{\mathcal{U}}, \tag{2.23}$$

corresponding to the Lagrangian decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \{0\} \end{smallmatrix}\right] \overset{\Psi}{\dotplus} \left[\begin{smallmatrix} \{0\} \\ \mathcal{U} \end{smallmatrix}\right]$ with $\Psi = 1_{\mathcal{U}}$, and defining

$$V := \left\{ \left[\begin{smallmatrix} \mathrm{i}z \\ x \\ \left[\begin{smallmatrix} u \\ \mathrm{i}y \end{smallmatrix}\right] \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \middle| \{ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \} \in \Gamma \right\}. \tag{2.24}$$

We will prove in Lemma 2.32 below that V is a Lagrangian subspace of the Kreĭn space $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ equipped with the inner product (2.13).

Thus, if we knew that also (2.4) holds, then $\Sigma = (V; \mathcal{X}, \mathcal{W})$ would be a conservative s/s system. However, conditions 2 and 3 of Definition 2.29 alone do not yet imply that V satisfies (2.4). Indeed, let $\mathcal{X}$ be an arbitrary nontrivial Hilbert space, and set $\mathcal{W} = \{0\}$ and $\Gamma := \left\{ \{ \left[\begin{smallmatrix} 0 \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \} \mid z \in \mathcal{X} \right\}$. Then $V = \left\{ \left[\begin{smallmatrix} z \\ 0 \\ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \end{smallmatrix}\right] \middle| z \in \mathcal{X} \right\} = V^{[\perp]}$ in $\mathfrak{K}$.

Fortunately, it is possible to meet condition (2.4) by replacing the state space $\mathcal{X}$ by a smaller space, and this can be done without essential loss of generality. The following proposition follows from [Kurula, 2010, Prop. 4.7].

Proposition 2.31 *Let V be a maximal non-negative subspace of $\mathfrak{K}$. Denote*

$$\widetilde{\mathcal{X}} := \mathcal{X} \ominus \left\{ z \,\middle|\, \left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \right\} \quad \text{and} \quad \widetilde{V} = V \cap \begin{bmatrix} \widetilde{\mathcal{X}} \\ \widetilde{\mathcal{X}} \\ \mathcal{W} \end{bmatrix}. \tag{2.25}$$

Then $\widetilde{\Sigma} := (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ is a passive s/s system and the sets of classical and generalized trajectories generated by V and $\widetilde{V}$ are the same.

The s/s system $\widetilde{\Sigma}$ is conservative if and only if $V = V^{[\perp]}$.

In this way every conservative boundary relation induces a unique conservative s/s system. See Theorem 2.33 below for an exact statement.

Conversely, let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a conservative s/s node, such that the signal space has a Lagrangian decomposition $\mathcal{W} = \mathcal{U} \stackrel{\Psi}{\dotplus} \mathcal{Y}$ with the inner product (2.17). Define a linear relation on $\mathcal{X}^2 \times \mathcal{U}^2$ by

$$\Gamma := \left\{ \left\{ \begin{bmatrix} x \\ -\mathrm{i} z \end{bmatrix}, \begin{bmatrix} P_{\mathcal{U}}^{\mathcal{Y}} w \\ -\mathrm{i}\Psi P_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} \right\} \mid \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \right\}. \tag{2.26}$$

In order to prove that Γ is a conservative boundary relation, we need to recall the *main transform* $\mathcal{J}(\Gamma)$ of Γ defined in [Derkach et al., 2006, Sect. 2.4] by

$$\mathcal{J}(\Gamma) := \left\{ \{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} z \\ -y \end{smallmatrix}\right] \} \mid \{ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \} \in \Gamma \right\}, \tag{2.27}$$

and to state the following lemma:

Lemma 2.32 *The space $\mathcal{U}^2$ with the indefinite inner product* (2.23) *is a Kreĭn space. Moreover, the following claims are equivalent for an arbitrary Lagrangian decomposition $\mathcal{W} = \mathcal{U} \stackrel{\Psi}{\dotplus} \mathcal{Y}$:*

1. *The subspace $V \subset \mathfrak{K}$ satisfies $V = V^{[\perp]}$.*
2. *The relation $\Gamma : \mathcal{X}^2 \to \mathcal{U}^2$ given by* (2.26) *satisfies conditions 2 and 3 of Definition 2.29.*
3. *The relation $\mathcal{J}(\Gamma)$ in $\mathcal{X} \times \mathcal{U}$ is self-adjoint.*

Proof The reader may verify that that $\mathcal{U}^2 = \left[\begin{smallmatrix} 1_{\mathcal{U}} \\ 1_{\mathcal{U}} \end{smallmatrix}\right] \mathcal{U} \boxplus - \left[\begin{smallmatrix} -1_{\mathcal{U}} \\ 1_{\mathcal{U}} \end{smallmatrix}\right] \mathcal{U}$ is a fundamental decomposition, and therefore $\mathcal{U}^2$ is a Kreĭn space with the given indefinite inner product.

In order to prove the equivalence of the three listed claims, first note that

$$\begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in V \iff \{[\begin{smallmatrix} x \\ -\mathrm{i}z \end{smallmatrix}],[\begin{smallmatrix} u \\ -\mathrm{i}\Psi y \end{smallmatrix}]\} \in \Gamma$$
$$\iff \{[\begin{smallmatrix} x \\ u \end{smallmatrix}],[\begin{smallmatrix} \mathrm{i}z \\ \mathrm{i}\Psi y \end{smallmatrix}]\} \in \mathcal{J}(\Gamma).$$

Moreover, $\begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in V^{[\perp]}$ if and only if

$$(u,\Psi\widetilde{y}) + (\Psi y,\widetilde{u}) - (z,\widetilde{x}) - (x,\widetilde{z}) = 0, \quad \begin{bmatrix} \widetilde{z} \\ \widetilde{x} \\ \left[\begin{smallmatrix} \widetilde{u} \\ \widetilde{y} \end{smallmatrix}\right] \end{bmatrix} \in V \iff$$
$$(-\mathrm{i}z,\widetilde{x}) - (x,-\mathrm{i}\widetilde{z}) = (-\mathrm{i}\Psi y,\widetilde{u}) - (u,-\mathrm{i}\Psi\widetilde{y}),$$
$$\{[\begin{smallmatrix} \widetilde{x} \\ -\mathrm{i}\widetilde{z} \end{smallmatrix}],[\begin{smallmatrix} \widetilde{u} \\ -\mathrm{i}\Psi\widetilde{y} \end{smallmatrix}]\} \in \Gamma \iff$$
$$([\begin{smallmatrix} -\mathrm{i}z \\ \mathrm{i}\Psi y \end{smallmatrix}],[\begin{smallmatrix} \widetilde{x} \\ \widetilde{u} \end{smallmatrix}]) = \left([\begin{smallmatrix} x \\ u \end{smallmatrix}],\left[\begin{smallmatrix} -\mathrm{i}\widetilde{z} \\ \mathrm{i}\Psi\widetilde{y} \end{smallmatrix}\right]\right), \quad \left\{[\begin{smallmatrix} \widetilde{x} \\ \widetilde{u} \end{smallmatrix}],\left[\begin{smallmatrix} -\mathrm{i}\widetilde{z} \\ \mathrm{i}\Psi\widetilde{y} \end{smallmatrix}\right]\right\} \in \mathcal{J}(\Gamma),$$

where the last line is equivalent to $\left\{[\begin{smallmatrix} x \\ u \end{smallmatrix}],[\begin{smallmatrix} -\mathrm{i}z \\ \mathrm{i}\Psi y \end{smallmatrix}]\right\} \in \mathcal{J}(\Gamma)^*$.

Thus $V \subset V^{[\perp]}$ if and only if condition 2 of Definition 2.29 holds, which in turn is true if and only if $\mathcal{J}(\Gamma) \subset \mathcal{J}(\Gamma)^*$. Analogously, $V^{[\perp]} \subset V$ if and only if condition 3 of Definition 2.29 holds, which in turn is true if and only if $\mathcal{J}(\Gamma)^* \subset \mathcal{J}(\Gamma)$. □

If the signal space $\mathcal{W}$ has no Lagrangian decomposition, which is the case, e.g., when the dimension of $\mathcal{W}$ is finite and odd, then Σ is not induced by any conservative boundary relation, cf. Example 1.9. We collect our observations in the following theorem:

Theorem 2.33 *The following claims are true:*

1. *Let $(V;\mathcal{X},\mathcal{W})$ be a* conservative *s/s node and assume that there exists a Lagrangian decomposition $\mathcal{W} = \mathcal{U} \overset{\Psi}{\dotplus} \mathcal{Y}$. Define Γ by* (2.26) *and set* $R := \ker(\Gamma)$.

   Then R is a closed symmetric operator *in $\mathcal{X}$, R^* is the closure of* $\operatorname{dom}(\Gamma)$ *in $\mathcal{X}^2$, Γ is a conservative boundary relation for R^*, and V can be recovered using the following expression, which reduces to* (2.24) *when $\mathcal{Y} = \mathcal{U}$ and $\Psi = 1_{\mathcal{U}}$:*
$$V = \left\{ \begin{bmatrix} \mathrm{i}z \\ x \\ \left[\begin{smallmatrix} u \\ \mathrm{i}\Psi^* y \end{smallmatrix}\right] \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \middle| \{[\begin{smallmatrix} x \\ z \end{smallmatrix}],[\begin{smallmatrix} u \\ y \end{smallmatrix}]\} \in \Gamma \right\}. \tag{2.28}$$

2. *Conversely, let R be a closed symmetric linear relation in the Hilbert space $\mathcal{X}$ and let $\Gamma : \mathcal{X}^2 \to \mathcal{U}^2$ be a conservative boundary relation for R^*. Let $\mathcal{W} := \mathcal{U}^2$ be the Kreĭn space with the indefinite inner product* (2.23) *(corresponding to $\Psi = 1_{\mathcal{U}}$). Define V by* (2.24), *and $\widetilde{\mathcal{X}}$ and $\widetilde{V}$ by* (2.25).

Then $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ *is a conservative s/s node with state space* $\widetilde{\mathcal{X}} = \mathcal{X} \ominus \operatorname{mul}(R)$*, where* $\operatorname{mul}(R) = \{z \mid \{0, z\} \in R\}$ *is the* multi-valued part *of* R*. Moreover, if we define* $\widetilde{\Gamma}$ *by the right-hand side of* (2.26) *with* V *replaced by* $\widetilde{V}$ *and* $\Psi = 1_{\mathcal{U}}$*, then*

$$\widetilde{\Gamma} = \Gamma\big|_{\operatorname{dom}(\Gamma)\cap\widetilde{\mathcal{X}}^2} = \Gamma\big|_{\operatorname{dom}(\Gamma)\cap\left[\begin{smallmatrix}\widetilde{\mathcal{X}}\\ \mathcal{X}\end{smallmatrix}\right]}. \tag{2.29}$$

3. *Let the conservative boundary relation* Γ *and the conservative s/s node* $\Sigma = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ *be related as in* (2.28) *and* (2.25)*. Denote the Weyl family and* γ*-field of* Γ *by* M *and* γ*, respectively, and let* $\widehat{\mathfrak{E}}$ *be the characteristic node bundle of* Σ*. Then*

$$\begin{aligned} M(\lambda) &= \left\{\{u, -\mathrm{i}\Psi y\} \,\middle|\, \left[\begin{smallmatrix} 0\\ x\\ \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]\end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\mathrm{i}\lambda)\right\} \quad \textit{and} \\ \gamma(\lambda) &= \left\{\{u, x\} \,\middle|\, \left[\begin{smallmatrix} 0\\ x\\ \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]\end{smallmatrix}\right] \in \widehat{\mathfrak{E}}(\mathrm{i}\lambda)\right\}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \end{aligned} \tag{2.30}$$

Proof First note that if $R = \ker(\Gamma)$ then (2.26) implies that

$$z \in \operatorname{mul}(R) \iff \{\left[\begin{smallmatrix}0\\ z\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]\} \in \Gamma \iff \left[\begin{smallmatrix}z\\ 0\\ 0\end{smallmatrix}\right] \in V. \tag{2.31}$$

1. Since every Lagrangian V is closed and neutral, Γ and its kernel R are also closed, and from (2.22) it follows that R is symmetric. By (2.31), R is single-valued. Lemma 2.32 yields that $\mathcal{J}(\Gamma)$ is self-adjoint, and applying [Derkach et al., 2006, Prop. 3.5], we obtain that Γ is a conservative boundary relation for R^*. Condition 1 of Definition 2.29 says that R^* is the closure of $\operatorname{dom}(\Gamma)$ in $\mathcal{X}^2$. It is easy to verify that (2.26) and (2.28) are equivalent.

2. Setting $\mathcal{Y} = \mathcal{U}$ and $\Psi = 1_{\mathcal{U}}$ in Lemma 2.32, we obtain that $\mathcal{W}$ is a Kreĭn space and that $V = V^{[\perp]}$, and according to Proposition 2.31, $\widetilde{\Sigma}$ is then a conservative s/s system. By [Derkach et al., 2006, Prop. 3.2], $R = \ker(\Gamma)$, and therefore (2.31) and (2.25) imply that $\widetilde{\mathcal{X}} = \mathcal{X} \ominus \operatorname{mul}(R)$.

The first equality in (2.29) follows by noting that

$$\begin{aligned} &\{\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right], \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]\} \in \widetilde{\Gamma} \iff \left[\begin{smallmatrix}\mathrm{i}z\\ x\\ \left[\begin{smallmatrix}u\\ \mathrm{i}y\end{smallmatrix}\right]\end{smallmatrix}\right] \in \widetilde{V} \iff \\ &\left[\begin{smallmatrix}z\\ x\\ \left[\begin{smallmatrix}u\\ \mathrm{i}y\end{smallmatrix}\right]\end{smallmatrix}\right] \in V,\ z, x \in \widetilde{\mathcal{X}} \iff \{\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right], \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]\} \in \Gamma,\ z, x \in \widetilde{\mathcal{X}}. \end{aligned}$$

The second equality holds, since we by (2.13) always have

$$\left[\begin{smallmatrix}\mathrm{i}z\\ x\\ \left[\begin{smallmatrix}u\\ \mathrm{i}y\end{smallmatrix}\right]\end{smallmatrix}\right] \in V = V^{[\perp]} \implies (x, \widetilde{z})_{\mathcal{X}} = 0,\ \left[\begin{smallmatrix}\widetilde{z}\\ 0\\ 0\end{smallmatrix}\right] \in V, \tag{2.32}$$

i.e., $x \in \widetilde{\mathcal{X}}$ automatically when $\begin{bmatrix} \mathrm{i}z \\ x \\ \left[\begin{smallmatrix} u \\ \mathrm{i}y \end{smallmatrix}\right] \end{bmatrix} \in V$ for a Lagrangian V.

3. The equalities (2.30) now follow from Definition 2.30 once we observe that

$$\begin{bmatrix} 0 \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in \widehat{\mathfrak{E}}(\mathrm{i}\lambda) \iff \begin{bmatrix} \mathrm{i}\lambda x \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in V,\ x \in \widetilde{\mathcal{X}} \iff \begin{bmatrix} \mathrm{i}\lambda x \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in V$$
$$\iff \left\{\left[\begin{smallmatrix} x \\ \lambda x \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ -\mathrm{i}\Psi y \end{smallmatrix}\right]\right\} \in \Gamma$$

where we used Definition 2.28, (2.32), and (2.28), respectively. □

Claim 3 of Theorem 2.33 shows that $\widehat{\mathfrak{E}} \cap \begin{bmatrix} \{0\} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ can be identified with the product of the γ-field and the Weyl family of the conservative boundary relation Γ in (2.26). Note, however, that there is an extra rotation of the complex plane in (2.30), due to the fact that in the boundary relation theory one works with self-adjoint operators that have $\mathbb{C} \setminus \mathbb{R}$ in their resolvent set, whereas in the s/s theory the convention is to use skew-adjoint operators whose resolvent sets contain $\mathbb{C} \setminus \mathrm{i}\mathbb{R}$. Also note that the ordering of the two internal variables z and x is different on the left-hand and the right-hand sides of (2.28), which is due to different conventions in different fields of mathematics.

A systems theory interpretation We now introduce dynamics to a conservative boundary relation by giving a systems and control theory interpretation. At the same time, the following results show that boundary relations, in spite of their name, are much more closely related to the general i/s/o systems in Section 2.2 than to the boundary control systems in Chapter 1.

Theorem 2.34 *Assume that $\Gamma \subset \mathcal{X}^2 \times \mathcal{U}^2$ is a conservative boundary relation with the following properties:*

1. *If $\left\{\left[\begin{smallmatrix} 0 \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right]\right\} \in \Gamma$ then $z = 0$ and $y = 0$.*
2. *The set $V_u := \left\{u \;\middle|\; \left\{\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\right\} \in \Gamma\right\}$ equals $\mathcal{U}$.*

Then Γ has the representation

$$\Gamma = \left\{\left\{\left[\begin{smallmatrix} x \\ -\mathrm{i}z \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ -\mathrm{i}y \end{smallmatrix}\right]\right\} \;\middle|\; \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}\,(S),\ \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = S\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right\}, \tag{2.33}$$

where $(S;\mathcal{X},\mathcal{U},\mathcal{U})$ is an impedance conservative i/s/o system node.

Moreover, $\Sigma := (V;\mathcal{X},\mathcal{W})$ is a conservative s/s node, where V is defined by (2.24) and $\mathcal{W} = \mathcal{U}^2$ with the inner product (2.23). The Lagrangian decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \{0\} \end{smallmatrix}\right] \dotplus^{\Psi} \left[\begin{smallmatrix} \{0\} \\ \mathcal{U} \end{smallmatrix}\right]$, $\Psi = 1_{\mathcal{U}}$, is admissible, and $(S;\mathcal{X},\mathcal{U},\mathcal{U})$ is the corresponding impedance representation.

Proof From claim 2 of Theorem 2.33 it follows that V defined in (2.24) generates a conservative s/s system. The representation (2.33) for some (single-valued) operator S follows from assumption 1. Then $V = V^{[\perp]}$, assumption 2, and [Ball and Staffans, 2006, Prop. 4.11] imply that S is an impedance conservative i/s/o system node operator, which is an impedance representation of V:

$$\begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{bmatrix} \in V \quad \Longleftrightarrow \quad \{[\begin{smallmatrix} x \\ -\mathrm{i}z \end{smallmatrix}], [\begin{smallmatrix} u \\ -\mathrm{i}y \end{smallmatrix}]\} \in \Gamma \quad \Longleftrightarrow \quad [\begin{smallmatrix} z \\ y \end{smallmatrix}] = S [\begin{smallmatrix} x \\ u \end{smallmatrix}],$$

where we have used Definition 2.6, (2.24) and (2.33). □

It follows from Theorems 2.21 and 2.34 that for every $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{U})$ and every initial state $x_0 \in \mathcal{X}$ with $\{[\begin{smallmatrix} x_0 \\ z \end{smallmatrix}], [\begin{smallmatrix} u(0) \\ y \end{smallmatrix}]\} \in \Gamma$, the system

$$\left\{ \begin{bmatrix} x(t) \\ -\mathrm{i}\dot{x}(t) \end{bmatrix}, \begin{bmatrix} u(t) \\ -\mathrm{i}y(t) \end{bmatrix} \right\} \in \Gamma, \quad t \in \mathbb{R}^+, \tag{2.34}$$

has a unique *classical* solution (u, x, y) with $x(0) = x_0$.

We have the following converse to Theorem 2.34:

Proposition 2.35 *If* $(S;\mathcal{X},\mathcal{U},\mathcal{U})$ *is an impedance-conservative i/s/o system node then* Γ *in* (2.33) *is a conservative boundary relation for* R^**, where* $R := \ker(\Gamma)$*. Moreover,* Γ *has properties 1 and 2 in Theorem 2.34.*

Proof If $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an impedance conservative i/s/o system node operator then $S = \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is skew-adjoint by [Staffans, 2002b, Thm 4.3], and this is equivalent to $S = \left[\begin{smallmatrix} -\mathrm{i}A\&B \\ \mathrm{i}C\&D \end{smallmatrix}\right]$ being self-adjoint. By (2.27) and (2.33),

$$\mathcal{J}(\Gamma) = \left\{ \{ [\begin{smallmatrix} x \\ u \end{smallmatrix}], \left[\begin{smallmatrix} -\mathrm{i}A\&B \\ \mathrm{i}C\&D \end{smallmatrix}\right] [\begin{smallmatrix} x \\ u \end{smallmatrix}] \} \;\middle|\; [\begin{smallmatrix} x \\ u \end{smallmatrix}] \in \operatorname{dom}\left(\left[\begin{smallmatrix} -\mathrm{i}A\&B \\ \mathrm{i}C\&D \end{smallmatrix}\right] \right) \right\},$$

and we obtain from [Derkach et al., 2006, Prop. 3.5] that Γ is a conservative boundary relation for R^*.

Moreover, by condition 4 of Definition 2.5, for all $u \in \mathcal{U}$ there exists some $x \in \mathcal{X}$ such that $[\begin{smallmatrix} x \\ u \end{smallmatrix}] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] \right)$. From (2.33) it now follows that condition 2 in Theorem 2.34 is met, and also that $\{[\begin{smallmatrix} 0 \\ z \end{smallmatrix}], [\begin{smallmatrix} 0 \\ y \end{smallmatrix}]\} \in \Gamma$ implies $[\begin{smallmatrix} z \\ y \end{smallmatrix}] = 0$. □

Using Proposition 2.31, one can reformulate Theorem 2.34 slightly in such a way that condition 1 is replaced by the weaker condition that

$$\{[\begin{smallmatrix} 0 \\ z \end{smallmatrix}], [\begin{smallmatrix} 0 \\ y \end{smallmatrix}]\} \in \Gamma \quad \Longrightarrow \quad y = 0. \tag{2.35}$$

Moreover, condition 1 implies that $\operatorname{dom}(S)$ is dense in $[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}]$ when Γ is a boundary relation, and therefore, condition 2 can be weakened to the condition that V_u is closed. We formulate the result but we leave the proof to the reader.

Corollary *Assume that* Γ *is a conservative boundary relation such that* (2.35) *holds and the set* V_u *in Theorem 2.34 is closed. Let* $\mathcal{W} := \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{U}\end{smallmatrix}\right]$ *with the indefinite inner product* (2.23)*, let* V *be given by* (2.24)*, and let* $\widetilde{\mathcal{X}}, \widetilde{V}$ *be given by* (2.25)*.*

Then Γ *has the representation*

$$\Gamma = \left\{ \left\{ \left[\begin{smallmatrix} x \\ -\mathrm{i}z \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ -\mathrm{i}y \end{smallmatrix}\right] \right\} \;\middle|\; \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S),\ z \in \mathcal{X},\ \left[\begin{smallmatrix} Pz \\ y \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\},$$

where P *is the orthogonal projection of* $\mathcal{X}$ *onto* $\widetilde{\mathcal{X}}$*, and* $(S; \mathcal{X}, \mathcal{U}, \mathcal{U})$ *is the impedance representation corresponding to the admissible Lagrangian decomposition* $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\ \{0\}\end{smallmatrix}\right] \stackrel{\Psi}{\dotplus} \left[\begin{smallmatrix}\{0\}\\ \mathcal{U}\end{smallmatrix}\right]$, $\Psi = 1_{\mathcal{U}}$*, of the conservative s/s system* $\Sigma = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$*.*

We could also have introduced dynamics to the boundary relation simply by considering the classical and generalized solutions of (2.34). However, without using claim 2 of Theorem 2.21, or changing to a scattering representation and using Theorem 2.12, we would not know that the sets of classical and generalized trajectories in fact are large.

2.6 Conclusions

We have presented the fundamentals of the state/signal approach to systems theory and we have made the basic connections between this theory and that of conservative boundary relations. We can conclude that the main objects of the two fields, namely the s/s system and the (conservative) boundary relation, are very closely related.

Sometimes technical complications arise from the way a s/s system is represented by an i/s/o system and not from the s/s system itself. For instance, the characteristic node bundle of a s/s system is much cleaner and more general than an i/s/o resolvent matrix. Moreover, in many cases it is useful to change from a impedance representation to a scattering representations in order to obtain a well-posed system which describes the dynamics of the system in a clear way; see Remark 2.22. The s/s formalism provides a firm basis for doing this. In particular, the families of all classical and generalized trajectories of a passive s/s system are in general more easily characterized by means of a scattering representation than by means of an impedance representation.

Passivity is a good example of a property which refers to a physical system, and not to any one of its input/output representations. Indeed, the property of passivity of a s/s system $(V; \mathcal{X}, \mathcal{W})$ simply means that the generating subspace V is maximal non-negative, whereas different i/s/o *representations* of the s/s system *are passive in different senses*, cf. Definitions 2.15 and 2.20.

For instance the flexibility in choosing i/s/o representations, the introduction of dynamics, the connection to control theory made in Theorem 2.34, and the work done on passive nonconservative s/s systems could potentially turn out to be useful for future research in the theory of boundary relations.

Conversely, it is interesting to look for new directions for the future development of the state/signal theory by studying the theory of boundary relations. In particular, the realization results [Derkach et al., 2006, Thm 3.9] and [Behrndt et al., 2009, Thm 6.1] can be utilized directly for conservative s/s systems in the case where a Lagrangian decomposition of the external signal space exists, i.e., when the signal space has equal positive and negative indices. An intriguing question is exactly how these realizations are related to those developed in [Arov et al., 2011] and their frequency-domain counterparts. A related question is to what extent the available results on Weyl families and their connections to the associated boundary relation can be employed in order to explore the properties of the characteristic node bundles of s/s systems.

It is our sincere hope that this exposition will increase the interaction between researchers of boundary relations and state/signal systems, thus preventing overlapping research, and that it gives rise to future cooperation on common research interests.

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