

# Admissible Factorizations of Hankel Operators Induce Well-Posed Linear Systems

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**Abstract.** One of the basic axioms of a well-posed linear system says that the Hankel operator of the input/output map of the system factors into the product of the input map and the output map. Here we prove the converse: every factorization of the Hankel operator of a bounded causal time-invariant map from  $L^2$  to  $L^2$  which satisfies a certain admissibility condition induces a stable well-posed linear system. In particular, there is a one-to-one correspondence between the set of all minimal stable well-posed realizations of a given stable causal time-invariant input/output map (or equivalently, of a given  $H^\infty$  transfer function) and all minimal stable admissible factorizations of the Hankel operator of this input/output map.

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## 1 The Main Result

Let  $U$  and  $Y$  be two Hilbert spaces, and let  $TIC(U; Y)$  denote the space of all bounded linear time-invariant and causal operators from  $L^2(\mathbf{R}; U)$  to

$L^2(\mathbf{R}; Y)$ , where  $\mathbf{R} = (-\infty, \infty)$ . The purpose of this article is to show that there is a one-to-one correspondence between the set of all minimal stable realizations of a given input/output map  $\mathfrak{D} \in TIC(U; Y)$  and the set of all minimal bounded factorizations of the Hankel operator of  $\mathfrak{D}$  which satisfy a certain admissibility condition. We begin by defining what we mean by a stable well-posed linear system in continuous time.

Let  $\mathbf{R}^- = (-\infty, 0]$ ,  $\mathbf{R}^+ = [0, \infty)$ , and for any function  $u$  defined on  $\mathbf{R}$ , let

$$\begin{aligned} (\tau^t u)(s) &= u(t + s), \quad t, s \in \mathbf{R}, \\ (\pi_- u)(s) &= \begin{cases} u(s), & s \in \mathbf{R}^-, \\ 0, & s \in \mathbf{R}^+, \end{cases} \\ (\pi_+ u)(s) &= \begin{cases} u(s), & s \in \mathbf{R}^+, \\ 0, & s \in \mathbf{R}^-, \end{cases} \end{aligned}$$

In particular, we can apply these operators to functions  $u \in L^2(\mathbf{R}; U)$  (the space of  $U$ -valued  $L^2$ -functions on  $\mathbf{R}$ ), where  $U$  is a Hilbert space. Then  $t \mapsto \tau^t$  is the (bilateral) left-shift group on  $L^2(\mathbf{R}; U)$ ,  $t \mapsto \tau_+^t = \pi_+ \tau^t$  is the (unilateral) left-shift semigroup on  $L^2(\mathbf{R}^+; U)$ , and  $\tau_-^t = \tau^t \pi_-$  is the (unilateral) left-shift semigroup on  $L^2(\mathbf{R}^-; U)$ .

**Definition 1.1.** *Let  $U$ ,  $H$  and  $Y$  be Hilbert spaces. A stable well-posed linear system  $\Psi$  on  $(Y, H, U)$  is a quadruple  $\Psi = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  of bounded linear operators satisfying the following conditions:*

- (i)  $t \mapsto \mathfrak{A}^t$  is a bounded strongly continuous semigroup on  $H$ ;
- (ii)  $\mathfrak{B}: L^2(\mathbf{R}^-; U) \rightarrow H$  satisfies  $\mathfrak{B}\tau_-^t = \mathfrak{A}^t \mathfrak{B}$  for all  $t \in \mathbf{R}^+$ ;
- (iii)  $\mathfrak{C}: H \rightarrow L^2(\mathbf{R}^+; Y)$  satisfies  $\mathfrak{C}\mathfrak{A}^t = \tau_+^t \mathfrak{C}$  for all  $t \in \mathbf{R}^+$ ;
- (iv)  $\mathfrak{D}: L^2(\mathbf{R}; U) \rightarrow L^2(\mathbf{R}; Y)$  satisfies  $\tau^t \mathfrak{D} = \mathfrak{D} \tau^t$  for all  $t \in \mathbf{R}$ ,  $\pi_- \mathfrak{D} \pi_+ = 0$ , and  $\pi_+ \mathfrak{D} \pi_- = \mathfrak{C} \mathfrak{B}$ .

The different components of  $\Psi$  are called as follows:  $U$  is the input space,  $H$  is the state space,  $Y$  is the output space,  $\mathfrak{A}$  is the semigroup,  $\mathfrak{B}$  is the input map,  $\mathfrak{C}$  is the output map, and  $\mathfrak{D}$  is the input-output map.

Thus, (ii) says that the input map  $\mathfrak{B}$  intertwines the left shift on  $\mathbf{R}^-$  with the basic state space semigroup  $\mathfrak{A}$ , and (iii) says that the output map  $\mathfrak{C}$  intertwines  $\mathfrak{A}$  with the left shift on  $\mathbf{R}^+$ . The condition  $\tau^t \mathfrak{D} = \mathfrak{D} \tau^t$  says that  $\mathfrak{D}$  is time-invariant, the condition  $\pi_- \mathfrak{D} \pi_+ = 0$  says that  $\mathfrak{D}$  is causal (thus,  $\mathfrak{D} \in TIC(U; Y)$ ), and the final condition  $\pi_+ \mathfrak{D} \pi_- = \mathfrak{C} \mathfrak{B}$  in (iv) says that the Hankel operator of the input/output map  $\mathfrak{D}$  factors into the product of the input map  $\mathfrak{B}$  and the output map  $\mathfrak{C}$ . For more details of this particular formulation of a well-posed linear system we refer the reader to Staffans [1995 1996 1997 1998 1999abcd]. Alternative (but more or less equivalent) formulations are given in Arov and Nudelman [1996], Curtain and Weiss [1989], Helton [1976], Jacob and Zwart [1998], Ober and Montgomery-Smith [1990], Ober and Wu [1996], Salamon [1987 1989], Weiss [1989abc 1991 1994ab], and Weiss and Weiss [1997].

Here we are primarily interested in the converse of part (iv) of Definition 1.1. Suppose that  $\mathfrak{D} \in TIC(U; Y)$ , i.e.,  $\mathfrak{D}: L^2(\mathbf{R}; U) \rightarrow L^2(\mathbf{R}; Y)$  satisfies both the time invariance requirement  $\tau^t \mathfrak{D} = \mathfrak{D} \tau^t$  and the causality requirement  $\pi_- \mathfrak{D} \pi_+ = 0$  in (iv). Suppose also that in one way or another we have succeeded to factor  $\pi_+ \mathfrak{D} \pi_-$  into  $\pi_+ \mathfrak{D} \pi_- = \mathfrak{C} \mathfrak{B}$ , where  $\mathfrak{B}: L^2(\mathbf{R}^-; U) \rightarrow H$  and  $\mathfrak{C}: H \rightarrow L^2(\mathbf{R}^+; Y)$  are bounded linear operators and  $H$  is an arbitrary Hilbert space. Is it then always possible to find a semigroup  $\mathfrak{A}$  on  $H$  such that the quadruple  $\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is a stable well-posed linear system?

Our answer to the preceding question, given in Theorem 1.3 below, is a qualified “yes”. One necessary restriction is that the factors in the factorization  $\pi_+ \mathfrak{D} \pi_- = \mathfrak{C} \mathfrak{B}$  satisfy some “hidden” regularity assumptions imposed on them by the fact that they also have to satisfy (ii) and (iii). To derive these hidden regularity assumptions we argue as follows: If (ii) holds then  $\mathfrak{B} \tau_-^t = \mathfrak{A}^t \mathfrak{B}$ , hence

$$\|\mathfrak{B} \tau_-^t u\|_H \leq K \|\mathfrak{B} u\|_H, \quad \forall t \in \mathbf{R}^+, \quad \forall u \in L^2(\mathbf{R}^-; U), \quad (1)$$

where  $K = \sup_{t \geq 0} \|\mathfrak{A}^t\|$ . In particular, for all  $u \in L^2(\mathbf{R}^-; U)$ ,

$$\mathfrak{B} u = 0 \Rightarrow \mathfrak{B} \tau_-^t u = 0, \quad \forall t \in \mathbf{R}^+. \quad (2)$$

The same computation applied to the adjoint of the output intertwining condition  $\mathfrak{C} \mathfrak{A}^t = \tau_+^t \mathfrak{C}$  gives an analogous condition for the output map, namely

$$\|(\tau_+^t \mathfrak{C})^* y\|_H \leq K \|\mathfrak{C}^* y\|_H, \quad \forall t \in \mathbf{R}^+, \quad \forall y \in L^2(\mathbf{R}^+; Y). \quad (3)$$

Another important property of the factorization  $\pi_+ \mathfrak{D} \pi_- = \mathfrak{C} \mathfrak{B}$  is related to the controllability and observability of the resulting system.

**Definition 1.2.**

- (i) A stable well-posed linear system  $\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  on  $(Y, H, U)$  is controllable if  $\mathfrak{B}$  has dense range, and exactly controllable in infinite time if the range of  $\mathfrak{B}$  is the whole state space  $H$ . The system is observable if  $\mathfrak{C}$  is one-to-one and exactly observable in infinite time if, in addition, the range of  $\mathfrak{C}$  is closed in  $L^2(\mathbf{R}^+; Y)$ . A system is minimal if it is both controllable and observable.
- (ii) By a stable factorization  $\pi_+ \mathfrak{D} \pi_- = \mathfrak{C} \mathfrak{B}$  of the Hankel operator of  $\mathfrak{D} \in TIC(U; Y)$  we mean a factorization where  $H$  is a Hilbert space, and  $\mathfrak{B}: L^2(\mathbf{R}^-; U) \rightarrow H$  and  $\mathfrak{C}: H \rightarrow L^2(\mathbf{R}^+; Y)$  are bounded linear operators. This factorization is minimal if, in addition, the range of  $\mathfrak{B}$  is dense in  $H$  and  $\mathfrak{C}$  is one-to-one.

It is well known that every well-posed linear system can be turned into a minimal system by factoring out the orthogonal complement of the range of the input map and projecting onto the orthogonal complement of the null space of the output map. See, for example, [Salamon 1989, p. 159] or [Arov and Nudelman 1996, Theorem 7.1] (the corresponding discrete time version is found in, e.g., [Helton 1974, Theorem 3a.1]).

The following is our main result:

**Theorem 1.3.** *Let  $\mathfrak{D} \in TIC(U; Y)$ , and suppose that the Hankel operator  $\pi_+ \mathfrak{D} \pi_-$  of  $\mathfrak{D}$  factors into  $\pi_+ \mathfrak{D} \pi_- = \mathfrak{C} \mathfrak{B}$ , where  $H$  is a Hilbert space, and  $\mathfrak{B}: L^2(\mathbf{R}^-; U) \rightarrow H$  and  $\mathfrak{C}: H \rightarrow L^2(\mathbf{R}^+; Y)$  are bounded linear operators (i.e.,  $\mathfrak{C} \mathfrak{B}$  is a stable factorization of  $\pi_+ \mathfrak{D} \pi_-$ ).*

- (i) If  $\mathfrak{B}$  has dense range then (1) implies (3), and if  $\mathfrak{C}$  is one-to-one, then (3) implies (1).
- (ii) Let  $H_{\mathfrak{B}}$  be the closure of the range of  $\mathfrak{B}$  in  $H$ . Then the following conditions are equivalent:
  - (a) condition (1) holds;

- (b) there is a (unique) semigroup  $\mathfrak{A}_{\mathfrak{B}}$  on  $H_{\mathfrak{B}}$  such that  $\begin{bmatrix} \mathfrak{A}_{\mathfrak{B}} & \mathfrak{B} \\ \mathfrak{C}_{\mathfrak{B}} & \mathfrak{D} \end{bmatrix}$  is a stable well-posed linear system on  $(Y, H_{\mathfrak{B}}, U)$ ; here  $\mathfrak{C}_{\mathfrak{B}}$  is the restriction of  $\mathfrak{C}$  to  $H_{\mathfrak{B}}$ .
- (iii) Let  $H_{\mathfrak{C}}$  be the orthogonal complement to the null space of  $\mathfrak{C}$  in  $H$ . Then the following conditions are equivalent:
- (a) condition (3) holds;
- (b) there is a (unique) semigroup  $\mathfrak{A}_{\mathfrak{C}}$  on  $H_{\mathfrak{C}}$  such that  $\begin{bmatrix} \mathfrak{A}_{\mathfrak{C}} & \mathfrak{B}_{\mathfrak{C}} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is a stable well-posed linear system on  $(Y, H_{\mathfrak{C}}, U)$ ; here  $\mathfrak{B}_{\mathfrak{C}} = P_{\mathfrak{C}}\mathfrak{B}$ , where  $P_{\mathfrak{C}}$  is the orthogonal projection of  $H$  onto  $H_{\mathfrak{C}}$ .
- (iv) If the factorization  $\pi_+\mathfrak{D}\pi_- = \mathfrak{C}\mathfrak{B}$  is minimal (i.e.,  $\mathfrak{B}$  has dense range and  $\mathfrak{C}$  is one-to-one), then the following conditions are equivalent:
- (a) condition (1) holds;
- (b) condition (3) holds;
- (c) there is a (unique) semigroup  $\mathfrak{A}$  on  $H$  such that  $\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is a stable well-posed linear system.

The proof of Theorem 1.3 is given in Section 6.

**Corollary 1.4.** *There is a one-to-one correspondence between the set of all minimal stable realizations of an input/output map  $\mathfrak{D} \in TIC(U; Y)$  and the set of all minimal stable factorizations of the Hankel operator of  $\mathfrak{D}$  satisfying the admissibility conditions (1) and (3).*

This follows from Definitions 1.1 and 1.2 and Theorem 1.3(iv). We remark that all the realizations in Corollary 1.4 are weakly similar (with a one-to-one, closed, possibly unbounded, densely defined similarity operator with dense range); see [Arov and Nudelman 1996, Proposition 7.10] or Staffans [1999d].

## 2 The History of the Problem

Theorem 1.3 is in the spirit [Kalman et al. 1969, Part 4] (although the setting is different). The importance of the Hankel operator of the input/output map in realization theory has long been recognized. There is some formal

resemblance between Theorem 1.3 and the factorizations results presented in [Kalman 1963, Theorem 1], [Kalman et al. 1969, Theorem (13.19)], and [Brockett 1970, Theorem 1, p. 93], but there is a very significant non-technical difference: the realization presented there is intrinsically time-dependent (and time-reversible), and its state space dynamics is trivial. A much more closely related result is found in [Kalman et al. 1969, Section 10.6] and [Fuhrmann 1981, pp. 31–32]: there we find the same algebraic construction (in discrete time), but without any continuity considerations of the type (1)–(3). Even closer to Theorem 1.3 is [Baras and Brockett 1975, Theorem 6], [Baras and Dewilde 1976, Theorem II.2.2] and [Fuhrmann 1981, Theorem 6-3, p. 293], which give sufficient conditions for the existence of a realization with bounded control and observation operators in the case of finite-dimensional  $U$  and  $Y$ . As a special case of a stable factorization we can take either  $\mathfrak{B}$  or  $\mathfrak{C}$  to be the identity operator; this leads to the exactly controllable (or restricted shift) and exactly observable (or restricted  $*$ -shift) realizations, respectively, different versions of which are found in, e.g., Baras and Dewilde [1976], [Fuhrmann 1974, Theorem 2.6], [Fuhrmann 1981, Section 3.2], [Helton 1974, p. 31], [Jacob and Zwart 1998, Theorem A.1], [Ober and Wu 1996, Sections 5.2–5.3], and [Salamon 1989, Theorem 4.3].

Various types of infinite-dimensional discrete and continuous time realizations have recently been studied in Ober and Montgomery-Smith [1990] and Ober and Wu [1993 1996] (the restricted shift and  $*$ -shift, input normal, output normal, and (par)balanced realizations, as well as their spectral and stability properties) and in Jacob and Zwart [1998] (minimal realizations of a scalar inner transfer function with an invertible or exponentially stable semigroup).

### 3 The Corresponding Frequency Domain Result

To derive a frequency domain analogue of Theorem 1.3 we first recall that the space  $TIC(U; Y)$  is isometrically isomorphic to the space  $H^\infty(U; Y)$  of  $\mathcal{L}(U; Y)$ -valued bounded analytic functions of the right half plane:

**Proposition 3.1.** *There is a one-to-one correspondence between  $TIC(U; Y)$  and  $H^\infty(U; Y)$  of the following type: To every  $\mathfrak{D} \in TIC(U; Y)$  there is a unique  $\widehat{\mathfrak{D}} \in H^\infty(U; Y)$ , and to every  $\widehat{\mathfrak{D}} \in H^\infty(U; Y)$  there is a unique*

$\mathfrak{D} \in TIC(U; Y)$  such that, for every  $u \in L^2(\mathbf{R}^+; Y)$ , the Laplace transform  $\widehat{\mathfrak{D}}u$  of  $\mathfrak{D}u$  is given by  $\widehat{\mathfrak{D}}u(z) = \widehat{\mathfrak{D}}(z)\hat{u}(z)$ ,  $\Re z > 0$ , where  $\hat{u}$  is the Laplace transform of  $u$ . Moreover, the operator norm of  $\mathfrak{D}$  in  $TIC(U; Y)$  is equal to the  $H^\infty(U; Y)$ -norm of  $\widehat{\mathfrak{D}}$  ( $= \sup_{\Re z > 0} \|\widehat{\mathfrak{D}}(z)\|$ ).

This result is well known. See, for example [Weiss 1991, Theorem 1.3 and Remark 1.6].

Thus, Theorem 1.3 may be interpreted as a realization result for the  $H^\infty$  transfer function  $\widehat{\mathfrak{D}}$ . Usually  $U$ ,  $H$  and  $Y$  are taken to be separable, in which case  $\widehat{\mathfrak{D}}$  has a well-defined boundary function on the imaginary axis, and the Hankel operator  $\pi_+ \mathfrak{D} \pi_-$  has a standard frequency domain interpretation (projection onto  $H^2(U)^\perp$  followed by multiplication by the boundary function followed by projection onto  $H^2(Y)$ ). However, in its present form Theorem 1.3 does not look like a “standard” realization result for an  $H^\infty$  transfer function  $\widehat{\mathfrak{D}}$ , which is typically expected to provide a representation of  $\widehat{\mathfrak{D}}$  of the form

$$\widehat{\mathfrak{D}}(z) = C(zI - A)^{-1}B + D, \quad \Re z > 0, \quad (4)$$

where  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  are the generators of the system  $\Psi$ . To get such a representation we have to write the system  $\Psi$  in “differential” form

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \end{aligned} \quad (5)$$

where  $A$  is the generator of the semigroup  $\mathfrak{A}$ ,  $B$  and  $C$  are the (unbounded) control and observation operator, determined by the fact that (in a well-defined sense)

$$\begin{aligned} \mathfrak{B}u &= \int_{-\infty}^0 \mathfrak{A}^{-s} Bu(s) ds, \\ (\mathfrak{C}x)(t) &= C\mathfrak{A}^t x, \quad t \geq 0, \end{aligned} \quad (6)$$

and  $D$  is the feedthrough operator. For this to be possible we need to restrict the set of permitted  $H^\infty$  functions slightly, and consider only functions  $\widehat{\mathfrak{D}}$  for which the (weak or strong) limit

$$Du = \lim_{\alpha \rightarrow +\infty} \widehat{\mathfrak{D}}(\alpha)u \quad u \in U, \quad (7)$$

exists in  $Y$ ; here  $\alpha \rightarrow +\infty$  along the *real axis*. Following Weiss [1994ab] and Weiss and Weiss [1997], we call such a transfer function (weakly or strongly) *regular*. By a *regular system* we mean a system with a regular transfer function. It has been known for roughly a decade how to construct the generators  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  of a regular system from the system operators  $\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ ; see Arov and Nudelman [1996], Salamon [1989], Weiss [1989ab 1994ab], and Staffans [1999d]. Our Theorem 1.3, combined with the general theory of regular systems, gives us a representation of the form (4) for a regular transfer function via the factorization of its Hankel operator. We refer the reader to the works cited above for details of how to construct the representation (4) of  $\widehat{\mathfrak{D}}$  from the system  $\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ . Even in the non-regular case it is possible to get a representation similar to (4) but slightly more complicated; see the cited references.

## 4 Discrete Time Realizations

Our Theorem 1.3 has an obvious discrete time counterpart. One way to formulate and prove this result is to use the Cayley transform, which often has been used to transfer results in the opposite direction from discrete to continuous time (see, e.g., Arov and Nudelman [1996] and Ober and Wu [1996]). However, it is easier to prove the corresponding discrete time result directly.

A discrete time system is usually written in difference form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \quad k \in \mathbf{Z}_+ = \{0, 1, 2, \dots\}, \end{aligned} \tag{8}$$

where  $A \in \mathcal{L}(H)$ ,  $B \in \mathcal{L}(U; H)$ ,  $C \in \mathcal{L}(H; Y)$ , and  $D \in \mathcal{L}(U; Y)$ ; here  $U$ ,  $H$ , and  $Y$  are Hilbert spaces and  $\mathcal{L}(U; Y)$  stands for the set of bounded linear operators from  $U$  to  $Y$ ; etc. The discrete time input map  $\mathfrak{B}$ , output map  $\mathfrak{C}$ , and input/output map  $\mathfrak{D}$  are given by

$$\begin{aligned} \mathfrak{B}u &= \sum_{k=0}^{\infty} A^k u_{-k-1}, \\ (\mathfrak{C}x)_k &= CA^k x, \quad k \in \mathbf{Z}_+, \\ (\mathfrak{D}u)_k &= \sum_{i=0}^{\infty} CA^i Bu_{k-i-1} + Du_k, \quad k \in \mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}, \end{aligned} \tag{9}$$



where  $u_k$  represents an  $U$ -valued sequence with finite support and  $x \in H$ . The system is *stable* if  $\sup_{k \in \mathbf{Z}_+} \|A^k\| < \infty$ ,  $\mathfrak{C} \in \mathcal{L}(H; l^2(\mathbf{Z}_+; Y))$ , and the operators  $\mathfrak{B}$  and  $\mathfrak{D}$  can be extended to (bounded) operators  $\mathfrak{B} \in \mathcal{L}(l^2(\mathbf{Z}_-; U); H)$  (where  $\mathbf{Z}_- = \{-1, -2, \dots\}$ ) and  $\mathfrak{D} \in \mathcal{L}(l^2(\mathbf{Z}; U); l^2(\mathbf{Z}; Y))$ . To get the *input-output representation* of the system in (8) we replace the operators  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  by the operators  $\begin{bmatrix} A & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ . This quadruple of operators satisfies a set of conditions similar to those listed in Definition 1.1. For each sequence  $u_k$ ,  $z \in \mathbf{Z}$  and each  $j \in \mathbf{Z}$  we define

$$\begin{aligned} (\pi_- u)_k &= \begin{cases} u_k, & k \in \mathbf{Z}_-, \\ 0, & k \in \mathbf{Z}_+, \end{cases} & (\pi_+ u)_k &= \begin{cases} u_k, & k \in \mathbf{Z}_-, \\ 0, & k \in \mathbf{Z}_+, \end{cases} \\ (\sigma u)_k &= u_{k+1}, \quad k \in \mathbf{Z}, & e_k^j &= \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \end{aligned}$$

Thus,  $\pi_+$  and  $\pi_-$  are complementary orthogonal projections operators in  $l^2(\mathbf{Z})$ ,  $\sigma$  is the (bilateral) left shift in  $l^2(\mathbf{Z})$ ,  $\sigma_+ = \pi_+ \sigma$  is the (unilateral) left-shift on  $l^2(\mathbf{Z}_+)$ , and  $\sigma_- = \sigma \pi_-$  is the (unilateral) left-shift semigroup on  $l^2(\mathbf{Z}_-)$ . The vectors  $e^j$  form an orthonormal basis in  $l^2(\mathbf{Z})$ . The operators  $\begin{bmatrix} A & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  arising from a stable discrete time system on  $(Y, H, U)$  are characterized by the fact that they satisfy the following four conditions:

- (i)  $A \in \mathcal{L}(H)$ , and  $\sup_{k \in \mathbf{Z}_+} \|A^k\| < \infty$ ;
- (ii)  $\mathfrak{B} \in \mathcal{L}(l^2(\mathbf{Z}_-; U); H)$  satisfies  $\mathfrak{B}\sigma_- = A\mathfrak{B}$ ;
- (iii)  $\mathfrak{C} \in \mathcal{L}(H; l^2(\mathbf{Z}_+; Y))$  satisfies  $\mathfrak{C}A = \sigma_+\mathfrak{C}$ ;
- (iv)  $\mathfrak{D} \in \mathcal{L}(l^2(\mathbf{Z}; U); l^2(\mathbf{Z}; Y))$  satisfies  $\sigma\mathfrak{D} = \mathfrak{D}\sigma$ ,  $\pi_-\mathfrak{D}\pi_+ = 0$ , and  $\pi_+\mathfrak{D}\pi_- = \mathfrak{C}\mathfrak{B}$ .

In particular,  $\mathfrak{D}$  is again time invariant and causal, and its Hankel operator  $\pi_+\mathfrak{D}\pi_-$  factors into  $\pi_+\mathfrak{D}\pi_- = \mathfrak{C}\mathfrak{B}$ . We call a quadruple of operators  $\begin{bmatrix} A & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  which satisfy (i)–(iv) a *stable discrete time well-posed linear system in input-output form* on  $(Y, H, U)$ . The corresponding operators  $B$ ,  $C$ , and  $D$  can be recovered from  $\mathfrak{B}$ ,  $\mathfrak{C}$  and  $\mathfrak{D}$  through

$$Bu = \mathfrak{B}(ue^{-1}), \quad Cx = (\mathfrak{C}x)_0, \quad Du = (\mathfrak{D}(ue^0))_0. \quad (10)$$

More details are given in Malinen [1997 1999]

The discrete time analogues of (1) and (3) are

$$\|\mathfrak{B}\sigma^k\pi_-u\|_H \leq K\|\mathfrak{B}u\|_H, \quad \forall k \in \mathbf{Z}_+, \quad \forall u \in l^2(\mathbf{Z}_-; U), \quad (11)$$

$$\|(\sigma^k\pi_+\mathfrak{C})^*y\|_H \leq K\|\mathfrak{C}^*y\|_H, \quad \forall k \in \mathbf{Z}_+, \quad \forall y \in l^2(\mathbf{Z}_+; Y), \quad (12)$$

where  $K = \sup_{k \in \mathbf{Z}_+} \|A^k\|$ . The discrete time version of Theorem 1.3 reads as follows:

**Theorem 4.1.** *Let  $\mathfrak{D} \in \mathcal{L}(l^2(\mathbf{Z}; U); l^2(\mathbf{Z}; Y))$  be time invariant and causal (i.e.,  $\sigma\mathfrak{D} = \mathfrak{D}\sigma$  and  $\pi_-\mathfrak{D}\pi_+ = 0$ ), and suppose that the Hankel operator  $\pi_+\mathfrak{D}\pi_-$  of  $\mathfrak{D}$  factors into  $\pi_+\mathfrak{D}\pi_- = \mathfrak{C}\mathfrak{B}$ , where  $H$  is a Hilbert space, and  $\mathfrak{B} \in \mathcal{L}(L^2(\mathbf{Z}_-; U); H)$  and  $\mathfrak{C} \in \mathcal{L}(H; L^2(\mathbf{Z}_+; Y))$ .*

- (i) *If  $\mathfrak{B}$  has dense range then (11) implies (12), and if  $\mathfrak{C}$  is one-to-one, then (12) implies (11).*
- (ii) *Let  $H_{\mathfrak{B}}$  be the closure of the range of  $\mathfrak{B}$  in  $H$ . Then the following conditions are equivalent:*
  - (a) *condition (11) holds;*
  - (b) *there is a (unique) operator  $A_{\mathfrak{B}} \in \mathcal{L}(H_{\mathfrak{B}})$  such that  $\begin{bmatrix} A_{\mathfrak{B}} & \mathfrak{B} \\ \mathfrak{C}_{\mathfrak{B}} & \mathfrak{D} \end{bmatrix}$  is a stable discrete time well-posed linear system on  $(Y, H_{\mathfrak{B}}, U)$ ; here  $\mathfrak{C}_{\mathfrak{B}}$  is the restriction of  $\mathfrak{C}$  to  $H_{\mathfrak{B}}$ .*
- (iii) *Let  $H_{\mathfrak{C}}$  be the orthogonal complement to the null space of  $\mathfrak{C}$  in  $H$ . Then the following conditions are equivalent:*
  - (a) *condition (12) holds;*
  - (b) *there is a (unique) operator  $A_{\mathfrak{B}} \in \mathcal{L}(H_{\mathfrak{B}})$  such that  $\begin{bmatrix} A_{\mathfrak{B}} & \mathfrak{B}_{\mathfrak{C}} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is a stable discrete time well-posed linear system on  $(Y, H_{\mathfrak{C}}, U)$ ; here  $\mathfrak{B}_{\mathfrak{C}} = P_{\mathfrak{C}}\mathfrak{B}$ , where  $P_{\mathfrak{C}}$  is the orthogonal projection of  $H$  onto  $H_{\mathfrak{C}}$ .*
- (iv) *If the factorization  $\pi_+\mathfrak{D}\pi_- = \mathfrak{C}\mathfrak{B}$  is minimal (i.e.,  $\mathfrak{B}$  has dense range and  $\mathfrak{C}$  is one-to-one), then the following conditions are equivalent:*
  - (a) *condition (11) holds;*
  - (b) *condition (12) holds;*
  - (c) *there is a (unique) operator  $A \in \mathcal{L}(H)$  such that  $\begin{bmatrix} A & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$  is a stable well-posed linear system.*

We leave the straightforward proof of this theorem to the reader (it is virtually identical to the proof of Theorem 1.3 given in Section 6).

## 5 Applications and Extensions

One possible way of factoring the Hankel operator  $\pi_+ \mathfrak{D} \pi_-$  is to factor the time-invariant operator  $\mathfrak{D}$  itself into  $\mathfrak{D} = \mathfrak{X} \mathfrak{Y}$ , where  $\mathfrak{Y}: L^2(\mathbf{R}; U) \rightarrow L^2(\mathbf{R}; Z)$  and  $\mathfrak{X}: L^2(\mathbf{R}; Z) \rightarrow L^2(\mathbf{R}; Y)$  are bounded and time-invariant (but not necessarily causal), and  $Z$  is some auxiliary Hilbert space. We can then take  $H = L^2(\mathbf{R}; Z)$ ,  $\mathfrak{A}^t = \tau^t$ ,  $\mathfrak{B} = \mathfrak{Y} \pi_-$ , and  $\mathfrak{C} = \pi_+ \mathfrak{X}$ . Strictly speaking, this is not a special case of Theorem 1.3 since this realization is, in general, neither controllable nor observable, but it is easy to see that this is a realization of  $\mathfrak{D}$  (to get into the context of Theorem 1.3 we have to factor out the orthogonal complement to the reachable subspace  $H_{\mathfrak{B}}$ , or project the state space  $H$  onto the orthogonal complement of the unobservable subspace, i.e., onto  $H_{\mathfrak{C}}$ ). In this realization all the information about the factor  $\mathfrak{Y}$  is contained in the input map  $\mathfrak{B}$ , and all the information about the factor  $\mathfrak{X}$  is contained in the output map  $\mathfrak{C}$ . In particular, we can let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be the factors in an inner-outer factorization of  $\mathfrak{D}$ , or the factors in a co-inner-outer factorization of  $\mathfrak{D}$ , or the factors in a Douglas-Shapiro-Shields factorization in the case where  $\mathfrak{D}$  is strictly noncyclic. (See, e.g., [Ober and Wu 1996, Theorem 4.8] for a description of the last factorization.) We shall return to this question elsewhere.

It is also easy to prove a version of Theorem 1.3 which applies to *unstable* systems: Instead of using the standard  $L^2$ -spaces we can use  $L^2$ -spaces with an exponential weight for the input and output functions. This method is useful also in the construction of an exponentially stable realization (whenever such a realization exists). In the case where  $\mathfrak{B}$ ,  $\mathfrak{C}$ , and  $\mathfrak{D}$  are stable, if we are willing to accept an unbounded semigroup  $\mathfrak{A}^t$  in Theorem 1.3 and an unbounded state trajectory  $A^k x$  in Theorem 4.1, then it suffices to take  $t \in [0, 1]$  in (1)–(3) and to take  $k = 1$  in (11)–(12). See Staffans [1999d] for details. There it is also shown how to extend Theorem 1.3 to the class of  $L^p$ -well-posed linear systems on a triple on Banach spaces  $(Y, X, U)$ , and an alternative version of (3) is given which refers directly to  $\mathfrak{C}$  instead of  $\mathfrak{C}^*$ .

## 6 Proof of Theorem 1.3

The proof of Theorem 1.3 is based on the following key identity, which is often taken as the definition of a Hankel operator (cf. [Fuhrmann 1981, p. 249]):

**Lemma 6.1.** *The Hankel operator  $\pi_+ \mathfrak{D} \pi_-$  of  $\mathfrak{D}$  satisfies*

$$\tau_+^t \pi_+ \mathfrak{D} \pi_- = \pi_+ \mathfrak{D} \pi_- \tau_-^t, \quad \forall t \in \mathbf{R}^+.$$

*In particular, if  $\pi_+ \mathfrak{D} \pi_-$  factors into  $\pi_+ \mathfrak{D} \pi_- = \mathfrak{C} \mathfrak{B}$ , then*

$$\tau_+^t \mathfrak{C} \mathfrak{B} = \mathfrak{C} \mathfrak{B} \tau_-^t, \quad \forall t \in \mathbf{R}^+.$$

*Proof.* Use the time invariance of  $\mathfrak{D}$  to get

$$\begin{aligned} \tau_+^t \pi_+ \mathfrak{D} \pi_- &= \pi_+ \tau^t \pi_+ \mathfrak{D} \pi_- = \pi_+ \tau^t \mathfrak{D} \pi_- \\ &= \pi_+ \mathfrak{D} \tau^t \pi_- = \pi_+ \mathfrak{D} \pi_- \tau^t \pi_- = \pi_+ \mathfrak{D} \pi_- \tau_-^t. \quad \square \end{aligned}$$

*Proof of Theorem 1.3.* (i) We get from Lemma 6.1 for all  $t \in \mathbf{R}^+$ , all  $u \in L^2(\mathbf{R}^-; U)$ , and all  $y \in L^2(\mathbf{R}^+; Y)$ ,

$$\langle (\tau_+^t \mathfrak{C})^* y, \mathfrak{B} u \rangle_H = \langle y, \tau_+^t \mathfrak{C} \mathfrak{B} u \rangle_H = \langle \mathfrak{C}^* y, \mathfrak{B} \tau_-^t u \rangle_H.$$

In particular, if (1) holds, then

$$\begin{aligned} |\langle (\tau_+^t \mathfrak{C})^* y, \mathfrak{B} u \rangle_H| &= |\langle \mathfrak{C}^* y, \mathfrak{B} \tau_-^t u \rangle_H| \leq \|\mathfrak{C}^* y\|_H \|\mathfrak{B} \tau_-^t u\|_H \\ &\leq K \|\mathfrak{C}^* y\|_H \|\mathfrak{B} u\|_H, \end{aligned}$$

which implies (3) whenever  $\mathfrak{B}$  has dense range. The other claim is proved in a similar way ( $\mathfrak{C}^*$  has dense range iff  $\mathfrak{C}$  is one-to-one).

(ii) The argument that we used above to derive (1) shows that (b)  $\Rightarrow$  (a).

Conversely, suppose that (a) holds. Without loss of generality, we may assume that  $H_{\mathfrak{B}} = H$  (otherwise we replace  $H$  by  $H_{\mathfrak{B}}$ ). The idea is to use the intertwining condition  $\mathfrak{A}^t \mathfrak{B} = \mathfrak{B} \tau_-^t$  in part (ii) of Definition 1.1 as a definition of  $\mathfrak{A}^t$ . Clearly, for this to be possible, the range of  $\mathfrak{B}$  must be dense in  $H$ .

Thus, for each  $x = \mathfrak{B} u \in \text{range}(\mathfrak{B})$  and  $t \in \mathbf{R}^+$ , we define

$$\mathfrak{A}^t x = \mathfrak{B} \tau_-^t u.$$

To see that this definition of  $\mathfrak{A}^t x$  does not depend on the particular choice of  $u$  we use the fact that (1) implies (2): if  $x = \mathfrak{B} u_1 = \mathfrak{B} u_2$  then  $\mathfrak{B}(u_1 - u_2) = 0$ , and  $\mathfrak{B} \tau_-^t(u_1 - u_2) = 0$  for all  $t \in \mathbf{R}^+$ .

We claim that  $\mathfrak{A}$  is a strongly continuous semigroup on  $\text{range}(\mathfrak{B})$ . Obviously  $\mathfrak{A}^0 = I$ . Let  $x = \mathfrak{B} u$  and  $\mathfrak{A}^t x = \mathfrak{B} \tau_-^t u$ . Then  $\mathfrak{A}^t x \in \text{range}(\mathfrak{B})$  and

$$\mathfrak{A}^s \mathfrak{A}^t x = \mathfrak{B} \tau_-^s \tau_-^t u = \mathfrak{B} \tau_-^{s+t} u = \mathfrak{A}^{s+t} x.$$

The strong continuity of  $\mathfrak{A}^t$  on  $\text{range}(\mathfrak{B})$  is obvious (the left-shift semigroup on  $L^2(\mathbf{R}^+; U)$  is strongly continuous and  $\mathfrak{B}$  is bounded). Thus,  $\mathfrak{A}$  is a strongly continuous semigroup on  $\text{range}(\mathfrak{B})$ .

Next we extend  $\mathfrak{A}$  to a strongly continuous semigroup on  $H$ . For each  $t$ ,  $\mathfrak{A}^t$  is densely defined, and condition (1) implies that  $\|\mathfrak{A}^t x\|_H \leq K\|x\|_H$  for each  $x \in \text{range}(\mathfrak{B})$ . By continuity,  $\mathfrak{A}$  has a unique extension to a bounded strongly continuous semigroup on  $H$  which satisfies the intertwining condition  $\mathfrak{A}^t \mathfrak{B} = \mathfrak{B} \tau_-^t$ .

It remains to show that this semigroup also satisfies the second intertwining condition  $\mathfrak{C} \mathfrak{A}^t = \tau_+^t \mathfrak{C}$  for all  $t \in \mathbf{R}^+$ . By the density of  $\text{range}(\mathfrak{B})$  in  $H$ , it suffices to show that  $\mathfrak{C} \mathfrak{A}^t \mathfrak{B} = \tau_+^t \mathfrak{C} \mathfrak{B}$ , and this is an immediate consequence of Lemma 6.1:  $\mathfrak{C} \mathfrak{A}^t \mathfrak{B} = \mathfrak{C} \mathfrak{B} \tau_-^t = \tau_+^t \mathfrak{C} \mathfrak{B}$ .

(iii) To prove (iii) it suffices to apply (ii) to the dual system.

(iv) This follows from (i)–(iii). □

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