The Dynamics Induced by a Boundary Relation

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Henk de Snoo Seminar, Dec 17, 2010

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Grandpa, where do they come from?

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Boundary Control I/S/O System

A boundary control input/state/output system can be written in the form

$$\Sigma_{i/s/o} : \begin{cases} \dot{x}(t) = Lx(t), & \\ u(t) = \Gamma_0 x(t), & t \geq 0 \\ y(t) = \Gamma_1 x(t), & \\ x(0) = x_0. & \end{cases} \qquad (1)$$

$\mathcal{X}$ is the state space, $x(t) \in \mathcal{X}$, $x_0 \in \mathcal{X}$,
$\mathcal{U}$ is the input space, $u(t) \in \mathcal{U}$,
$\mathcal{Y}$ is the output space, $y(t) \in \mathcal{Y}$ (these are Hilbert spaces),
L is the main operator (always unbounded),
Γ_0 is the boundary control operator (surjective and unbounded),
Γ_1 is the observation operator (can be bounded or unbounded).

Boundary Control State/Signal System

A boundary control state/signal system is similar to a boundary control i/s/o system, but we no longer specify which part of the "boundary signal" $w(t) := \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$ is the input, and which part is the output. After replacing $\begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}$ by Γ we get an equation of the type

$$\Sigma : \begin{cases} \dot{x}(t) = Lx(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \geq 0; \qquad x(0) = x_0. \tag{2}$$

$\mathcal{X}$ is the *state space*, $x(t) \in \mathcal{X}$, $x_0 \in \mathcal{X}$, $\mathcal{X}$ is a Hilbert space,
$\mathcal{W}$ is the *signal space*, $w(t) \in \mathcal{W}$, $\mathcal{W}$ is a Kreĭn space,
L is the main operator (always unbounded),
Γ is the boundary operator (also unbounded),
L and Γ have the *same domain*
$\mathrm{Dom}\,(L) = \mathrm{Dom}\,(\Gamma) = \mathrm{Dom}\,\left(\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]\right) \subset \mathcal{X}$.

Boundary Control Systems $\leftrightarrow$ Boundary Triplets

There is an almost one-to-one correspondence between
conservative boundary control s/s systems $\leftrightarrow$
(conservative) boundary triplets

However, today I want to talk about the dynamics of boundary relations and not the dynamics of boundary triplets. To do this I have to go beyond the class of boundary s/s systems.

The Generating Subspace

Given a boundary control s/s system

$$\Sigma : \begin{cases} \dot{x}(t) = Lx(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \geq 0; \qquad x(0) = x_0. \qquad (2)$$

we can rewrite it in the graph form

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \right. \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \qquad (3)$$

where

$$V := \left\{ \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \in \mathfrak{K} \;\middle|\; x \in \mathrm{Dom}\left(\begin{bmatrix} L \\ \Gamma \end{bmatrix} \right) \right\}. \qquad (4)$$

Here V is the generating subspace, which is a subspace of the node space $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$.

State/Signal System

A general state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is of the form

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \right. \qquad ((3))$$

where $\mathcal{X}$ is the state space (a Hilbert space), and
$\mathcal{W}$ is the signal space (a Kreĭn space).
The generating subspace V is a closed subspace of the
node space $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix} \right]$.
$x(t) \in \mathcal{X}$ is the state at time $t \in \mathbb{R}^+$,
$x_0 \in \mathcal{X}$ is the initial state at time zero,
$w(t) \in \mathcal{W}$ is the signal at time $t \in \mathbb{R}^+$.

Example: A System Node

A system node is a construction used in the theory of well-posed (and non-wellposed) linear systems. It has a
state space $\mathcal{X}$ (a Hilbert space),
input space $\mathcal{U}$ (a Hilbert space),
output space $\mathcal{Y}$ (a Hilbert space).
It is a closed operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$. The dynamics of a system node is described by

$$\Sigma : \begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = S\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{5}$$

We can rewrite this as a state/signal system by taking $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ and defining

$$V := \left\{ \begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \end{bmatrix} \subset \begin{bmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{bmatrix} \;\middle|\; \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\}. \tag{6}$$

Example: Classical I/S/O System

Consider the classical input/state/output system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \qquad (7)$$

Here A, B, C, and D are bounded linear operators.
We can rewrite this as a state/signal system by taking
$\mathcal{W} = \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ $(= \mathcal{Y} \times \mathcal{U})$ and defining

$$V := \left\{ \begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{aligned} z &= Ax + Bu \\ y &= Cx + Du \end{aligned} \right\}. \qquad (8)$$

Thus, state/signal systems need not have anything to do with boundary control!
However, there is an almost one-to-one correspondence between conservative state/signal systems ↔ (conservative) boundary relations!
Thus, boundary relations do not necessarily have anything to do with boundary control!

Classical and Generalized Trajectories

We recall the equation describing the dynamics:

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \right. \tag{3}$$

- $\left[\begin{smallmatrix} x \\ w \end{smallmatrix} \right]$ is a classical trajectory of Σ if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix} \right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{X}) \end{smallmatrix} \right]$ and (3) holds for all $t \in \mathbb{R}^+$.
- $\left[\begin{smallmatrix} x \\ w \end{smallmatrix} \right]$ is a generalized trajectory of Σ if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix} \right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix} \right]$ and there exists a sequence of classical trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix} \right]$ such that $x_n \to x$ uniformly on bounded intervals and $w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W})$.

Simplifying Assumption

In this talk I focus on state/signal systems which are conservative, as studied in (Kur10).
They are well-posed in the sense of (KS09).
Simplifying Assumption: In the equation describing the dynamics

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \right. \tag{3}$$

I throughout make the simplifying assumption that the present state $x(t)$ and the present signal $w(t)$ determine the value of $\dot{x}(t)$ uniquely. To guarantee this I assume (for simplicity) that

$$\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0. \tag{9}$$

The assumption can always be made "without loss of generality" (by factoring out an unreachable and unobservable part of the state space).

Power Balance Equation

A conservative s/s system

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \right. \tag{3}$$

preserves energy, and so does the *dual system*. Preservation of energy means that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|_{\mathcal{X}}^2 = [w(t), w(t)]_{\mathcal{W}}. \tag{10}$$

Here $\frac{1}{2}\|x(t)\|_{\mathcal{X}}^2$ is the internal energy stored state at time t (= the Hamiltonian), and $\frac{1}{2}[w(t), w(t)]_{\mathcal{W}}$ represents the power entering into the system from the outside world. Thus, if we want to allow the energy to flow in both directions, then we must allow the right-hand side to take both positive and negative values, and we cannot replace the indefinite inner product $[\cdot, \cdot]_{\mathcal{W}}$ in $\mathcal{W}$ by a positive definite Hilbert space inner product $(\cdot, \cdot)_{\mathcal{W}}$ in $\mathcal{W}$.

The Node Space $\mathfrak{K}$

By carrying out the differentiation in the power balance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} = [w(t), w(t)]_{\mathcal{W}} \tag{10}$$

we get the *Lagrangian identity*

$$-(\dot{x}(t), x(t))_{\mathcal{X}} - (x(t), \dot{x}(t))_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} = 0. \tag{11}$$

At $t = 0$ the vector $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix}$ can be an arbitrary vector in V, and hence (11) with $t = 0$ implies

$$-(z, x)_{\mathcal{X}} - (x, z)_{\mathcal{X}} + [w, w]_{\mathcal{W}} = 0, \qquad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V. \tag{12}$$

This inequality says that V is a neutral subspace of the node space $\mathfrak{K}$ with respect to a suitable indefinite inner product!

Define

$$\left[\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix}\right]\right]_{\mathfrak{K}} = \left(\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, J_{\mathfrak{K}} \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix}\right)_{\mathfrak{K}}, \quad J_{\mathfrak{K}} := \begin{bmatrix} 0 & -1_{\mathcal{X}} & 0 \\ -1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix}. \tag{13}$$

Then

$$-(z,x)_{\mathcal{X}} - (x,z)_{\mathcal{X}} + [w,w]_{\mathcal{W}} = 0, \qquad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \tag{12}$$

says that

$$\left[\left[\begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix}\right]\right]_{\mathfrak{K}} = 0, \qquad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V. \tag{14}$$

In other words, V is a neutral subspace of the node space $\mathfrak{K}$ with respect to the inner product (13).
Equivalently, $V \subset V^{[\perp]}$!

We get the dual system by replacing V by $V^{[\perp]}$. The duals system preserves energy if $V^{[\perp]}$ is neutral, i.e., if $V^{[\perp]} \subset V$.

Definition

The state/signal system

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \right. \tag{3}$$

is conservative if V is Lagrangian, i.e., if $V = V^{[\perp]}$.

By a Lagrangian decomposition of the Kreĭn signal space $\mathcal{W}$ we mean a direct sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ where both $\mathcal{U}$ and $\mathcal{Y}$ are Lagrangian subspaces of $\mathcal{W}$, i.e., $\mathcal{U} = \mathcal{U}^{[\perp]}$ and $\mathcal{Y} = \mathcal{Y}^{[\perp]}$. With suitable choices of norms in $\mathcal{U}$ and $\mathcal{Y}$ we can write the inner product in $\mathcal{W}$ in the form

$$[y_1 + u_1, y_2 + u_2]_{\mathcal{W}} = (\Psi y_1, u_2)_{\mathcal{U}} + (u_1, \Psi y_2)_{\mathcal{U}}, \qquad (15)$$

for all u_1, $u_2 \in \mathcal{U}$, and y_1, $y_2 \in \mathcal{Y}$, and for some unitary operator $\Psi : \mathcal{U} \to \mathcal{Y}$. We then write $\mathcal{W} = \mathcal{U} \overset{\Psi}{+} \mathcal{Y}$.

Boundary Relation = Generating Subspace

Answer to question "Where do they come from"?:
A boundary relation $\simeq$ the generating subspace V of a conservative s/s system which has been reinterpreted as a relation.

Theorem

Let $(V;\mathcal{X},\mathcal{W})$ be a conservative s/s node and assume that there exists a Lagrangian decomposition $\mathcal{W}=\mathcal{U}\overset{\Psi}{\dot{+}}\mathcal{Y}$. Interpret V as the (slightly modified) graph of a relation $\Gamma\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{U}\end{smallmatrix}\right]$:

$$V=\left\{\left[\begin{smallmatrix} iz \\ x \\ \left[\begin{smallmatrix} u \\ i\Psi^* y\end{smallmatrix}\right]\end{smallmatrix}\right]\in\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]\;\middle|\;\{\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right],\left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]\}\in\Gamma\right\}, \tag{16}$$

and set $R:=\operatorname{Ker}(\Gamma)$. Then R is a closed symmetric operator in $\mathcal{X}$, R^* is the closure of $\operatorname{dom}(\Gamma)$ in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\end{smallmatrix}\right]$, and Γ is a conservative boundary relation for R^*.

Boundary control s/s system $\Rightarrow$ Γ is an operator.

The Characteristic Manifold

Taking Laplace transforms in the formula $\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ for all $t > 0$, we get

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x(0) \\ \hat{x}(\lambda) \\ \widehat{w}(\lambda) \end{bmatrix} \in V, \quad \lambda \in \mathbb{C}^+. \tag{17}$$

Definition

The characteristic manifold of the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is the family of subspaces $\widehat{\mathfrak{V}}(\lambda)$ defined by

$$\widehat{\mathfrak{V}}(\lambda) = \left\{ \begin{bmatrix} x \\ x_0 \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{bmatrix} \lambda x - x_0 \\ x \\ w \end{bmatrix} \in V \right\}. \tag{18}$$

The domain of $\widehat{\mathfrak{V}}(\lambda)$ consists of all those points $\lambda \in \mathbb{C}$ where this manifold is analytic.

Here $\widehat{\mathfrak{V}}$ is analytic at a point λ_0 if $\widehat{\mathfrak{V}}(\lambda)$ has a graph representation in some neighborhood of λ_0 with an analytic angle operator.

The Weyl Family and the Gamma Field

Theorem

1. *The characteristic manifold $\widehat{\mathfrak{V}}$ is defined and analytic (at least) in the open right-half plane.*
2. *The Weyl family and the Gamma field can be obtained from the characteristic manifold by first intersecting $\widehat{\mathfrak{V}}(\lambda)$ with $\begin{bmatrix} \mathcal{X} \\ 0 \\ \mathcal{W} \end{bmatrix}$, then projecting it onto either $\begin{bmatrix} 0 \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ or $\begin{bmatrix} \mathcal{X} \\ 0 \\ \mathcal{U} \end{bmatrix}$, and finally interpreting the result as a relation.*

Here $\mathcal{U}$ is one of the two components in the Lagrangian decomposition $\mathcal{W} = \mathcal{U} \overset{\Psi}{+} \mathcal{Y}$.

Non-Conservative State/Signal Systems

Above I only discussed conservative state/signal systems.
Question: What happens when the state/signal system is well-posed but not conservative?
Answer:

- We will then have to deal with two different generating subspaces V and $V^{[\perp]} \neq V$, and two different s/s systems $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma^{[\perp]} = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$.
- To each of these s/s systems corresponds a "non-conservative boundary relation".
- Thus, we end up with pairs of boundary relations instead of just one boundary relation.
- In this case the "Lagrangian identity" simply says that the two systems are dual to each other.
- Details will be worked out later.

Conclusion

- Boundary relations = generating subspaces of conservative state/signal systems, reinterpreted as relations.
- The Weyl family and the Gamma fields are obtained from the characteristic manifold of the state/signal system by intersections and projections.
- Pairs of boundary relations are related to non-conservative state/signal systems.
- Boundary relations do not in reality have much to do with boundary control, only historically.

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