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# Lax–Phillips Scattering and Well-Posed Linear Systems

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**Abstract**

We discuss the connection between Lax–Phillips scattering theory and the theory of well-posed linear systems, and show that the latter theory is a natural extension of the former. As a consequence of this, there is a close connection between the Lax–Phillips generator and the generators of the corresponding well-posed linear system. All the essential information about these two systems is contained in the system operator $S = \left[\begin{smallmatrix} A & B \\ \multicolumn{2}{c}{N} \end{smallmatrix}\right]$, where $A$ is the generator of the (central) semigroup, $B$ is the control operator, and $N$ is the combined observation/feedthrough operator. If the system is compatible in the sense of Helton or regular in the sense of Weiss, then this system operator can be written in the more familiar form $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where $C$ is the observation operator and $D$ is the (generalized) feedthrough operator. We show that $S$ is closed and densely defined. In the reflexive case the adjoint of $S$ is the system operator of the dual system. We give formulas for the Lax–Phillips generator and resolvent in terms of the system operator. By applying the Hille–Yoshida theorem to the Lax–Phillips semigroup we get necessary and sufficient conditions for the $L^p$-admissibility or joint $L^p$-admissibility of a control operator $B$ and an observation operator $C$.

## 1 Introduction

Many infinite-dimensional systems can be described by the equations

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

on a triple of Banach spaces, namely, the input space $U$, the state space $X$, and the output space $Y$. We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operators $A$, $B$, and $C$ are usually unbounded. It is often convenient to use the "integral" representation of the system, which consists of the four operators from the initial state $x_0$ and the input function $u$ to the final state $x(t)$ and the output function $y$:

$$\begin{aligned} x(t) &= \mathfrak{A}^t x_0 + \mathfrak{B}_0^t u, \\ y &= \mathfrak{C} x_0 + \mathfrak{D}_0 u. \end{aligned} \tag{1.2}$$

Here, $\mathfrak{A}^t$ is the semigroup generated by $A$ (which maps the initial state $x_0$ into the final state $x(t)$), $\mathfrak{B}_0^t$ is the map from the input $u$ (restricted to the interval $(0,t)$) to the final state $x(t)$,

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$\mathfrak{C}$ is the map from the initial state $x_0$ to the output $y$, and $\mathfrak{D}_0$ is the input-output map from $u$ (restricted to $(0,\infty)$) to $y$.

The *well-posedness* assumption is that (1.2) behaves well in an $L^p$-setting, $1 \le p < \infty$, i.e., $x(t) \in X$ and $y \in L^p_{\text{loc}}(\mathbf{R}^+;Y)$ depend continuously on $x_0 \in X$ and on $u \in L^p_{\text{loc}}(\mathbf{R}^+;U)$. If this is the case, we call the operators $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ a *well-posed linear system*, where $\mathfrak{B} = \lim_{t\to\infty} \mathfrak{B}_0^t \tau^{-t}$, $\mathfrak{D} = \lim_{t\to\infty} \tau^t \mathfrak{D}_0 \tau^{-t}$, and $(\tau^t u)(s) = u(s+t)$, $-\infty < s,\ t < \infty$. See Section 2 for details.

The classical *Lax–Phillips model* was developed by Lax and Phillips (1967) (conservative systems) and Lax and Phillips (1973) (nonconservative systems) to provide a mathematical description of a scattering process where an incoming wave hits an obstacle and is scattered into an outgoing wave. An extension to well-posed unstable system can be obtained through a simple rescaling, as described by Helton (1976). In this extended formulation the Lax–Phillips model is a semigroup with a particular structure: it acts as an exponentially weighted incoming shift on the incoming subspace, as an exponentially weighted outgoing shift on the outgoing subspace, and the central part of the semigroup describes "a generalized scattering process". As we shall see in Section 3, this central part can be taken to be an arbitrary well-posed linear system. Thus there is a one-to-one correspondence between the class of all well-posed linear systems and all extended Lax–Phillips models. This note is devoted to a study of this correspondence.

We begin by presenting the most basic results about a $L^p$-well-posed linear system (Section 2) and the corresponding Lax–Phillips model (Section 3). We proceed in Section 4 to show that there is a close connection between the Lax–Phillips generator and the generators of the corresponding well-posed linear system. All the essential information about these two systems is contained in the *system operator* $S = \left[\begin{smallmatrix}A & B\\ \multicolumn{2}{c}{N}\end{smallmatrix}\right]$ where $A$ is the generator of the (central) semigroup, $B$ is the control operator, and $N$ is the combined observation/feedthrough operator. If the system is compatible in the sense of Helton (1976) or regular in the sense of Weiss (1994a), then this system operator can be written in the more familiar form $S = \left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$, where $C$ is the observation operator and $D$ is the (generalized) feedthrough operator. We show that $S$ is closed and densely defined from $X \times U$ to $X \times Y$. In the reflexive case the adjoint of $S$ is the system operator of the dual system. We give formulas for the Lax–Phillips generator and resolvent in terms of the system operator. Finally, in the last section we apply the Hille–Yoshida theorem to the Lax–Phillips semigroup and get necessary and sufficient conditions for the admissibility or joint admissibility of a control operator $B$ and an observation operator $C$.

## 2 Well-posed linear systems

As already outlined in Section 1, it is possible to define a well-posed linear system $\Psi = \left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ without any reference to the system of equations (1.1). For this, we have to introduce some spaces and some simple operators. We denote $\mathbf{R}^+ = [0,\infty)$, $\mathbf{R}^- = (-\infty,0)$,

$$
\begin{aligned}
(\tau^t u)(s) &= u(t+s), \quad -\infty < s,\ t < \infty,\\
(\pi_J u)(s) &= \begin{cases} u(s), & s \in J,\\ 0, & s \notin J,\end{cases} \quad \text{for all } J \subset \mathbf{R},\\
\pi_+ u &= \pi_{\mathbf{R}^+}, \qquad \pi_- u = \pi_{\mathbf{R}^-},\\
\tau_+^t &= \pi_+ \tau^t, \qquad \tau_-^t = \tau^t \pi_-.
\end{aligned}
$$

The space $L^p_{c,\text{loc}}(\mathbf{R};U)$ consists of all the functions $u : \mathbf{R} \to U$ that are locally in $L^p$ and whose support is bounded to the left. We interpret $L^p_{\text{loc}}(\mathbf{R}^+;U)$ as the subspace of functions in $L^p_{c,\text{loc}}(\mathbf{R};U)$ which vanish on $\mathbf{R}^-$. A sequence of functions $u_n$ converges in $L^p_{c,\text{loc}}(\mathbf{R};U)$ to a function $u$ if the common support of all the functions $u_n$ is bounded to the left and $u_n$ converges to $u$ in the $L^p$ sense on every bounded time interval. The continuity of $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ in the following definition is with respect to this convergence.

**2.1. Definition.** *Let $U$, $X$, and $Y$ be Banach spaces, and let $1 \le p < \infty$. An* $L^p$-well-posed linear system $\Psi$ *on $(Y, X, U)$ is a quadruple $\Psi = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ of continuous linear operators satisfying the following conditions:*

*(i)* $t \mapsto \mathfrak{A}^t$ *is a strongly continuous semigroup of operators on $X$;*

*(ii)* $\mathfrak{B} \colon L^p_{c,\mathrm{loc}}(\mathbf{R}; U) \to X$ *satisfies* $\mathfrak{A}^t \mathfrak{B} u = \mathfrak{B} \tau^t_- u$, *for all* $u \in L^p_{c,\mathrm{loc}}(\mathbf{R}; U)$ *and all* $t \in \mathbf{R}^+$*;*

*(iii)* $\mathfrak{C} \colon X \to L^p_{c,\mathrm{loc}}(\mathbf{R}; Y)$ *satisfies* $\mathfrak{C}\mathfrak{A}^t x = \tau^t_+ \mathfrak{C} x$, *for all* $x \in X$ *and all* $t \in \mathbf{R}^+$*;*

*(iv)* $\mathfrak{D} \colon L^p_{c,\mathrm{loc}}(\mathbf{R}; U) \to L^p_{c,\mathrm{loc}}(\mathbf{R}; Y)$ *satisfies* $\tau^t \mathfrak{D} u = \mathfrak{D} \tau^t u$, $\pi_- \mathfrak{D} \pi_+ u = 0$, *and* $\pi_+ \mathfrak{D} \pi_- u = \mathfrak{C}\mathfrak{B} u$, *for all* $u \in L^p_{c,\mathrm{loc}}(\mathbf{R}; U)$ *and all* $t \in \mathbf{R}$.

*The different components of $\Psi$ are called as follows: $U$ is the input space, $X$ is the state space, $Y$ is the output space, $\mathfrak{A}$ is the semigroup, $\mathfrak{B}$ is the input map, $\mathfrak{C}$ is the output map, and $\mathfrak{D}$ is the input-output map. The state $x(t) \in X$ at time $t \in \mathbf{R}^+$ and the output $y \in L^p_{\mathrm{loc}}(\mathbf{R}^+; Y)$ of $\Psi$ with initial time zero, initial state $x_0 \in X$ and input function $u \in L^p_{\mathrm{loc}}(\mathbf{R}^+; U)$ are given by (1.2) with $\mathfrak{B}^t_0 = \mathfrak{B}\tau^t \pi_{(0,t)} u$ and $\mathfrak{D}_0 u = \mathfrak{D}\pi_+ u$.*

For more details, explanations and examples we refer the reader to Arov and Nudelman (1996), Curtain and Weiss (1989), Salamon (1987, 1989), Staffans (1997, 1998a,c,b, 1999a,b), Weiss (1989a,b,c, 1991, 1994a,b), Weiss and Weiss (1997) (and the references therein). Most of the available literature deals with Hilbert spaces and $p = 2$.

Before introducing the operators $B$ and $C$ in (1.1), we need two auxiliary spaces $X_1$ and $X_{-1}$. Choose any $\gamma$ in the resolvent set of the generator $A$ of $\mathfrak{A}$. We let $X_1$ be the domain of $A$, with the norm $\|x\|_{X_1} = \|(\gamma I - A)x\|_X$, and $X_{-1}$ is the completion of $X$ with the norm $\|x\|_{X_{-1}} = \|(\gamma I - A)^{-1} x\|_X$. The semigroup $\mathfrak{A}$ can be extended to a strongly continuous semigroup on $X_{-1}$, which we denote by the same symbol. We denote the space of bounded linear operators from $U$ to $Y$ by $\mathcal{L}(U;Y)$, and let $L^p_\omega(\mathbf{R}^+; U)$ represent the space of functions $u \colon \mathbf{R}^+ \to U$ for which $t \mapsto \mathrm{e}^{-\omega t} u(t)$ belongs to $L^p(\mathbf{R}^+; U)$.

**2.2. Proposition.** *Let $\Psi = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ be a $L^p$-well-posed linear system on $(Y, X, U)$. Denote the growth rate of $\mathfrak{A}$ by $\omega_{\mathfrak{A}}$.*

*(i)* $\Psi = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *has a unique* control operator $B \in \mathcal{L}(U; X_{-1})$, *determined by the fact that the input term $\mathfrak{B}^t_0 u$ in (1.2) is given by the standard variation of constants formula (the function inside the integral takes it values in $X_{-1}$, but the final result belongs to $X$)*

$$\mathfrak{B}^t_0 = \mathfrak{B}\tau^t \pi_{(0,t)} u = \int_0^t \mathfrak{A}^{t-s} B u(s)\, ds, \quad \forall t \in \mathbf{R}^+, \quad \forall u \in L^p_{\mathrm{loc}}(\mathbf{R}^+; U).$$

*(ii)* $\Psi$ *has a unique* observation operator $C \in \mathcal{L}(X_1; Y)$, *determined by the fact that the output term $\mathfrak{C} x_0$ in (1.2) is given by (for almost all $t \in \mathbf{R}^+$)*

$$(\mathfrak{C} x_0)(t) = C \mathfrak{A}^t x_0, \qquad \forall x_0 \in X_1.$$

*(iii)* $\Psi$ *has a unique analytic* $\mathcal{L}(U;Y)$*-valued* transfer function $\widehat{\mathfrak{D}}$ *defined on* $\Re z > \omega_{\mathfrak{A}}$*, determined by the fact that the Laplace transform* $\widehat{\mathfrak{D}_0 u}$ *of the input-output term* $\mathfrak{D}_0 u$ *in* (1.2) *is given by*

$$\widehat{\mathfrak{D}_0 u} = \widehat{\mathfrak{D}}(z)\hat{u}(z), \quad \forall \Re z > \omega_{\mathfrak{A}}, \quad \forall u \in L^p_{\omega_{\mathfrak{A}}}(\mathbf{R}^+;U),$$

*where* $\hat{u}$ *is the Laplace transform of* $u$*.*

The existence of a control operator $B$ is proved in Weiss (1989a), the existence of an observation operator is proved in Weiss (1989b), and the existence of a transfer function is proved in Curtain and Weiss (1989) and Weiss (1991). (See also Salamon (1989) and (Weiss, 1994a, Remark 2.4).) The control operator $B$ is said to be *bounded* if the range of $B$ lies in $X$, in which case $B \in \mathcal{L}(U;X)$. The observation operator $C$ is said to be *bounded* if it is continuous with respect to the norm of $X$, i.e., if it can be extended to an operator in $\mathcal{L}(X;Y)$.

To get a time-domain representation for the output $y$ of an $L^p$-well-posed linear system similar to the second equation in (1.1) we introduce the subspace $V$ of $X \times U$ define by

$$V = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X \times U \;\middle|\; Ax + Bu \in X \right\}. \tag{2.1}$$

Also this space is a Banach space with the norm

$$|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]|_V = \left\{ |x|_X^2 + |u|_U^2 + |Ax + Bu|_X^2 \right\}^{1/2}.$$

If $X$ and $U$ are Hilbert spaces, then so is $V$.

**2.3. Proposition.** *Let* $\Psi = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *be a* $L^p$*-well-posed linear system on* $(Y,X,U)$*. Denote the growth rate of* $\mathfrak{A}$ *by* $\omega_{\mathfrak{A}}$*. For each* $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in V$ *we define* $N \in \mathcal{L}(V;Y)$ *by*

$$\begin{aligned} N \begin{bmatrix} x \\ u \end{bmatrix} &= C(\alpha I - A)^{-1}\big[\alpha x - (Ax + Bu)\big] + \widehat{\mathfrak{D}}(\alpha)u \\ &= C\big[x - (\alpha I - A)^{-1}Bu\big] + \widehat{\mathfrak{D}}(\alpha)u, \end{aligned} \tag{2.2}$$

*where* $\alpha \in \mathbf{C}$ *with* $\Re\alpha > \omega_{\mathfrak{A}}$ *can be chosen in an arbitrary way (i.e., the the result is independent of* $\alpha$ *as long as* $\Re\alpha > \omega_{\mathfrak{A}}$*). We call* $N$ *the* combined observation/feedthrough operator *of* $\Psi$*.*

*(i) The output* $y = \mathfrak{C}x_0 + \mathfrak{D}_0 u$ *of* $\Psi$ *defined in* (1.2) *is given by*

$$y(t) = N \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = N \begin{bmatrix} \mathfrak{A}^t x_0 + \mathfrak{B}_0^t u \\ u(t) \end{bmatrix}, \quad t \geq 0, \tag{2.3}$$

*for all* $x_0 \in X$ *and all* $u \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^+;U)$ *satisfying* $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in V$*. In particular,* $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in V$ *for all* $t \geq 0$*.*

*(ii) The transfer function* $\widehat{\mathfrak{D}}$ *of* $\Psi$ *is given by*

$$\widehat{\mathfrak{D}}(z) = N \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}, \qquad \Re z > \omega_{\mathfrak{A}}. \tag{2.4}$$

See Arov and Nudelman (1996), Curtain and Weiss (1989), Salamon (1987, 1989), and Weiss (1989a,b) (or Staffans (1999b)) for the proof.

If either $B$ or $C$ is bounded then it is possible to split $N$ into

$$N \begin{bmatrix} x \\ u \end{bmatrix} = Cx + Du, \tag{2.5}$$

where $D = \widehat{\mathfrak{D}}(\alpha) - C(\alpha I - A)^{-1}B$ belongs to $\mathcal{L}(U;Y)$ (and does not depend on $\alpha$). More generally, this can be done whenever the system is compatible in the sense of Helton (1976). To describe this property we introduce one more subspace $Z$ of $X$. We choose any $\gamma$ in the resolvent set of $A$, and define

$$Z = \{ z \in X \mid z = (\gamma I - A)^{-1}(x + Bu) \text{ for some } x \in X \text{ and } u \in U \}. \tag{2.6}$$

Then $Z$ is a Banach space with the norm

$$|z|_Z = \inf_{(\gamma I - A)^{-1}(x+Bu)=z} \left( |x|_X^2 + |u|_U^2 \right)^{1/2},$$

satisfying $X_1 \subset Z \subset X$, and it is a Hilbert space if both $X$ and $U$ are Hilbert spaces.

**2.4. Definition.** *The $L^p$-well-posed linear system $\Psi = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ is* compatible *if its observation operator can be extended to an operator in $\mathcal{L}(Z;Y)$.*

This extension need not be unique since $X_1$ need not be dense in $Z$. This means that $D$ need not be unique either. As Helton (1976) comments, most physically motivated systems seem to be compatible. For example all systems which are (weakly) regular in the sense of Weiss (1994a) and Weiss and Weiss (1997) are compatible, and so are all systems with a finite-dimensional input space (Staffans, 1999b). A reasonably complete theory for compatible systems is presented in Staffans (1999b).

# 3 The Lax–Phillips Scattering Model

Instead of using a $L^p$-well-posed linear system to formalize the idea of having an output and state at time $t > 0$ which depend continuously on the input and the initial state we can proceed in a different way which leads to a generalized *Lax–Phillips scattering model.* This is a semigroup $\mathbb{T}$ defined on $\mathcal{Y} \times X \times \mathcal{U} = L^p_\omega(\mathbf{R}^-;Y) \times X \times L^p_\omega(\mathbf{R}^+;U)$ with certain additional properties. (Here $L^p_\omega(\mathbf{R}^-;Y)$ consists of all the functions $y\colon \mathbf{R}^- \to Y$ for which $t \mapsto \mathrm{e}^{-\omega t}y(t)$ belongs to $L^p(\mathbf{R}^-;Y)$ and similarly for $L^p_\omega(\mathbf{R}^+;U)$.) We call $\mathcal{U}$ the *incoming subspace*, $X$ the *central state space*, and $\mathcal{Y}$ the *outgoing subspace*. In the classical cases treated in Lax and Phillips (1967, 1973) $\omega$ is taken to be zero and $\mathbb{T}$ is required to be unitary (the conservative case) or a contraction semigroup (the nonconservative case).

We claim that there is a one-to-one correspondence between the class of all well-posed linear systems and the class of all Lax–Phillips models. The parameter $\omega \in \mathbf{R}$ can be chosen in an arbitrary way (the best choice depends on the particular application).

Let $\Psi = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ be a given $L^p$-well-posed linear system. To each such system we construct a Lax–Phillips model $\mathbb{T}$ on $\mathcal{Y} \times X \times \mathcal{U}$ as follows. The initial data consists of the initial incoming state $u_0 \in \mathcal{U}$ representing the future values of the input, the initial central state $x_0 \in X$ is identical to the initial state of $\Psi$, and the initial outgoing state $y_0 \in \mathcal{Y}$ represents the past values of the output. At time $t \geq 0$ the incoming state $u_t$ is the left-shifted input $\tau_+^t u_0$ (the unused part of the input). The central state $x_t$ at time $t$ is equal to the state $x(t) = \mathfrak{A}^t x_0 + \mathfrak{B}_0^t u$ of $\Psi$ at time $t$ with initial time zero, initial state $x_0$, and input $u_0$ (it depends only on $x_0$ and on the discarded part $\pi_{(0,t)}u$ of $u$). The outgoing state $y_t$ at time $t$ consists of two parts: it is the sum of $\tau_-^t y_0$ (the left-shifted original outgoing state) and $\tau^t\pi_{(0,t)}(\mathfrak{C}x_0 + \mathfrak{D}_0 u_0)$ (the restriction of the output $\mathfrak{C}x_0 + \mathfrak{D}_0 u_0$ of $\Psi$ to the interval $(0,t)$ shifted to the left by $\tau^t$ so that the shifted and truncated output is supported on $(-t,0)$). Formalizing this idea we get the following theorem, where we use the notations

$$\mathfrak{B}_0^t = \mathfrak{B}\tau^t\pi_{(0,t)}, \qquad \mathfrak{C}_0^t = \pi_{(0,t)}\mathfrak{C}, \qquad \mathfrak{D}_0^t = \pi_{(0,t)}\mathfrak{D}\pi_{(0,t)}.$$

**3.1. Theorem.** *Let $\Psi = \left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ be a $L^p$-well-posed linear system on $(Y,X,U)$ with $1 \le p < \infty$. Let $\omega \in \mathbf{R}$, $\mathcal{Y} = L^p_\omega(\mathbf{R}^-;Y)$ and $\mathcal{U} = L^p_\omega(\mathbf{R}^+;U)$. For each $(y_0,x_0,u_0) \in \mathcal{Y}\times X\times\mathcal{U}$ and $t \ge 0$, define*

$$\mathbb{T}^t\begin{bmatrix} y_0\\ x_0\\ u_0\end{bmatrix} = \begin{bmatrix}\tau_-^t & \tau^t\mathfrak{C}_0^t & \tau^t\mathfrak{D}_0^t\\ 0 & \mathfrak{A}^t & \mathfrak{B}_0^t\\ 0 & 0 & \tau_+^t\end{bmatrix}\begin{bmatrix} y_0\\ x_0\\ u_0\end{bmatrix}. \tag{3.1}$$

*Then $\mathbb{T}$ is a strongly continuous semigroup on $\mathcal{Y}\times X\times U$.*

Here the strong continuity of $\mathbb{T}$ is obvious, and so is the property $\mathbb{T}(0) = I$. The proof of the semigroup property $\mathbb{T}(s+t) = \mathbb{T}(s)\mathbb{T}(t)$ for $s$, $t \ge 0$ is a short algebraic computation based on Definition 2.1 (see Staffans (1999b) for details).

The semigroup $\mathbb{T}$ in Theorem 3.1 has an additional 'causality' property, which in the Hilbert space case where $p = 2$ and $U$, $X$, and $Y$ are Hilbert spaces can be described as follows: for all $t \ge 0$, the images of the central and incoming states under $\mathbb{T}^t$ are orthogonal to the image of the outgoing state, and the null space of $\mathbb{T}^t$ projected onto the central and outgoing spaces is orthogonal to the null space of $\mathbb{T}^t$ projected onto the incoming space. In the general case these properties can easiest be characterized in the following way.

**3.2. Definition.** *A* Lax–Phillips model *of type $L^p_\omega$ is a semigroup on $\mathcal{Y}\times X\times\mathcal{U} = L^p_\omega(\mathbf{R}^-;Y)\times X\times L^p_\omega(\mathbf{R}^+;U)$ with the structure*

$$\mathbb{T}^t = \begin{bmatrix}\tau_-^t & \mathbb{C}^t & \mathbb{D}^t\\ 0 & \mathbb{A}^t & \mathbb{B}^t\\ 0 & 0 & \tau_+^t\end{bmatrix}, \tag{3.2}$$

*where $\mathbb{A}$ is strongly continuous and $\mathbb{B}^t$, $\mathbb{C}^t$, and $\mathbb{D}^t$ satisfy the* causality conditions

$$\begin{aligned} \mathbb{C}^t &= \pi_{(-t,0)}\mathbb{C}^t, &\qquad \mathbb{D}^t &= \pi_{(-t,0)}\mathbb{D}^t,\\ \mathbb{D}^t &= \mathbb{D}^t\pi_{(0,t)}, & \mathbb{B}^t &= \mathbb{B}^t\pi_{(0,t)}. \end{aligned} \tag{3.3}$$

This set of conditions is a rewritten version of conditions (1.2) in Lax and Phillips (1973). Helton (1976) uses the name *inertness* for this additional causality property.

**3.3. Corollary.** *The semigroup $\mathbb{T}$ constructed in Theorem 3.1 is a Lax–Phillips model of type $L^p_\omega$.*

This is immediate from Theorem 3.1 and Definition 3.2. We call the semigroup $\mathbb{T}$ in Theorem 3.1 the *Lax–Phillips model (of type $L^p_\omega$) induced by $\Psi$*.

It is only slightly more difficult to prove a converse to Corollary 3.3: To every Lax–Phillips model there corresponds a well-posed linear system which induces this Lax–Phillips model:

**3.4. Theorem.** *Let $\mathbb{T}$ be a Lax–Phillips model of type $L^p_\omega$. With the notations of Definition 3.2, let*

$$\begin{aligned} \mathfrak{A} &= \mathbb{A}, & \mathfrak{B} &= \lim_{s\to\infty} \mathbb{B}^s \tau^{-s}, \\ \mathfrak{C} &= \lim_{t\to\infty} \tau^{-t}\mathbb{C}^t, & \mathfrak{D} &= \lim_{\substack{t\to\infty \\ s\to\infty}} \tau^{-s}\mathbb{D}^{s+t}\tau^{-t}. \end{aligned} \tag{3.4}$$

*Then $\Psi = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ is an $L^p$-well-posed linear system on $(Y, X, U)$, and $\mathbb{T}$ is the Lax–Phillips model induced by this system.*

The proof of Theorem 3.4 is another algebraic computation given in Staffans (1999b).

**3.5. Corollary.** *For each $\omega \in \mathbf{R}$ and $1 \le p < \infty$, there is a one-to-one correspondence between the class of all $L^p$-well-posed linear systems and all Lax–Phillips models of type $L^p_\omega$: every $L^p$-well-posed linear system $\Psi$ induces a unique Lax–Phillips model $\mathbb{T}$ of type $L^p_\omega$, and conversely, every Lax–Phillips model $\mathbb{T}$ of type $L^p_\omega$ induces a unique $L^p$-well-posed linear system $\Psi$.*

This is a union of Corollary 3.3 and Theorem 3.4. Parts of this corollary (where either the input operator or output operator vanishes) were proved by Grabowski and Callier (1996) and Engel (1997). It is also (implicitly) contained in Arov and Nudelman (1996).

There are a number of important ingredients in the Lax–Phillips scattering theory, such as the backward and forward wave operators, the scattering operator, and the scattering matrix. All of these have natural analogies in the theory of well-posed linear systems. In the discussion below we choose $\omega > \omega_{\mathfrak{A}}$, where $\omega_{\mathfrak{A}}$ is the growth rate of $\mathfrak{A}$.

The *backward wave operator* $W_-$ (denoted by $W_2$ in (Lax and Phillips, 1973, Theorem 1.2)) is the limit of the last column of $\mathbb{T}\tau^{-t}$ as $t \to \infty$. It maps $L^p_\omega(\mathbf{R}; U)$ into $L^p_\omega(\mathbf{R}^-; Y) \times X \times L^p_\omega(\mathbf{R}^+; U)$, and it is given by (cf. Theorem 3.4)

$$W_- u = \begin{bmatrix} \pi_-\mathfrak{D} \\ \mathfrak{B} \\ \pi_+ \end{bmatrix} u. \tag{3.5}$$

Thus, it keeps the future input $\pi_+ u$ intact, and maps the past input $\pi_- u$ into the past output $\pi_-\mathfrak{D}u$ and the present central state $\mathfrak{B}u$.

The *forward wave operator* $W_+$ (denoted by $W_1$ in (Lax and Phillips, 1973, Theorem 1.2)) is the limit of the first row of $\tau^{-t}\mathbb{T}$ as $t \to \infty$. It maps $L^p_\omega(\mathbf{R}^-; Y) \times X \times L^p_\omega(\mathbf{R}^+; U)$ into $L^p_\omega(\mathbf{R}; Y)$, and it is given by (cf. Theorem 3.4)

$$W_+ \begin{bmatrix} y \\ x_0 \\ u \end{bmatrix} = \begin{bmatrix} \pi_- & \mathfrak{C} & \mathfrak{D}\pi_+ \end{bmatrix} \begin{bmatrix} y \\ x_0 \\ u \end{bmatrix}. \tag{3.6}$$

Thus, it keeps the past output $\pi_- y$ intact, and maps the present central state $x_0$ and the future input $\pi_+ u$ into the future output $\mathfrak{C}x_0 + \mathfrak{D}\pi_+ u$.

The *scattering operator* in Lax–Phillips theory is the product $W_+W_-$, and it is given by

$$W_+W_- = \begin{bmatrix} \pi_- & \mathfrak{C} & \mathfrak{D}\pi_+ \end{bmatrix} \begin{bmatrix} \pi_-\mathfrak{D} \\ \mathfrak{B} \\ \pi_+ \end{bmatrix} = \pi_-\mathfrak{D} + \mathfrak{C}\mathfrak{B} + \pi_+\mathfrak{D} = \mathfrak{D}. \tag{3.7}$$

Thus, the scattering operator is nothing but the (bilaterally shift-invariant) input-output map $\mathfrak{D}$ of the corresponding well-posed linear system.

To get the *scattering matrix* of the Lax–Phillips system we apply the scattering operator $\mathfrak{D}$ to an input of the form $u(t) = \mathrm{e}^{zt}u_0$, where $z \in \mathbf{C}$ has a sufficiently large real part and $u_0 \in U$ is fixed; see (Lax and Phillips, 1973, pp. 187–188). Because of the shift-invariance of $\mathfrak{D}$, the resulting output is of the type $y(t) = \mathrm{e}^{zt}y_0$ for some $y_0 \in Y$. The scattering matrix (evaluated at $z$) is defined to be the operator that maps $u_0 \in U$ into $y_0 \in Y$. It follows from (Weiss, 1991, p. 194) that the scattering matrix of a Lax–Phillips system is equal to the transfer function $\widehat{\mathfrak{D}}$ of the corresponding well-posed linear system.

## 4 The System Operator

All the necessary information about a well-posed linear system $\Psi = \left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ is contained in the *system operator* $S = \left[\begin{smallmatrix}A\ B\\ N\end{smallmatrix}\right]$ which in the compatible case can be written in the form $S = \left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$. This is an unbounded operator from $X \times U$ to $X \times Y$, whose domain is the space $V$ introduced in Section 2. It has the following properties.

**4.1. Proposition.** *The system operator $S = \left[\begin{smallmatrix}A\ B\\ N\end{smallmatrix}\right] : X \times U \supset V \to X \times Y$ is closed and densely defined. In the reflexive case where $1 < p < \infty$ and $U$, $X$, and $Y$ are reflexive Banach spaces the system operator of the dual system is equal to the adjoint $S^*$ of $S$.*

We refer the reader to Staffans (1999b) for the proof of this result. The definition of the dual of a well-posed linear system can be found in, e.g., Staffans (1997, 1999b) and in Weiss and Weiss (1997).

The generator of the Lax–Phillips model can be characterized as follows:

**4.2. Theorem.** *Let $1 \le p < \infty$ and $\omega \in \mathbf{R}$, let $\Psi = \left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ be an $L^p$-well-posed linear system on $(Y, X, U)$ with semigroup generator $A$, control operator $B$, and combined observation/feedthrough operator $N$, and let $T$ be the generator of the corresponding Lax–Phillips model $\mathbb{T}$ of type $L^p_\omega$ defined in Definition 3.2.*

*(i) The domain of $T$ consists of all the vectors $\left[\begin{smallmatrix}y_0\\ x_0\\ u_0\end{smallmatrix}\right] \in W^{1,p}_\omega(\mathbf{R}^-;Y) \times X \times W^{1,p}_\omega(\mathbf{R}^+;U)$ which satisfy $Ax_0 + Bu_0(0) \in X$ and $y_0(0) = N\left[\begin{smallmatrix}x_0\\ u_0(0)\end{smallmatrix}\right]$, and on its domain $T$ is given by*

$$T\begin{bmatrix}y_0\\ x_0\\ u_0\end{bmatrix} = \begin{bmatrix}y_0'\\ Ax_0 + Bu_0(0)\\ u_0'\end{bmatrix}.$$

*Thus, the following two conditions are equivalent:*

*(a) $\left[\begin{smallmatrix}y_0\\ x_0\\ u_0\end{smallmatrix}\right] \in \mathcal{D}(T)$ and $\left[\begin{smallmatrix}y\\ x\\ u\end{smallmatrix}\right] = T\left[\begin{smallmatrix}y_0\\ x_0\\ u_0\end{smallmatrix}\right]$;*

*(b) $y_0 \in W^{1,p}_\omega(\mathbf{R}^-;Y)$, $x_0 \in X$, $x \in X$, $u_0 \in W^{1,p}_\omega(\mathbf{R}^+;U)$, and*

$$\begin{bmatrix}x\\ y_0(0)\end{bmatrix} = \begin{bmatrix}A & B\\ & N\end{bmatrix}\begin{bmatrix}x_0\\ u_0(0)\end{bmatrix}, \qquad \begin{bmatrix}y\\ u\end{bmatrix} = \begin{bmatrix}y_0'\\ u_0'\end{bmatrix}.$$

*(ii) The spectrum of $T$ contains the vertical line $\{\Re\alpha = \omega\}$, and*

$$\sigma(T) \cap \{\Re\lambda > \omega\} = \sigma(A) \cap \{\Re\lambda > \omega\}.$$

*For each $\alpha \in \rho(T) \cap \{\Re\alpha > \omega\}$ and $\left[\begin{smallmatrix}y\\ x\\ u\end{smallmatrix}\right] \in L^p_\omega(\mathbf{R}^-;Y) \times X \times L^p_\omega(\mathbf{R}^+;U)$ the following conditions are equivalent:*

*(a)* $\begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} = (\alpha I - T)^{-1} \begin{bmatrix} y \\ x \\ u \end{bmatrix}$;

$$\text{(b)} \begin{cases} \begin{bmatrix} x_0 \\ y_0(0) \end{bmatrix} = \begin{bmatrix} (\alpha I - A)^{-1} & (\alpha I - A)^{-1}B \\ C(\alpha I - A)^{-1} & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \begin{bmatrix} x \\ \hat{u}(\alpha) \end{bmatrix}, \\ y_0(t) = \mathrm{e}^{\alpha t} y_0(0) + \int_t^0 \mathrm{e}^{\alpha(t-s)} y(s)\, ds, \qquad t < 0, \\ u_0(t) = \int_t^\infty \mathrm{e}^{\alpha(t-s)} u(s)\, ds, \qquad t \geq 0. \end{cases}$$

Theorem 4.2 does not say anything about the part of the spectrum of $T$ which lies in the half plane $\{\Re\alpha < \omega\}$. A result similar to the one above is true for this part of the spectrum if $\Psi$ is *time-reversible*. We shall return to this question elsewhere.

# 5 Admissibility

According to Corollary 3.5, there is a one-to-one correspondence between the class of all $L^p$-well-posed linear systems and all Lax–Phillips scattering models of type $L^p$. This means that we can reduce the study of the generators of a well-posed linear system to the study of the generators the Lax–Phillips semigroup. This way we can obtain necessary and sufficient conditions for the admissibility or joint admissibility of a control operator $B$ and an observation operator $C$. These notions are defined as follows.

As always we let $U$, $X$, and $Y$ be Banach spaces and let $1 \leq p < \infty$. We let $\mathfrak{A}$ be a strongly continuous semigroup on the Banach space $X$ with generator $A$, and define the spaces $X_1$ and $X_{-1}$ as in Section 2. This time we specify, in addition, some $\omega \in \mathbf{R}$, and suppose that $\mathfrak{A}$ is $\omega$-bounded, i.e., $\sup_{t>0} \mathrm{e}^{-\omega t}\|\mathfrak{A}^t\| < \infty$.

An operator $B \in \mathcal{L}(U; X_{-1})$ is an *$L^p$-admissible control operator for $A$* if for some $t > 0$ (hence for all $t > 0$) the operator

$$\mathfrak{B}_0^t u = \int_0^t \mathfrak{A}^{t-s} B u(s)\, ds, \qquad u \in L^p((0,t); U), \tag{5.1}$$

maps $L^p((0,t);U)$ into $X$ (instead of $X_{-1}$). (This operator is then bounded with values in $X$). We call $B$ *$\omega$-bounded* if the resulting input map

$$\mathfrak{B}u = \lim_{v \to -\infty} \int_v^0 \mathfrak{A}^{-s} B u(s)\, ds, \qquad u \in L^p_\omega(\mathbf{R}^-; U) \tag{5.2}$$

is $\omega$-bounded, i.e., it defines a bounded linear operator from $L^p_\omega(\mathbf{R}^-;U)$ to $X$.

The operator $C \in \mathcal{L}(X_1; Y)$ is an *$L^p$-admissible observation operator for $A$* if the map

$$(\mathfrak{C}x)(t) = C\mathfrak{A}^t x, \qquad x \in X_1, \qquad t \geq 0, \tag{5.3}$$

can be extended to a bounded linear operator $X \to L^p_{\text{loc}}(\mathbf{R}^+; Y)$, and it is *$\omega$-bounded* if the resulting output map $\mathfrak{C}$ is $\omega$-bounded, i.e., it maps $X$ into $L^p_\omega(\mathbf{R}^+;Y)$.

The operators $B \in \mathcal{L}(U; X_{-1})$ and $C \in \mathcal{L}(X_1; Y)$ are *jointly $L^p$-admissible for $A$* if $B$ is an $L^p$-admissible control operator for $A$, $C$ is an $L^p$-admissible observation operator for $A$, and the operator $\mathfrak{D}\colon W^{1,p}_{c,\text{loc}}(\mathbf{R}; U) \to C_c(\mathbf{R}; Y)$ defined by

$$(\mathfrak{D}u)(t) = C\big[\mathfrak{B}\tau^t u - (\alpha I - A)^{-1} B u(t)\big] + D_\alpha u(t), \qquad t \in \mathbf{R}, \tag{5.4}$$

can be extended to a continuous operator $L^p_{c,\mathrm{loc}}(\mathbf{R};U) \to L^p_{c,\mathrm{loc}}(\mathbf{R};Y)$. Here $\alpha \in \rho(A)$ and $D_\alpha \in \mathcal{L}(U;Y)$ can be chosen in an arbitrary way. By introducing the combined observation/feedthrough operator

$$N\begin{bmatrix} x \\ u \end{bmatrix} = C\big[x - (\alpha I - A)^{-1}Bu\big] + D_\alpha u. \tag{5.5}$$

we can simplify the formula for $(\mathfrak{D}u)(t)$ into

$$(\mathfrak{D}u)(t) = N\begin{bmatrix} \mathfrak{B}\tau^t u \\ u(t) \end{bmatrix}, \qquad t \in \mathbf{R}. \tag{5.6}$$

We call $B$ and $C$ *jointly $\omega$-bounded* if both $B$ and $C$ are $\omega$-bounded and, in addition, the operator $\mathfrak{D}$ can be extended to a bounded linear operator from $L^p_\omega(\mathbf{R};U)$ to $L^p_\omega(\mathbf{R};Y)$. If (and only if) $B$ and $C$ are jointly admissible, then the four operator $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ can be combined into a $L^p$-well-posed linear system $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ with system operator $\left[\begin{smallmatrix} A & B \\ \multicolumn{2}{c}{N} \end{smallmatrix}\right]$. (Here $\mathfrak{D}$ is determined by $A$, $B$, and $C$ only modulo a constant static term.)

**5.1. Theorem.** *Let $\omega \in \mathbf{R}$, $1 \le p < \infty$, and let $A$ be the generator of an $\omega$-bounded $C_0$ semigroup on $X$.*

*(i) $B \in \mathcal{L}(U;X_{-1})$ is an $L^p$-admissible $\omega$-bounded control operator for $A$ if and only if there is a constant $M > 0$ such that, for all $u \in L^p_\omega(\mathbf{R}^+;U)$, $\lambda > \omega$, and $n = 0,1,2,\dots$,*

$$\left|\frac{\partial^n}{\partial\lambda^n}(\lambda I - A)^{-1}B\hat{u}(\lambda)\right|_X \le \frac{Mn!}{(\lambda-\omega)^{n+1}}\|u\|_{L^p_\omega(\mathbf{R}^+;U)}. \tag{5.7}$$

*(ii) $C \in \mathcal{L}(X_1;Y)$ is an $L^p$-admissible $\omega$-bounded observation operator for $A$ if and only if there is a constant $M > 0$ such that, for all $x_0 \in X$, $\lambda > \omega$, and $n = 0,1,2,\dots$,*

$$\left(\int_0^\infty \left|\frac{\partial^n}{\partial\lambda^n}\mathrm{e}^{-(\lambda-\omega)t}C(\lambda I - A)^{-1}x_0\right|_Y^p dt\right)^{1/p} \le \frac{Mn!}{(\lambda-\omega)^{n+1}}|x_0|_X. \tag{5.8}$$

*(iii) The operators $B \in \mathcal{L}(U;X_{-1})$ and $C \in \mathcal{L}(X_1;U)$ are jointly $L^p$ admissible and $\omega$-bounded iff $B$ is an $L^p$-admissible $\omega$-bounded control operator for $A$ (cf. (i)), $C$ is an admissible $\omega$-bounded observation operator for $A$ (cf. (ii)) and there is a constant $M > 0$ such that, for all $u \in L^p_\omega(\mathbf{R}^+;U)$, $\lambda > \omega$, and $n = 0,1,2,\dots$,*

$$\left(\int_0^\infty \left|\frac{\partial^n}{\partial\lambda^n}\mathrm{e}^{-(\lambda-\omega)t}\widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda)\right|_Y^p dt\right)^{1/p} \le \frac{Mn!}{(\lambda-\omega)^{n+1}}\|u\|_{L^p_\omega(\mathbf{R}^+;U)}, \tag{5.9}$$

*where*

$$\widehat{\mathfrak{D}}(\lambda) = (\alpha-\lambda)C(\lambda I - A)^{-1}(\alpha I - A)^{-1}B + D_\alpha; \tag{5.10}$$

*here $\alpha$ with $\Re\alpha > \omega$ and $D_\alpha \in \mathcal{L}(U;Y)$ can be chosen in an arbitrary manner.*

Part (ii) of this theorem was proved by Grabowski and Callier (1996) in the exponentially stable Hilbert space case (i.e., $\mathfrak{A}$ is exponentially stable, $\omega = 0$, $p = 2$, and $U$, $X$, and $Y$ are Hilbert spaces), and the corresponding case of part (i) can be derived from (ii) by duality. The general case of parts (i) and (ii) was proved by Engel (1997). Part (iii) appears to be new. A proof of the full theorem is given in Staffans (1999b).

Condition (5.9) does not depend on the particular *realization* $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ of $\mathfrak{D}$, i.e., it does not contain any direct references to $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, but only to $\widehat{\mathfrak{D}}$ which is completely determined by $\mathfrak{D}$. This indicates that the following conjecture may be true:

**5.2. Conjecture.** *An analytic $\mathcal{L}(U;Y)$-valued function $\widehat{\mathfrak{D}}$ on $\Re\lambda > \omega$ is the transfer function of an $\omega$-bounded $L^p$-well-posed linear system if and only if there is a constant $M > 0$ such that* (5.9) *holds for all $u \in L^p_\omega(\mathbf{R}^+;U)$, $\lambda > \omega$, and $n = 0, 1, 2, \ldots$.*

Clearly, by Theorem 5.1, condition (5.9) is necessary for $\widehat{\mathfrak{D}}$ to be a $L^p$-well-posed $\omega$-bounded transfer function, and we conjecture that it is also sufficient. This would give us a necessary and sufficient condition for an $H^\infty$ function to be an $L^p$-multiplier.

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