Well-Posed Linear Systems, Lax–Phillips Scattering, and L^p-Multipliers

Olof J. Staffans

Abstract. We discuss the connection between Lax–Phillips scattering theory and the theory of well-posed linear systems, and show that the latter theory is a natural extension of the former. As a consequence of this, there is a close connection between the Lax–Phillips generator and the generators of the corresponding well-posed linear system. All the essential information about these two systems is contained in the system operator $S_\Sigma = \left[\begin{smallmatrix} A & B \\ C\&D \end{smallmatrix}\right]$, where A is the generator of the (central) semigroup, B is the control operator, and $C\&D$ is the combined observation/feedthrough operator. In the important Hilbert space case this system operator can be written in the more familiar form $S_\Sigma = \left[\begin{smallmatrix} A & B \\ \overline{C} & D \end{smallmatrix}\right]$, where $\overline{C}$ is a (not necessarily uniquely determined) observation operator and D is the corresponding (generalized) feedthrough operator. The system operator is closed and densely defined. In the reflexive case the adjoint of S_Σ is the system operator of the dual system. We give formulas for the Lax–Phillips generator and resolvent in terms of the system operator. By applying the Hille–Yoshida theorem to the Lax–Phillips semigroup we get necessary and sufficient conditions for the L^p-admissibility or joint L^p-admissibility of a control operator B and an observation operator C. This leads to a criterion for an H^∞-function to be an L^p-multiplier.

1. Background

This review of the relationship between the Lax–Phillips scattering theory on one hand and the theory of well-posed linear systems on the other hand has a very definite *date of conception*: the talk on '*Passive Linear Systems and Scattering Theory*' by Prof. *D. Z. Arov* given at MTNS in Padova in 1998. He said:

'In the connection with Lax–Phillips scattering scheme *Yu Smulijan [1986] proposed the following definition* of a linear continuous time-invariant system. It is *a little bit different from the one proposed in control theory by D. Salamon.*'

This gave me the motivation to take a closer look at the scattering theory by Lax and Phillips to find out how the theory developed by Arov and Smulijan differs from the one developed by Salamon. After studying [4] and [14, 15] for some time

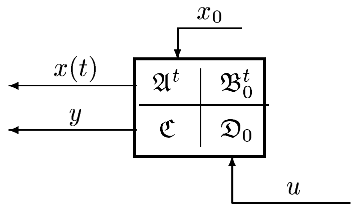

FIGURE 1. Input/State/Output Diagram of Σ

I found the rather surprising answer: there is *no real difference* (although different people tend to *emphasize* different aspects of the theory.)

Thus, this is a presentation of some of the basic notions of the general theory of well-posed linear systems developed by, among others, Adamjan, Arov, Lax, Helton, Nudelman, Ober, Phillips, Salamon, Smulijan, and Weiss. To me *Arov's notation* which he (at least partially) inherited from Lax and Phillips felt quite cumbersome, since I am used to a control theory type notation. Therefore I use a set of notations which is an extension of the standard control theory type notations (and which resemble those used by Salamon and by Weiss).

I apologize for the fact that I do not in all instances know *which results should be credited to whom*. Many of these results have been *discovered and then rediscovered*, maybe even several times.

A preliminary version of this review was presented in [24]. Details and proofs are given in the paper [26] by George Weiss and myself, and in the book manuscript [25] available (in postscript form) at http://www.abo.fi/˜staffans/.

2. Introduction

Many infinite-dimensional systems can be described by the equations

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= \overline{C}x(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

on a triple of Banach spaces, namely, the input space U, the state space X, and the output space Y. We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operator A is the generator of a strongly continuous semigroup, and it is usually unbounded. Also B and $\overline{C}$ are usually unbounded, whereas D is bounded.

Because of the presence of the unboudned operators A, B, and $\overline{C}$ it is often convenient to use the 'integral' representation of the system, which consists of the four operators from the initial state x_0 and the input function u to the final state $x(t)$ and the output function y:

$$\begin{aligned} x(t) &= \mathfrak{A}^t x_0 + \mathfrak{B}_0^t u, \\ y &= \mathfrak{C} x_0 + \mathfrak{D}_0 u. \end{aligned} \tag{2}$$

Here, $\mathfrak{A}^t$ is the semigroup generated by A (which maps the initial state x_0 into the final state $x(t)$), $\mathfrak{B}_0^t$ is the map from the input u (restricted to the interval $[0,t]$) to the final state $x(t)$, $\mathfrak{C}$ is the map from the initial state x_0 to the output y, and $\mathfrak{D}_0$ is the input-output map from u (restricted to $[0,\infty)$) to y. If the operators B, $\overline{C}$, and D in (1) are bounded, then we get formulas for the corresponding operators $\mathfrak{B}_0^t$, $\mathfrak{C}$, and $\mathfrak{D}_0$ in (2) by using the standard 'variation of parameters formula' (recall that $\mathfrak{A}^t$ is the semigroup generated by A):

$$\begin{aligned} \mathfrak{B}_0^t u &= \int_0^t \mathfrak{A}^{t-v} Bu(v)\,dv, && t \ge 0, \\ (\mathfrak{C}x_0)(t) &= \overline{C}\mathfrak{A}^t x_0, && t \ge 0, \\ (\mathfrak{D}_0 u)(t) &= \overline{C}\int_0^t \mathfrak{A}^{t-v} Bu(v)\,dv + Du(t), && t \ge 0. \end{aligned} \tag{3}$$

As we shall see later, these formulas remain valid also for certain classes of unbounded operators B and $\overline{C}$.

For the moment, let us ignore (1) and instead focus on the *well-posedness* of the system (2). The standard well-posedness assumption is that (2) behaves well in an L^p-setting, where $1 \le p < \infty$, i.e., $x(t) \in X$ and $y \in L^p_{\mathrm{loc}}(\mathbf{R}^+;Y)$ depend continuously on $x_0 \in X$ and on $u \in L^p_{\mathrm{loc}}(\mathbf{R}^+;U)$. If this is the case, we call the operators $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ a *L^p-well-posed linear system*, where

$$\mathfrak{B}u = \lim_{t\to\infty} \mathfrak{B}_0^t \tau^{-t} u, \qquad \mathfrak{D}u = \lim_{t\to\infty} \tau^t \mathfrak{D}_0 \tau^{-t} u,$$

each defined for those $u \in L^p_{\mathrm{loc}}(\mathbf{R};U)$ for which the respective limit exists; here $(\tau^t u)(s) = u(s+t)$, $-\infty < s,\ t < \infty$, is the bilateral left shift by t. In the case where (2) is induced by the system (1) with bounded B, $\overline{C}$, and D, we have

$$\begin{aligned} \mathfrak{B}u &= \int_{-\infty}^0 \mathfrak{A}^{-v} Bu(v)\,dv, && \\ (\mathfrak{C}x_0)(t) &= \overline{C}\mathfrak{A}^t x_0, && t \ge 0, \\ \mathfrak{D}u &= \overline{C}\int_{-\infty}^t \mathfrak{A}^{t-v} Bu(v)\,dv + Du(t), && t \in \mathbf{R}, \end{aligned} \tag{4}$$

at least for those u whose support is bounded to the left.

As we shall see in Section 3, it is possible to define a well-posed linear system $\Sigma = \left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ without any reference to the system of equations (1).

The classical *Lax–Phillips model* was developed by Lax and Phillips in [14] (conservative systems) and [15] (dissipative systems) to provide a mathematical description of a scattering process where an incoming wave hits an obstacle and is scattered into an outgoing wave. It was soon realized (see [1], [4], and [10]) that it is possible to extend the classical Lax–Phillips model into a more general model of a well-posed infinite-dimensional system by relaxing some of the original assumptions on the incoming and outgoing subspaces, and by replacing the standard dissipativity assumption by a well-posedness assumption. In this extended formulation the Lax–Phillips model is a semigroup with a particular structure: it acts as an exponentially weighted incoming shift on the incoming subspace, as an exponentially weighted outgoing shift on the outgoing subspace, and the central part of the semigroup describes 'a generalized scattering process'. As we shall see in Section 4, this central part can be taken to be an arbitrary well-posed linear system. Thus there is a one-to-one correspondence between the class of all well-posed linear systems and all extended Lax–Phillips models.

We begin by presenting the most basic results about a L^p-well-posed linear system (Section 3) and the corresponding Lax–Phillips model (Section 4). We proceed in Section 5 to show that there is a close connection between the Lax–Phillips generator and the generators of the corresponding well-posed linear system. All the essential information about these two systems is contained in the *system operator* $S_\Sigma = \left[\begin{smallmatrix} A & B \\ C\&D \end{smallmatrix}\right]$ where A is the generator of the (central) semigroup, B is the control operator, and $C\&D$ is the combined observation/feedthrough operator. In the important case where X and U are Hilbert spaces this system operator can be written in the more familiar form $S_\Sigma = \left[\begin{smallmatrix} A & B \\ \overline{C} & D \end{smallmatrix}\right]$, where $\overline{C}$ is a (not necessarily unique) observation operator and D is the corresponding (generalized) feedthrough operator. We system operator is closed and densely defined from $X \times U$ to $X \times Y$. In the reflexive case the adjoint of S_Σ is the system operator of the dual system. We give formulas for the Lax–Phillips generator and resolvent in terms of the system operator. Finally, in the last section we apply the Hille–Yoshida theorem to the Lax–Phillips semigroup and get necessary and sufficient conditions for the admissibility or joint admissibility of a control operator B and an observation operator C. This leads to a criterion for an H^∞-function to be an L^p-multiplier.

3. Well-posed linear systems

As already outlined in Section 2, it is possible to define a well-posed linear system $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ without any reference to the system of equations (1). For this, we have to introduce some spaces and some simple operators. We denote $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}^+ = [0, \infty)$, $\mathbf{R}^- = (-\infty, 0)$,

$$(\pi_J u)(s) = \begin{cases} u(s), & s \in J, \\ 0, & s \notin J, \end{cases} \quad \text{for all } J \subset \mathbf{R},$$

$$(\pi_+ u)(s) = \pi_{\mathbf{R}^+} u = \begin{cases} u(s), & s \in \mathbf{R}^+, \\ 0, & s \in \mathbf{R}^-, \end{cases},$$

$$(\pi_- u)(s) = \pi_{\mathbf{R}^-} u = \begin{cases} 0, & s \in \mathbf{R}^+, \\ u(s), & s \in \mathbf{R}^-, \end{cases},$$

$$(\tau^t u)(s) = u(t+s), \quad s,\ t \in \mathbf{R}.$$

Thus π_+ is 'restriction to $\mathbf{R}^+$', π_- is the 'restriction to $\mathbf{R}^-$', and τ^t shift to the left for $t > 0$ and to the right for $t < 0$. Moreover, we define

$$\begin{aligned} \tau_+^t &= \pi_+\tau^t, \qquad t \geq 0, \\ \tau_-^t &= \tau^t\pi_-, \qquad t \geq 0, \end{aligned}$$

so that τ_+^t is the left shift by $t \geq 0$ on $\mathbf{R}^+$ and τ_-^t is the left shift by $t \geq 0$ on $\mathbf{R}^-$. Thus, τ_+ is an 'incoming left shift' and τ_- is an 'outgoing left shift'.

The space $L_c^p(\mathbf{R}^-;U)$ consists of all the functions $u \in L^p(\mathbf{R}^-;U)$ with a bounded support. The space $L_{c,\mathrm{loc}}^p(\mathbf{R};U)$ consists of all the functions $u : \mathbf{R} \to U$ that are locally in L^p and whose support is bounded to the left. We interpret $L_c^p(\mathbf{R}^-;U)$ as the subspace of functions in $L_{c,\mathrm{loc}}^p(\mathbf{R};U)$ which vanish on $\mathbf{R}^+$, and $L_{\mathrm{loc}}^p(\mathbf{R}^+;U)$ as the subspace of functions in $L_{c,\mathrm{loc}}^p(\mathbf{R};U)$ which vanish on $\mathbf{R}^-$. A sequence of functions u_n converges in $L_{c,\mathrm{loc}}^p(\mathbf{R};U)$ to a function u if the common support of all the functions u_n is bounded to the left and u_n converges to u in the L^p sense on every bounded time interval. The continuity of $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ in the following definition is with respect to this convergence.

Definition 3.1. *Let U, X, and Y be Banach spaces, and let $1 \leq p < \infty$. An L^p-*well-posed linear system Σ *on (Y,X,U) is a quadruple $\Sigma = \left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ of continuous linear operators satisfying the following conditions:*

- (i) *$t \mapsto \mathfrak{A}^t$ is a strongly continuous semigroup of operators on X;*
- (ii) *$\mathfrak{B}\colon L_c^p(\mathbf{R}^-;U) \to X$ satisfies $\mathfrak{A}^t\mathfrak{B}u = \mathfrak{B}\tau_-^t u$, for all $u \in L_c^p(\mathbf{R}^-;U)$ and all $t \in \mathbf{R}^+$;*
- (iii) *$\mathfrak{C}\colon X \to L_{\mathrm{loc}}^p(\mathbf{R};Y)$ satisfies $\mathfrak{C}\mathfrak{A}^t x = \tau_+^t\mathfrak{C}x$, for all $x \in X$ and all $t \in \mathbf{R}^+$;*
- (iv) *$\mathfrak{D}\colon L_{c,\mathrm{loc}}^p(\mathbf{R};U) \to L_{c,\mathrm{loc}}^p(\mathbf{R};Y)$ satisfies $\tau^t\mathfrak{D}u = \mathfrak{D}\tau^t u$, $\pi_-\mathfrak{D}\pi_+u = 0$, and $\pi_+\mathfrak{D}\pi_-u = \mathfrak{C}\mathfrak{B}u$, for all $u \in L_{c,\mathrm{loc}}^p(\mathbf{R};U)$ and all $t \in \mathbf{R}$.*

The different components of Σ are called as follows: U is the input space, X is the state space, Y is the output space, $\mathfrak{A}$ is the semigroup, $\mathfrak{B}$ is the input (or reachability, or controllability) map, $\mathfrak{C}$ is the output (or observability) map, and $\mathfrak{D}$ is the input-output map. The state $x(t) \in X$ at time $t \in \mathbf{R}^+$ and the output $y \in L_{\mathrm{loc}}^p(\mathbf{R}^+;Y)$ of Σ with initial time zero, initial state $x_0 \in X$ and input function $u \in L_{\mathrm{loc}}^p(\mathbf{R}^+;U)$ are given by (2) with $\mathfrak{B}_0^t = \mathfrak{B}\tau^t\pi_{[0,t]}u$ and $\mathfrak{D}_0 u = \mathfrak{D}\pi_+u$.

It is easy to see that the operators defined in (4) (with bounded B, C, and D) satisfy these conditions. Moreover, we only have to integrate over a finite interval since the support of u is bounded to the left. (There is also a similar theory for the case $p = \infty$; see [25].)

Every well-posed linear system has as a finite exponential *growth bound*. By the growth bound $\omega_{\mathfrak{A}}$ of a system $\Sigma = \left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ we understand the growth bound of its semigroup $\mathfrak{A}$:

$$\omega_{\mathfrak{A}} = \lim_{t\to\infty} \frac{\log(\|\mathfrak{A}^t\|)}{t} = \inf_{t>0} \frac{\log(\|\mathfrak{A}^t\|)}{t}.$$

As is well known, we always have $\omega_{\mathfrak{A}} < \infty$, but possibly $\omega_{\mathfrak{A}} = -\infty$. To explain in which sense the other operators $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ are exponentially bounded we introduce exponentially weighted L^p-spaces of the following type: we let $L^p_\omega(\mathbf{R}^+;U)$ represent the space of functions $u\colon \mathbf{R}^+ \to U$ for which $t \mapsto \mathrm{e}^{-\omega t}u(t)$ belongs to $L^p(\mathbf{R}^+;U)$.

Theorem 3.2. *Let $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ be a L^p-well-posed linear system, $1 \le p < \infty$, on (Y, X, U) with growth bound $\omega_{\mathfrak{A}}$, and let $\omega > \omega_{\mathfrak{A}}$. Then $\mathfrak{B}$ has a unique extension to an bounded linear operator from $L^p_\omega(\mathbf{R}^-;U)$ to X, $\mathfrak{C}$ is a bounded linear operator from X to $L^p_\omega(\mathbf{R}^+;Y)$, and $\mathfrak{D}$ has a unique extension to a bounded linear operator from $L^p_\omega(\mathbf{R};U)$ to $L^p_\omega(\mathbf{R};Y)$.*

Every well-posed linear system also has a *transfer function*:

Theorem 3.3. *Let $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ be a L^p-well-posed linear system, $1 \le p < \infty$, on (Y, X, U) with growth bound $\omega_{\mathfrak{A}}$. Then there is a unique analytic $\mathcal{L}(U;Y)$-valued* transfer function $\widehat{\mathfrak{D}}$ *defined (at least) on $\Re z > \omega_{\mathfrak{A}}$ determined by the fact that the Laplace transform $\widehat{\mathfrak{D}_0 u}$ of the input-output term $\mathfrak{D}_0 u$ in (2) is given by, for all $u \in L^p_{\omega_{\mathfrak{A}}}(\mathbf{R}^+;U)$,*

$$\widehat{\mathfrak{D}_0 u} = \widehat{\mathfrak{D}}(z)\hat{u}(z), \quad \Re z > \omega_{\mathfrak{A}},$$

where $\hat{u}$ is the Laplace transform of u.

Thus, $\widehat{\mathfrak{D}}$ can be interpreted as an '$L^p_\omega(U;Y)$-multiplier' for every $\omega > \omega_{\mathfrak{A}}$.

The following theorem gives us a first connection between an arbitrary L^p-well-posed linear system and a system of equations of the type (1):

Theorem 3.4. *Every L^p-well-posed linear system $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ with $1 \le p < \infty$ has a unique closed (unbounded) densely defined* system operator

$$S_\Sigma\colon X \times U \supset \mathcal{D}(S_\Sigma) \to X \times Y$$

with the following properties. If $x_0 \in X$, $u \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^+;U)$ and $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma)$, then the state $x(t)$ and the output $y(t)$ of Σ with initial state x_0, and input u satisfies $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma)$ for all $t \ge 0$, and

$$\begin{aligned} \begin{bmatrix} x'(t) \\ y(t) \end{bmatrix} &= S_\Sigma \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t \ge 0, \\ x(0) &= x_0. \end{aligned} \tag{5}$$

The proof of this theorem is given in [26] (and also in [25]).

Note that (5) reduces to (1) for *smooth input functions* and *compatible initial conditions* provided S_Σ can be written in the form $S_\Sigma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Is this always possible?

Before giving a (partial) answer to this question we need to introduce two auxiliary spaces X_1 and X_{-1}. Choose any γ in the resolvent set of the generator A of $\mathfrak{A}$. We let $X_1 = \mathcal{D}(A)$, with the norm $\|x\|_{X_1} = \|(\gamma I - A)x\|_X$, and X_{-1} is the completion of X with the norm $\|x\|_{X_{-1}} = \|(\gamma I - A)^{-1}x\|_X$. We have $X_1 \subset X \subset X_{-1}$ with continuous and dense imbeddings. The semigroup $\mathfrak{A}$ can be restricted to to a strongly continuous semigroup on X_1 and extended to a strongly continuous semigroup on X_{-1} (which still we denote by the same symbol). We denote the space of bounded linear operators from U to Y by $\mathcal{L}(U;Y)$.

Theorem 3.5. *Every L^p-well-posed linear system $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ with $1 \le p < \infty$ has a unique* control operator $B \in \mathcal{L}(U;X_{-1})$ *and a unique combined* observation/feedthrough operator $C\&D\colon \mathcal{D}(S_\Sigma) \to X \times Y$, *such that S_Σ can be written in the form*

$$S_\Sigma \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} A & B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \qquad \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(S_\Sigma).$$

Thus, the state $x(t)$ and the output $y(t)$ of Σ in Theorem 3.4 satisfy

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= C\&D \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t \ge 0, \\ x(0) &= x_0, \end{aligned}$$

where the equation for x' is valid in X_{-1}. Moreover, $\mathcal{D}(S_\Sigma)$ is given by

$$\mathcal{D}(S_\Sigma) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X \times U \;\middle|\; Ax + Bu \in X \right\}.$$

In particular, if $x \in X_1$, then $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma)$, and we can define the observation operator $C \in \mathcal{L}(X_1;Y)$ *by*

$$Cx = C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad x \in X_1.$$

This theorem is actually older than Theorem 3.4; see [4], [5], [17, 18], and [27, 28] (or [25]) for the proof. In [4] the combined observation/feedthrough operator is denoted by N. The control operator B is said to be *bounded* if the range of B lies in X, in which case $B \in \mathcal{L}(U;X)$. The observation operator C is said to be *bounded* if it can be extended to an operator in $\mathcal{L}(X;Y)$.

There is an simple connection between the transfer function introduced in Theorem 3.3 and the operators introduced in Theorem 3.5.

Theorem 3.6. *With the notation of Theorems 3.3 and 3.5, the transfer function $\widehat{\mathfrak{D}}$ of Σ is given by*

$$\widehat{\mathfrak{D}}(z) = (C\&D) \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}, \quad \Re z > \omega_{\mathfrak{A}}.$$

Conversely, for all $z \in \mathbf{C}$ with $\Re z > \omega_{\mathfrak{A}}$ and for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma)$ we have

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C[x + (zI - A)^{-1}Bu] + \widehat{\mathfrak{D}}(z)u.$$

For more details, explanations and examples we refer the reader to [1], [2, 3], [4], [5], [17, 18], [19, 20, 22, 21, 23, 25], [27, 28, 29, 30, 32, 33], [36] (and the references therein). Most of the available literature deals with Hilbert spaces and $p = 2$.

Let us now return to the question of the possibility to split S_Σ even further into $S_\Sigma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. For the purpose of the following discussion, let us temporarily split S_Σ into $S_\Sigma = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, where, with the notation of Theorem 3.4, $A\&B \colon \mathcal{D}(S_\Sigma) \to X$ maps $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ into x' and (as in Theorem 3.5) $C\&D \colon \mathcal{D}(S_\Sigma) \to Y$ maps $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ into y. According to Theorem 3.5, it is always possible to extend the domain of $A\&B$ to all of $X \times U$ by allowing the values of $A\&B$ to belong to the larger space X_{-1}. This extension is unique since $\mathcal{D}(A\&B) = \mathcal{D}(S_\Sigma)$ is dense in $X \times U$. If we denote the extended operator by $\overline{A\&B}$, then

$$\overline{A\&B} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = Ax + Bu, \qquad x \in X, \qquad u \in U,$$

where

$$Ax = \overline{A\&B} \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad Bu = \overline{A\&B} \begin{bmatrix} 0 \\ u \end{bmatrix};$$

here A represents the extension of the original generator A to an operator $X \to X_{-1}$.

In order to get a similar splitting of $C\&D$ into $C\&D = \begin{bmatrix} \overline{C} & D \end{bmatrix}$ we need to extend $C\&D$ in a similar fashion. This extension is more difficult since we cannot, in general, replace the original range space Y of $C\&D$ be a larger space Y_{-1}. For example, if Y is finite-dimensional, then there is no natural candidate for the space Y_{-1}. The smallest possible domain of the extended operator $\overline{C\&D}$ is $Z \times U$, where Z is defined as follows. We choose any γ in the resolvent set of A, and let

$$\begin{aligned} Z = \big\{ z \in X \bigm| z = (\gamma I - A)^{-1}(x + Bu) \\ \text{for some } x \in X \text{ and } u \in U \big\}. \end{aligned} \tag{6}$$

This is a Banach space with the norm

$$|z|_Z = \inf_{(\gamma I - A)^{-1}(x+Bu) = z} \left(|x|_X^2 + |u|_U^2 \right)^{1/2},$$

satisfying $X_1 \subset Z \subset X$, and it is a Hilbert space if both X and U are Hilbert spaces. It is easy to see that $\mathcal{D}(S_\Sigma) \subset Z \times U$, but the embedding $\mathcal{D}(S_\Sigma) \subset Z \times U$ need not be dense.

Definition 3.7. *The well-posed linear system* $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *is* compatible *if its combined observation/feedthrough operator* $C\&D$ *can be extended to an operator* $\overline{C\&D} \in \mathcal{L}(Z \times U; Y)$. *We define the corresponding extended observation operator*

$\overline{C} \in \mathcal{L}(Z;Y)$ *and feedthrough operator* $D \in \mathcal{L}(U;Y)$ *by*

$$\begin{aligned} \overline{C} &= \overline{C\&D} \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad x \in Z, \\ Du &= \overline{C\&D} \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad u \in U. \end{aligned} \tag{7}$$

The extension of $C\&D$ to $Z \times U$ need not be unique since $\mathcal{D}(S_\Sigma)$ not be dense in $Z \times U$. This means that $\overline{C}$ and D need not be unique either. However, there is a one-to-one correspondence between $\overline{C\&D}$, $\overline{C}$ and D, i.e., any one of these three operators determine the other two uniquely.

In spite of the possible non-uniqueness of the extended observation operator $\overline{C}$ and the corresponding feedthrough operator D, independently of how these operators are chosen, it is still true that the formula for the output y in Theorem 3.5 simplifies into

$$y(t) = \overline{C}x(t) + Du(t), \quad t \geq 0,$$

and the formula for the transfer function given in Theorem 3.3 simplifies into

$$\widehat{\mathfrak{D}}(z) = \overline{C}(zI - A)^{-1}B + D. \qquad \Re z > \omega_{\mathfrak{A}}.$$

In particular, the formula (3) holds whenever $u \in W^{1,p}([0,t];U)$ and $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma)$.

It has for some time been considered an open question among specialists whether every L^p-well-posed linear system is compatible. Recently it was discovered that the answer to this question is positive, at least in the Hilbert space case.

Theorem 3.8. *Let* Σ *be a* L^p*-well-posed linear system on* (Y,X,U). *Then* Σ *is compatible in (at least) the following cases:*

(i) X *and* U *are Hilbert spaces;*
(ii) *At least one of the spaces* X, U, *or* Y *is finite-dimensional.*

The more difficult part (i) of this theorem was proved in [26], and (the easy) part (ii) in [25].

4. The Lax–Phillips Scattering Model

Instead of using a L^p-well-posed linear system to formalize the idea of having an output and state at time $t > 0$ which depend continuously on an input and the initial state we can proceed in a different way which leads to a generalized *Lax–Phillips scattering model.* This is a semigroup $\mathbb{T}$ defined on $\mathcal{Y} \times X \times \mathcal{U} = L^p_\omega(\mathbf{R}^-;Y) \times X \times L^p_\omega(\mathbf{R}^+;U)$ with certain additional properties. (Here $L^p_\omega(\mathbf{R}^-;Y)$ consists of all the functions $y\colon \mathbf{R}^- \to Y$ for which $t \mapsto \mathrm{e}^{-\omega t}y(t)$ belongs to $L^p(\mathbf{R}^-;Y)$ and similarly for $L^p_\omega(\mathbf{R}^+;U)$.) We call $\mathcal{U}$ the *incoming subspace*, X the *central state space*, and $\mathcal{Y}$ the *outgoing subspace.* In the classical cases treated in [14, 15] ω is taken to be zero and $\mathbb{T}$ is required to be unitary (the conservative case) or a contraction semigroup (the nonconservative case).

We claim that there is a one-to-one correspondence between the class of all well-posed linear systems and the class of all Lax–Phillips models. The parameter $\omega \in \mathbf{R}$ can be chosen in an arbitrary way (the best choice depends on the particular application).

Let $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ be a given L^p-well-posed linear system. To each such system we construct a Lax–Phillips model $\mathbb{T}$ on $\mathcal{Y} \times X \times \mathcal{U}$ as follows. The initial data consists of the initial incoming state $u_0 \in \mathcal{U}$ representing the future values of the input, the initial central state $x_0 \in X$ is identical to the initial state of Σ, and the initial outgoing state $y_0 \in \mathcal{Y}$ represents the past values of the output. At time $t \geq 0$ the incoming state u_t is the left-shifted input $\tau_+^t u_0$ (the unused part of the input). The central state x_t at time t is equal to the state $x(t) = \mathfrak{A}^t x_0 + \mathfrak{B}_0^t u$ of Σ at time t with initial time zero, initial state x_0, and input u_0 (it depends only on x_0 and on the discarded part $\pi_{[0,t]}u$ of u). The outgoing state y_t at time t consists of two parts: it is the sum of $\tau_-^t y_0$ (the left-shifted original outgoing state) and $\tau^t \pi_{[0,t]}(\mathfrak{C}x_0 + \mathfrak{D}_0 u_0)$ (the restriction of the output $\mathfrak{C}x_0 + \mathfrak{D}_0 u_0$ of Σ to the interval $[0,t]$ shifted to the left by τ^t so that the support of the shifted and truncated output is $(-t,0)$). Formalizing this idea we get the following theorem, where we use the notations

$$\mathfrak{B}_0^t = \mathfrak{B}\tau^t \pi_{[0,t]}, \qquad \mathfrak{C}_0^t = \pi_{[0,t]}\mathfrak{C}, \qquad \mathfrak{D}_0^t = \pi_{[0,t]}\mathfrak{D}\pi_{[0,t]}.$$

Theorem 4.1. *Let $\omega \in \mathbf{R}$, $\mathcal{Y} = L^2_\omega(\mathbf{R}^-;Y)$ and $\mathcal{U} = L^2_\omega(\mathbf{R}^+;U)$. For all $t \geq 0$ we define on $\mathcal{Y} \times X \times \mathcal{U}$ the operator $\mathbb{T}^t$ by*

$$\mathbb{T}^t = \begin{bmatrix} \tau^t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tau_+^t \end{bmatrix} \begin{bmatrix} I & \mathfrak{C}_0^t & \mathfrak{D}_0^t \\ 0 & \mathfrak{A}^t & \mathfrak{B}_0^t \\ 0 & 0 & I \end{bmatrix}.$$

Then $\mathbb{T}$ is a strongly continuous semigroup. If x and y are the state trajectory and the output function of Σ corresponding to the initial state $x_0 \in X$ and the input function $u_0 \in \mathcal{U}$, and if we define $y(t) = y_0(t)$ for $t < 0$, then for all $t \geq 0$,

$$\begin{bmatrix} \pi_{(-\infty,t]}y \\ x(t) \\ \pi_{[t,\infty)}u_0 \end{bmatrix} = \begin{bmatrix} \tau^{-t} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tau^{-t} \end{bmatrix} \mathbb{T}^t \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix}. \tag{8}$$

Formula (8) shows that at any time $t \geq 0$, the *first component* of $\mathbb{T}_t \left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right]$ represents the *past output*, the *second component* represents the *present state* and the *third component* represents the *future input.*

Here the strong continuity of $\mathbb{T}$ is obvious, and so is the property $\mathbb{T}(0) = I$. The proof of the semigroup property $\mathbb{T}(s+t) = \mathbb{T}(s)\mathbb{T}(t)$ for $s,\ t \geq 0$ is a short algebraic computation based on Definition 3.1 (see [26] or [25] for details).

The semigroup $\mathbb{T}$ in Theorem 4.1 has an additional 'causality' property, which in the Hilbert space case where $p = 2$ and U, X, and Y are Hilbert spaces can be described as follows: for all $t \geq 0$, the images of the central and incoming states under $\mathbb{T}^t$ are orthogonal to the image of the outgoing state, and the null space of $\mathbb{T}^t$ projected onto the central and outgoing spaces is orthogonal to the null space of $\mathbb{T}^t$ projected onto the incoming space. In the general case these properties can easiest be characterized in the following way.

Definition 4.2. *A* Lax–Phillips model *of type L^p_ω is a semigroup on $\mathcal{Y} \times X \times \mathcal{U} = L^p_\omega(\mathbf{R}^-;Y) \times X \times L^p_\omega(\mathbf{R}^+;U)$ with the structure*

$$\mathbb{T}^t = \begin{bmatrix} \tau^t_- & \mathbb{C}^t & \mathbb{D}^t \\ 0 & \mathbb{A}^t & \mathbb{B}^t \\ 0 & 0 & \tau^t_+ \end{bmatrix}, \tag{9}$$

where $\mathbb{A}$ is strongly continuous and $\mathbb{B}^t$, $\mathbb{C}^t$, and $\mathbb{D}^t$ satisfy the causality conditions

$$\begin{aligned} \mathbb{C}^t &= \pi_{(-t,0)}\mathbb{C}^t, & \mathbb{D}^t &= \pi_{(-t,0)}\mathbb{D}^t, \\ \mathbb{D}^t &= \mathbb{D}^t\pi_{[0,t]}, & \mathbb{B}^t &= \mathbb{B}^t\pi_{[0,t]}. \end{aligned} \tag{10}$$

This set of conditions is a rewritten version of conditions (1.2) in [15]. Helton [10] uses the name *inertness* for this additional causality property.

Corollary 4.3. *The semigroup $\mathbb{T}$ constructed in Theorem 4.1 is a Lax–Phillips model of type L^p_ω.*

This is immediate from Theorem 4.1 and Definition 4.2. We call the semigroup $\mathbb{T}$ in Theorem 4.1 the *Lax–Phillips model (of type L^p_ω) induced by Σ.*

It is only slightly more difficult to prove a converse to Corollary 4.3: To every Lax–Phillips model there corresponds a well-posed linear system which induces this Lax–Phillips model:

Theorem 4.4. *Let $\mathbb{T}$ be a Lax–Phillips model of type L^p_ω. With the notations of Definition 4.2, let*

$$\begin{aligned} \mathfrak{A} &= \mathbb{A}, & \mathfrak{B} &= \lim_{s\to\infty} \mathbb{B}^s\tau^{-s}, \\ \mathfrak{C} &= \lim_{t\to\infty} \tau^{-t}\mathbb{C}^t, & \mathfrak{D} &= \lim_{\substack{t\to\infty \\ s\to\infty}} \tau^{-s}\mathbb{D}^{s+t}\tau^{-t}. \end{aligned} \tag{11}$$

Then $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ is an L^p-well-posed linear system on (Y,X,U), and $\mathbb{T}$ is the Lax–Phillips model induced by this system.

The proof of Theorem 4.4 is another algebraic computation given in [25].

Corollary 4.5. *For each $\omega \in \mathbf{R}$ and $1 \le p < \infty$, there is a one-to-one correspondence between the class of all L^p-well-posed linear systems and all Lax–Phillips models of type L^p_ω: every L^p-well-posed linear system Σ induces a unique Lax–Phillips model $\mathbb{T}$ of type L^p_ω, and conversely, every Lax–Phillips model $\mathbb{T}$ of type L^p_ω induces a unique L^p-well-posed linear system Σ.*

This is a union of Corollary 4.3 and Theorem 4.4. Parts of this corollary (where either the input operator or output operator vanishes) were proved by

Grabowski and Callier [7] and by Engel [6]. It is also (implicitly) contained in [4] and mentioned in [3].

Our next theorem describes the generator of the Lax–Phillips semigroup:

Theorem 4.6. *Let* $1 \le p < \infty$ *and* $\omega \in \mathbf{R}$, *let* $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *be a* L^p*-well-posed linear system on* (Y, X, U) *with system operator* $S_\Sigma = \left[\begin{smallmatrix} A & B \\ C\&D \end{smallmatrix}\right]$, *and let* T *be the generator of the corresponding Lax–Phillips model* $\mathbb{T}$ *of type* L^p_ω *defined in Definition 4.2.*

(i) *The domain of* T *consists of all the vectors* $\left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right] \in W^{1,p}_\omega(\mathbf{R}^-; Y) \times X \times W^{1,p}_\omega(\mathbf{R}^+; U)$ *which satisfy* $\left[\begin{smallmatrix} x_0 \\ u_0(0) \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma)$ *and* $y_0(0) = C\&D \left[\begin{smallmatrix} x_0 \\ u_0(0) \end{smallmatrix}\right]$, *and on its domain* T *is given by*

$$T \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} y_0' \\ Ax_0 + Bu_0(0) \\ u_0' \end{bmatrix}.$$

Thus, the following three conditions are equivalent (here $\hat{u}$ *and* $\hat{u}_0$ *are the Laplace transforms of* u *and* u_0, *and* $\hat{y}$ *and* $\hat{y}_0$ *are the left-sided Laplace transforms of* y *and* y_0*):*

(a) $\left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(T)$ *and* $\left[\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right] = T \left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right]$;

(b) $y_0 \in W^{1,p}_\omega(\mathbf{R}^-; Y)$, $x_0 \in X$, $u_0 \in W^{1,p}_\omega(\mathbf{R}^+; U)$, $\left[\begin{smallmatrix} x_0 \\ u_0(0) \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma)$ *and*

$$\begin{bmatrix} x \\ y_0(0) \end{bmatrix} = S_\Sigma \begin{bmatrix} x_0 \\ u_0(0) \end{bmatrix}, \qquad \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} y_0' \\ u_0' \end{bmatrix}.$$

(c) $y_0 \in W^{1,p}_\omega(\mathbf{R}^-; Y)$, $x_0 \in X$, $u_0 \in W^{1,p}_\omega(\mathbf{R}^+; U)$, $\left[\begin{smallmatrix} x_0 \\ u_0(0) \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma)$ *and*

$$\begin{aligned} \begin{bmatrix} x \\ y_0(0) \end{bmatrix} &= S_\Sigma \begin{bmatrix} x_0 \\ u_0(0) \end{bmatrix}, \\ \hat{y}(z) &= z\hat{y}_0(z) - y_0(0), \qquad \Re z < \omega, \\ \hat{u}(z) &= z\hat{u}_0(z) + u_0(0), \qquad \Re z > \omega. \end{aligned}$$

(ii) *The spectrum of* T *contains the vertical line* $\{\Re\alpha = \omega\}$. *A point* α *with* $\Re\alpha > \omega$ *belongs to the spectrum of* T *iff it belongs to the spectrum of* A, *and a point* α *whith* $\Re\alpha < \omega$ *belongs to the spectrum of* T *iff* $\left[\begin{smallmatrix} \alpha I - A & -B \\ -C\&D \end{smallmatrix}\right]$ *is not invertible.*

(iii) *Let* $\alpha \in \rho(T)$ *with* $\Re\alpha > \omega$ *and let* $\left[\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right] \in L^2_\omega(\mathbf{R}^-; Y) \times X \times L^2_\omega(\mathbf{R}^+; U)$. *Denote* $\widehat{\mathfrak{D}}(\alpha) = C\&D \left[\begin{smallmatrix} (\alpha I - A)^{-1} B \\ I \end{smallmatrix}\right]$. *Then the following three conditions are equivalent:*

(a) $\left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right] = (\alpha I - T)^{-1} \left[\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right]$;

$$
\text{(b)} \quad \begin{cases} \begin{bmatrix} x_0 \\ y_0(0) \end{bmatrix} = \begin{bmatrix} (\alpha I - A)^{-1} & (\alpha I - A)^{-1}B \\ C(\alpha I - A)^{-1} & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \begin{bmatrix} x \\ \hat{u}(\alpha) \end{bmatrix}, \\ y_0(t) = e^{\alpha t} y_0(0) + \int_t^0 e^{\alpha(t-s)} y(s)\, ds, \qquad t \le 0, \\ u_0(t) = \int_t^\infty e^{\alpha(t-s)} u(s)\, ds, \qquad t \ge 0. \end{cases}
$$

$$
\text{(c)} \quad \begin{cases} \begin{bmatrix} x_0 \\ y_0(0) \end{bmatrix} = \begin{bmatrix} (\alpha I - A)^{-1} & (\alpha I - A)^{-1}B \\ C(\alpha I - A)^{-1} & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \begin{bmatrix} x \\ \hat{u}(\alpha) \end{bmatrix}, \\ \hat{y}_0(z) = \dfrac{\hat{y}(z) + y_0(0)}{\alpha - z}, \qquad \Re z < \omega, \\ \hat{u}_0(z) = \dfrac{\hat{u}(z) - \hat{u}(\alpha)}{\alpha - z}, \qquad \Re z > \omega. \end{cases}
$$

(iv) *Let* $\alpha \in \rho(T)$ *with* $\Re\alpha < \omega$ *and let* $\left[\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right] \in L^2_\omega(\mathbf{R}^-;Y) \times X \times L^2_\omega(\mathbf{R}^+;U)$. *Then the following three conditions are equivalent:*

(a) $\left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right] = (\alpha I - T)^{-1} \left[\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right]$;

$$
\text{(b)} \quad \begin{cases} \begin{bmatrix} x_0 \\ u_0(0) \end{bmatrix} = \begin{bmatrix} \alpha I - A & -B \\ & -C\&D \end{bmatrix}^{-1} \begin{bmatrix} x \\ \hat{y}(\alpha) \end{bmatrix}, \\ y_0(t) = -\int_{-\infty}^t e^{\alpha(t-s)} y(s)\, ds, \qquad t \le 0, \\ u_0(t) = e^{\alpha t} u_0(0) - \int_0^t e^{\alpha(t-s)} u(s)\, ds, \qquad t \ge 0. \end{cases}
$$

$$
\text{(c)} \quad \begin{cases} \begin{bmatrix} x_0 \\ u_0(0) \end{bmatrix} = \begin{bmatrix} \alpha I - A & -B \\ & -C\&D \end{bmatrix}^{-1} \begin{bmatrix} x \\ \hat{y}(\alpha) \end{bmatrix}, \\ \hat{y}_0(z) = \dfrac{\hat{y}(z) - \hat{y}(\alpha)}{\alpha - z}, \qquad \Re z < \omega, \\ \hat{u}_0(z) = \dfrac{\hat{u}(z) + u_0(0)}{\alpha - z}, \qquad \Re z > \omega. \end{cases}
$$

The proof of this theorem is given in [26] (and also in [25]).

There are a number of important ingredients in the Lax–Phillips scattering theory, such as the backward and forward wave operators, the scattering operator, and the scattering matrix. All of these have natural analogies in the theory of well-posed linear systems. In the discussion below we choose $\omega > \omega_{\mathfrak{A}}$, where $\omega_{\mathfrak{A}}$ is the growth rate of $\mathfrak{A}$.

The *backward wave operator* W_- (denoted by W_2 in [15, Theorem 1.2]) is the limit of the last column of $\mathbb{T}\tau^{-t}$ as $t \to \infty$. It maps $L^p_\omega(\mathbf{R};U)$ into $L^p_\omega(\mathbf{R}^-;Y) \times X \times L^p_\omega(\mathbf{R}^+;U)$, and it is given by (cf. Theorem 4.4)

$$
W_- u = \begin{bmatrix} \pi_- \mathfrak{D} \\ \mathfrak{B} \\ \pi_+ \end{bmatrix} u. \tag{12}
$$

Thus, it keeps the future input $\pi_+ u$ intact, and maps the past input $\pi_- u$ into the past output $\pi_- \mathfrak{D} u$ and the present central state $\mathfrak{B} u$.

The *forward wave operator* W_+ (denoted by W_1 in [15, Theorem 1.2]) is the limit of the first row of $\tau^{-t}\mathbb{T}$ as $t \to \infty$. It maps $L^p_\omega(\mathbf{R}^-;Y) \times X \times L^p_\omega(\mathbf{R}^+;U)$ into $L^p_\omega(\mathbf{R};Y)$, and it is given by (cf. Theorem 4.4)

$$W_+ \begin{bmatrix} y \\ x_0 \\ u \end{bmatrix} = \begin{bmatrix} \pi_- & \mathfrak{C} & \mathfrak{D}\pi_+ \end{bmatrix} \begin{bmatrix} y \\ x_0 \\ u \end{bmatrix}. \tag{13}$$

Thus, it keeps the past output $\pi_- y$ intact, and maps the present central state x_0 and the future input $\pi_+ u$ into the future output $\mathfrak{C} x_0 + \mathfrak{D}\pi_+ u$.

The *scattering operator* in Lax–Phillips theory is the product $W_+ W_-$, and it is given by

$$W_+ W_- = \begin{bmatrix} \pi_- & \mathfrak{C} & \mathfrak{D}\pi_+ \end{bmatrix} \begin{bmatrix} \pi_- \mathfrak{D} \\ \mathfrak{B} \\ \pi_+ \end{bmatrix} = \pi_- \mathfrak{D} + \mathfrak{C}\mathfrak{B} + \pi_+ \mathfrak{D} = \mathfrak{D}. \tag{14}$$

Thus, the scattering operator is nothing but the (bilaterally shift-invariant) input-output map $\mathfrak{D}$ of the corresponding well-posed linear system.

To get the *scattering matrix* of the Lax–Phillips system we apply the scattering operator $\mathfrak{D}$ to an input of the form $u(t) = \mathrm{e}^{zt} u_0$, where $z \in \mathbf{C}$ has a sufficiently large real part and $u_0 \in U$ is fixed; see [15, pp. 187–188]. Because of the shift-invariance of $\mathfrak{D}$, the resulting output is of the type $y(t) = \mathrm{e}^{zt} y_0$ for some $y_0 \in Y$. The scattering matrix (evaluated at z) is defined to be the operator that maps $u_0 \in U$ into $y_0 \in Y$. It follows from [30, p. 194] that the scattering matrix of a Lax–Phillips system is equal to the transfer function $\widehat{\mathfrak{D}}$ of the corresponding well-posed linear system.

In their study of the *conservative* case, Lax and Phillips [14] assume some additional *controllability* and *observability* properties of the system:

(i) The *image of the incoming subspace* $\mathcal{U}$ under $\mathbb{T}^t$, $0 \le t < \infty$, is *dense* in the state space $\mathcal{Y} \times X \times \mathcal{U}$.
(ii) If the *projection* of a trajectory onto the *outgoing subspace* $\mathcal{Y}$ *vanishes*, then the trajectory is *identically zero*.

These additional controllability and observability assumptions imply the following additional conclusions (in the conservative case):

(i) Both $\mathfrak{A}^t$ and $\mathfrak{A}^{*t}$ *tend strongly to zero* as $t \to \infty$,
(ii) $\widehat{\mathfrak{D}}$ is *inner from both sides*, i.e., $\widehat{\mathfrak{D}}(i\omega)$ is unitary for almost all real ω,
(iii) Both the backward and the forward *wave operators are unitary*,
(iv) The system is *exactly controllable and exactly observable* in infinite time.

Without the additional controllability and observability assumptions none of the additional conclusions listed above need hold.

According to [4], *every contractive H^∞-function over $\mathbf{C}_0$* (= the right half-plane) is the *transfer function of some unitary system* (which need not be controllable or observable). It is also the transfer function of some *controllable and observable dissipative system*. For more details on conservative and dissipative systems, see [4], [14, 15], and [35]

5. Admissibility

According to Corollary 4.5, there is a one-to-one correspondence between the class of all L^p-well-posed linear systems and all Lax–Phillips scattering models of type L^p. This means that we can reduce the study of the generators of a well-posed linear system to the study of the generators the Lax–Phillips semigroup. This way we can obtain necessary and sufficient conditions for the admissibility or joint admissibility of a control operator B and an observation operator C. These notions are defined as follows.

As always we let U, X, and Y be Banach spaces and let $1 \le p < \infty$. We let $\mathfrak{A}$ be a strongly continuous semigroup on the Banach space X with generator A, and define the spaces X_1 and X_{-1} as in Section 2. This time we specify, in addition, some $\omega \in \mathbf{R}$, and suppose that $\mathfrak{A}$ is ω-bounded, i.e., $\sup_{t>0} \mathrm{e}^{-\omega t}\|\mathfrak{A}^t\| < \infty$.

An operator $B \in \mathcal{L}(U; X_{-1})$ is an *L^p-admissible control operator for A* if for some $t > 0$ (hence for all $t > 0$) the operator

$$\mathfrak{B}_0^t u = \int_0^t \mathfrak{A}^{t-s} B u(s)\, ds, \qquad u \in L^p([0,t];U), \tag{15}$$

maps $L^p([0,t];U)$ into X (instead of X_{-1}). (This operator is then bounded with values in X). We call B *ω-bounded* if the resulting input map

$$\mathfrak{B} u = \lim_{v \to -\infty} \int_v^0 \mathfrak{A}^{-s} B u(s)\, ds, \qquad u \in L^p_\omega(\mathbf{R}^-;U) \tag{16}$$

is ω-bounded, i.e., it defines a bounded linear operator from $L^p_\omega(\mathbf{R}^-;U)$ to X.

The operator $C \in \mathcal{L}(X_1;Y)$ is an *L^p-admissible observation operator for A* if the map

$$(\mathfrak{C}x)(t) = C\mathfrak{A}^t x, \qquad x \in X_1, \qquad t \ge 0, \tag{17}$$

can be extended to a bounded linear operator $X \to L^p_{\mathrm{loc}}(\mathbf{R}^+;Y)$, and it is ω-bounded if the resulting output map $\mathfrak{C}$ is ω-bounded, i.e., it maps X into $L^p_\omega(\mathbf{R}^+;Y)$.

The operators $B \in \mathcal{L}(U;X_{-1})$ and $C \in \mathcal{L}(X_1;Y)$ are *jointly L^p-admissible for A* if B is an L^p-admissible control operator for A, C is an L^p-admissible observation operator for A, and the operator $\mathfrak{D}\colon W^{1,p}_{c,\mathrm{loc}}(\mathbf{R};U) \to C_c(\mathbf{R};Y)$ defined by

$$(\mathfrak{D}u)(t) = C\big[\mathfrak{B}\tau^t u - (\alpha I - A)^{-1} B u(t)\big] + D_\alpha u(t), \qquad t \in \mathbf{R}, \tag{18}$$

can be extended to a continuous operator $L^p_{c,\mathrm{loc}}(\mathbf{R};U) \to L^p_{c,\mathrm{loc}}(\mathbf{R};Y)$. Here $\alpha \in \rho(A)$ and $D_\alpha \in \mathcal{L}(U;Y)$ can be chosen in an arbitrary way. By introducing the combined observation/feedthrough operator

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C\big[x - (\alpha I - A)^{-1}Bu\big] + D_\alpha u. \tag{19}$$

we can simplify the formula for $(\mathfrak{D}u)(t)$ into

$$(\mathfrak{D}u)(t) = C\&D \begin{bmatrix} \mathfrak{B}\tau^t u \\ u(t) \end{bmatrix}, \qquad t \in \mathbf{R}. \tag{20}$$

We call B and C *jointly ω-bounded* if both B and C are ω-bounded and, in addition, the operator $\mathfrak{D}$ can be extended to a bounded linear operator from $L^p_\omega(\mathbf{R};U)$ to $L^p_\omega(\mathbf{R};Y)$. If (and only if) B and C are jointly admissible, then the four operator $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ can be combined into a L^p-well-posed linear system $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ with system operator $\left[\begin{smallmatrix} A & B \\ C\&D \end{smallmatrix}\right]$. (Here $\mathfrak{D}$ is determined by A, B, and C only modulo a constant static term.)

Before looking at the general case of L^p-admissibility, let us treat the important special case where $p = 2$ and U, X, and Y are Hilbert spaces. In this case there is a very simple characterization of the class of all L^2-well-posed ω-bounded transfer function:

Proposition 5.1. *Let U and Y be Hilbert spaces. A $\mathcal{L}(U;Y)$-valued function $\widehat{\mathfrak{D}}$ defined on $\Re\lambda > \omega$ is the transfer function of an ω-bounded L^2-well-posed linear system if and only if it is analytic and bounded on $\Re\lambda > \omega$ (i.e., it belongs to H^∞).*

This was proved independently by (at least) Salamon [18] and Curtain and Weiss [5].

The admissibility of a control operator B or an observation operator C is much more delicate in this case. In 1990 George Weiss [31] made the following conjecture:

Conjecture 5.2. *Let U, X, and Y be Hilbert spaces, and let A generate a C_0 semigroup on X. Then*

(i) *$B \in \mathcal{L}(U;X_{-1})$ is an L^2-admissible ω-bounded control operator for A if and only if there is a constant $K > 0$ such that*

$$\|(\lambda I - A)^{-1}B\| \le \frac{K}{\sqrt{\Re\lambda - \omega}}, \qquad \Re\lambda > \omega.$$

(ii) *$C \in \mathcal{L}(X_1;Y)$ is an L^2-admissible ω-bounded observation operator for A if and only if there is a constant $K > 0$ such that*

$$\|C(\lambda I - A)^{-1}\| \le \frac{K}{\sqrt{\Re\lambda - \omega}}, \qquad \Re\lambda > \omega.$$

It is easy to see that the given conditions are necessary. These two conjectures are dual of each other, so it suffices to prove or disprove one of them.

It was discovered recently by Zwart and Jacob [37] that Weiss' conjecture is false in general. It is not true even if we restrict the dimensions of U and Y to be one (see [13]) or if we require the semigroup to be a contraction semigroup (see [12]). However, it is true in several special cases. For example, the second conjecture about the observation operator is known to be true in the following special cases (here we denote the semigroup generated by A by $t \mapsto \mathfrak{A}^t$ and take $\omega = 0$):

(i) Y is finite-dimensional and $\mathfrak{A}$ is normal [8], [9], [31], [34],
(ii) Y is finite-dimensional and $\mathfrak{A}$ is the right-shift on $L^2(\mathbf{R}^+)$ [16],
(iii) Y is finite-dimensional and $\mathfrak{A}$ is a contraction semigroup [11],
(iv) $\mathfrak{A}$ is exponentially stable and $\mathfrak{A}^t$ is right-invertible for some (hence all) $t > 0$ [31] [34].

Let us now return to the general case of L^p-admissibility and Banach spaces. By applying the Hille–Yoshida theorem to the semigroup in Corollary 4.5 we get the following necessary and sufficient conditions for admissibility:

Theorem 5.3. *Let $\omega \in \mathbf{R}$, $1 \le p < \infty$, and let A be the generator of an ω-bounded C_0 semigroup on X.*

(i) *$B \in \mathcal{L}(U; X_{-1})$ is an L^p-admissible ω-bounded control operator for A if and only if there is a constant $M > 0$ such that, for all $u \in L^p_\omega(\mathbf{R}^+; U)$, $\lambda > \omega$, and $n = 0, 1, 2, \dots$,*

$$\left| \frac{\partial^n}{\partial \lambda^n} (\lambda I - A)^{-1} B \hat{u}(\lambda) \right|_X \le \frac{M n!}{(\lambda - \omega)^{n+1}} \|u\|_{L^p_\omega(\mathbf{R}^+; U)}. \tag{21}$$

(ii) *$C \in \mathcal{L}(X_1; Y)$ is an L^p-admissible ω-bounded observation operator for A if and only if there is a constant $M > 0$ such that, for all $x_0 \in X$, $\lambda > \omega$, and $n = 0, 1, 2, \dots$,*

$$\left(\int_0^\infty \left| \frac{\partial^n}{\partial \lambda^n} \mathrm{e}^{-(\lambda - \omega)t} C (\lambda I - A)^{-1} x_0 \right|_Y^p dt \right)^{1/p} \le \frac{M n!}{(\lambda - \omega)^{n+1}} |x_0|_X. \tag{22}$$

(iii) *The operators $B \in \mathcal{L}(U; X_{-1})$ and $C \in \mathcal{L}(X_1; U)$ are jointly L^p admissible and ω-bounded iff B is an L^p-admissible ω-bounded control operator for A (cf. (i)), C is an admissible ω-bounded observation operator for A (cf. (ii)) and there is a constant $M > 0$ such that, for all $u \in L^p_\omega(\mathbf{R}^+; U)$, $\lambda > \omega$, and $n = 0, 1, 2, \dots$,*

$$\left(\int_0^\infty \left| \frac{\partial^n}{\partial \lambda^n} \mathrm{e}^{-(\lambda - \omega)t} \widehat{\mathfrak{D}}(\lambda) \hat{u}(\lambda) \right|_Y^p dt \right)^{1/p} \le \frac{M n!}{(\lambda - \omega)^{n+1}} \|u\|_{L^p_\omega(\mathbf{R}^+; U)}, \tag{23}$$

where

$$\widehat{\mathfrak{D}}(\lambda) = (\alpha - \lambda) C (\lambda I - A)^{-1} (\alpha I - A)^{-1} B + D_\alpha; \tag{24}$$

here α with $\Re \alpha > \omega$ and $D_\alpha \in \mathcal{L}(U; Y)$ can be chosen in an arbitrary manner.

Part (ii) of this theorem was proved by Grabowski and Callier [7] in the exponentially stable Hilbert space case (i.e., $\mathfrak{A}$ is exponentially stable, $\omega = 0$, $p = 2$, and U, X, and Y are Hilbert spaces), and the corresponding case of part (i) can be derived from (ii) by duality. The general case of parts (i) and (ii) was proved by Engel [6]. Part (iii) may (or may not) be new. A proof of the full theorem is given in [25].

Condition (23) does not depend on the particular *realization* $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ of $\mathfrak{D}$, i.e., it does not contain any direct references to $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, but only to $\widehat{\mathfrak{D}}$ which is completely determined by $\mathfrak{D}$. This indicates that the following conjecture may be true:

Conjecture 5.4. *An analytic $\mathcal{L}(U;Y)$-valued function $\widehat{\mathfrak{D}}$ on $\Re\lambda > \omega$ is the transfer function of an ω-bounded L^p-well-posed linear system if and only if there is a constant $M > 0$ such that* (23) *holds for all $u \in L^p_\omega(\mathbf{R}^+;U)$, $\lambda > \omega$, and $n = 0, 1, 2, \ldots$.*

Clearly, by Theorem 5.3, condition (23) is necessary for $\widehat{\mathfrak{D}}$ to be a L^p-well-posed ω-bounded transfer function, and we conjecture that it is also sufficient. This would give us a necessary and sufficient condition for an H^∞ function to be an L^p-multiplier.

References

[1] A. Adamajan and Damir Z. Arov. On unitary couplings of semiunitary operators. In *Eleven Papers in Analysis*, volume 95 of *American Mathematical Society Translations*, pages 75–129, Providence, R.I., 1970. American Mathematical Society.

[2] Damir Z. Arov. Passive linear stationary dynamic systems. *Siberian Math. J.*, 20:149–162, 1979.

[3] Damir Z. Arov. Passive linear systems and scattering theory. In *Dynamical Systems, Control Coding, Computer Vision*, volume 25 of *Progress in Systems and Control Theory*, pages 27–44, Basel Boston Berlin, 1999. Birkhäuser Verlag.

[4] Damir Z. Arov and Michael A. Nudelman. Passive linear stationary dynamical scattering systems with continuous time. *Integral Equations Operator Theory*, 24:1–45, 1996.

[5] Ruth F. Curtain and George Weiss. Well posedness of triples of operators (in the sense of linear systems theory). In *Control and Optimization of Distributed Parameter Systems*, volume 91 of *International Series of Numerical Mathematics*, pages 41–59, Basel, 1989. Birkhäuser-Verlag.

[6] Klaus-J. Engel. On the characterization of admissible control- and observation operators. *Systems Control Lett.*, 34:225–227, 1998.

[7] Piotr Grabowski and Frank M. Callier. Admissible observation operators. Semigroup criteria of admissibility. *Integral Equations Operator Theory*, 25:182–198, 1996.

[8] Scott Hansen and George Weiss. The operator Carleson measure criterion for admissibility of control operators for diagonal semigroups on l^2. *Systems Control Lett.*, 16:219–227, 1991.

[9] Scott Hansen and George Weiss. New results on the operator Carleson measure criterion. *IMA J. Math. Control Inform.*, 14:3–32, 1997.

[10] J. William Helton. Systems with infinite-dimensional state space: the Hilbert space approach. *Proceedings of the IEEE*, 64:145–160, 1976.

[11] Birgit Jacob and Jonathan R. Partington. The Weiss conjecture on admissibility of observation operators for contraction semigroups. To appear in Integral Equations and Operator Theory, 2000.

[12] Birgit Jacob, Jonathan R. Partington, and Sandra Pott. Admissible and weakly admissible observation operators for the right shift semigroup. Manuscript, 2000.

[13] Birgit Jacob and Hans Zwart. Disproof of two conjectures of George Weiss. Memorandum 1546, Faculty of Mathematical Sciences, University of Twente, The Netherlands, 2000.

[14] P. D. Lax and R. S. Phillips. *Scattering Theory.* Academic Press, New York, 1967.

[15] P. D. Lax and R. S. Phillips. Scattering theory for dissipative hyperbolic systems. *J. Funct. Anal.*, 14:172–235, 1973.

[16] Jonathan R. Partington and George Weiss. Admissible observation operators for the right shift semigroup. *Math. Control Signals Systems*, 13:179–192, 2000.

[17] Dietmar Salamon. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. *Trans. Amer. Math. Soc.*, 300:383–431, 1987.

[18] Dietmar Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, 21:147–164, 1989.

[19] Olof J. Staffans. Quadratic optimal control of stable well-posed linear systems. *Trans. Amer. Math. Soc.*, 349:3679–3715, 1997.

[20] Olof J. Staffans. Coprime factorizations and well-posed linear systems. *SIAM J. Control Optim.*, 36:1268–1292, 1998a.

[21] Olof J. Staffans. Quadratic optimal control of well-posed linear systems. *SIAM J. Control Optim.*, 37:131–164, 1998b.

[22] Olof J. Staffans. Feedback representations of critical controls for well-posed linear systems. *Internat. J. Robust Nonlinear Control*, 8:1189–1217, 1998c.

[23] Olof J. Staffans. Admissible factorizations of Hankel operators induce well-posed linear systems. *Systems Control Lett.*, 37, 1999.

[24] Olof J. Staffans. Lax–Phillips scattering and well-posed linear systems. In *Proceedings of the 7th IEEE Mediterranean Conference on Control and Systems*, CD-ROM, Haifa, Israel, July 28–30 1999.

[25] Olof J. Staffans. *Well-Posed Linear Systems.* Book manuscript, 2000.

[26] Olof J. Staffans and George Weiss. Transfer functions of regular linear systems. Part II: the system operator and the Lax-Phillips semigroup. Manuscript, 2000.

[27] George Weiss. Admissibility of unbounded control operators. *SIAM J. Control Optim.*, 27:527–545, 1989a.

[28] George Weiss. Admissible observation operators for linear semigroups. *Israel J. Math.*, 65:17–43, 1989b.

[29] George Weiss. The representation of regular linear systems on Hilbert spaces. In *Control and Optimization of Distributed Parameter Systems*, volume 91 of *International Series of Numerical Mathematics*, pages 401–416, Basel, 1989c. Birkhäuser-Verlag.

[30] George Weiss. Representations of shift-invariant operators on L^2 by H^∞ transfer functions: an elementary proof, a generalization to L^p, and a counterexample for L^∞. *Math. Control Signals Systems*, 4:193–203, 1991.

[31] George Weiss. Two conjectures on the admissibility of control operators. In *Control and Optimization of Distributed Parameter Systems*, pages 367–378, Basel, 1991. Birkhäuser-Verlag.

[32] George Weiss. Transfer functions of regular linear systems. Part I: characterizations of regularity. *Trans. Amer. Math. Soc.*, 342:827–854, 1994a.

[33] George Weiss. Regular linear systems with feedback. *Math. Control Signals Systems*, 7:23–57, 1994b.

[34] George Weiss. A powerful generalization of the Carleson measure theorem? In *Open Problems in Mathematical Systems and Control Theory*, pages 267–272, London, 1999. Spinger-Verlag.

[35] George Weiss, Olof J. Staffans, and Marius Tuchnak. Well-posed linear systems – a survay with emphasis on conservative systems. Manuscript, 2000.

[36] Martin Weiss and George Weiss. Optimal control of stable weakly regular linear systems. *Math. Control Signals Systems*, 10:287–330, 1997.

[37] Hans Zwart and Birgit Jacob. Disproof of an admissibility conjecture of Weiss. Memorandum 1539, Faculty of Mathematical Sciences, University of Twente, The Netherlands, 2000.

Åbo Akademi University
Department of Mathematics
FIN-20500 Åbo, Finland
http://www.abo.fi/˜staffans/
E-mail address: `Olof.Staffans@abo.fi`