Feedback Representations of Critical Controls for Well-Posed Linear Systems

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Abstract. This is the first part in a three part study on the suboptimal full information H^∞ problem for a well-posed linear system with input space U, state space H, and output space Y. We define a cost function $Q(x_0, u) = \int_{\mathbf{R}^+} \langle y(s), Jy(s)\rangle_Y \, ds$, where $y \in L^2_{\mathrm{loc}}(\mathbf{R}^+; Y)$ is the output of the system with initial state $x_0 \in H$ and control $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+; U)$, and J is a self-adjoint operator on Y. The cost function Q is quadratic in x_0 and u, and we suppose (in the stable case) that the second derivative of $Q(x_0, u)$ with respect to u is nonsingular. This implies that, for each $x_0 \in H$, there is unique critical control u^{crit} such that the derivative of $Q(x_0, u)$ with respect to u vanishes at $u = u^{\mathrm{crit}}$. We show that u^{crit} can be written in feedback form whenever the input/output map of the system has a coprime factorization with a (J, S)-inner numerator; here S is a particular self-adjoint operator on U. A number of properties of this feedback representation are established, such as the equivalence of the (J, S)-losslessness of the factorization and the positivity of the Riccati operator on the reachable subspace.

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1 Introduction

In three earlier papers Staffans [1997 1998ab] we have studied the standard and nonstandard quadratic cost minimization problems for well-posed linear systems. This is the first out of three papers where we develop a similar theory for the minimax setting on which the standard full information H^∞-results are based. Many of our proofs rely on results proved in Staffans [1997 1998ab], and we refer the readers to these papers for some of the details.

We briefly recapitulate what was done in Staffans [1997 1998ab]. Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a well-posed linear system with input space U, state space H, and output space Y (see Section 2 for a short review of these systems.) In the standard quadratic cost minimization problem the standard cost function

$$Q(x_0, u) = \int_{\mathbf{R}^+} \langle y(s), y(s) \rangle_Y \; ds = \|y\|^2_{L^2(\mathbf{R}^+;Y)} \tag{1.1}$$

is minimized with respect to $u \in L^2(\mathbf{R}^+; U)$; here

$$y = \mathcal{C}x_0 + \mathcal{D}\pi_+ u$$

is the observation of Ψ with initial value $x_0 \in H$ and control u.[1] The stable case where the mapping from x_0 and u into the output $y = \mathcal{C}x_0 + \mathcal{D}\pi_+ u$ is continuous from $H \times L^2(\mathbf{R}^+; U)$ into $L^2(\mathbf{R}^+; Y)$ is discussed in Staffans [1997]. In this case Q is a bounded quadratic function of x_0 and u, and it is assumed that Q is uniformly convex with respect to u. This implies that, for each $x_0 \in H$, there is a unique minimizing control $u^{\min}(x_0)$. We can find $u^{\min}(x_0)$ by differentiating $Q(x_0, u)$ with respect to u and setting the result equal to zero. As shown in Staffans [1997], the solution to this problem is closely related to a canonical spectral factorization problem.

To get the nonstandard minimization problem we replace the cost function Q by the more general cost function

$$Q(x_0, u) = \int_{\mathbf{R}^+} \langle y(s), Jy(s) \rangle_Y \; ds, \tag{1.2}$$

where $J \in \mathcal{L}(Y)$ is self-adjoint (but not positive in general). Basically, this amounts to replacing the standard inner product in Y by a nonstandard

[1] Usually an additional term $\int_{\mathbf{R}^+} \langle u(s), Ru(s) \rangle_U \; ds$ is added to (1.1), but we shall not do so here since it is possible to absorb this term into the integral $\int_{\mathbf{R}^+} \langle y(s), y(s) \rangle_Y \; ds$ by adding a copy of the control to the output; cf. formula (1.3) below.

(non-definite) inner product $\langle \cdot, J\cdot \rangle_Y$. However, it is assumed that, in spite of the fact that J need not be positive, the functional Q is still strictly convex with respect to u. This is the situation encountered in, e.g., the bounded real and positive real lemmas. Since Q is strictly convex, the same technique that we use to solve the standard problem applies to the nonstandard problem as well. See [Staffans 1998b, Sections 2 and 8].

The unstable case is treated in Staffans [1998ab] through the use of a preliminary stabilizing feedback that reduces the unstable case to the stable one. In that case $Q(x_0, u)$ is still strictly convex with respect to u, but it is now unbounded. Instead of solving a canonical spectral factorization problem we have to solve a canonical coprime factorization problem.

In the full information H^∞ case we use the same cost function (1.2), but Q is no longer required to be strictly convex. Instead it is assumed that $Q(u, x_0)$ has a saddle point with respect to u for each x_0 in H. In the standard formulation of the full information H^∞ problem the input space U is separated into its positive part U^+ (governed by the minimizing player, the control engineer) and its negative part U^- (governed by the maximizing player, nature). In addition, it is customary to add a copy of nature's part of the control to the output, and to write the self-adjoint operator J in (1.2) as an operator $J = \left[\begin{smallmatrix} Q & L^* \\ L & R \end{smallmatrix}\right]$ mapping $Y \times U^-$ into itself. In this formulation we have to replace $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ by the bigger system

$$\Psi = \begin{bmatrix} \mathcal{A} & \begin{bmatrix} \mathcal{B}^+ & \mathcal{B}^- \end{bmatrix} \\ \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} & \begin{bmatrix} \mathcal{D}^+ & \mathcal{D}^- \\ 0 & I \end{bmatrix} \end{bmatrix} \tag{1.3}$$

with two independent (vector) inputs and two (vector) outputs. The formulas become more clumsy, and much of the simplicity of the solutions of the standard and nonstandard quadratic cost minimization problems in Staffans [1997 1998ab] is lost. Since the minimax structure of the problem is important only in certain parts of our solution to the full information H^∞ problem, and since it actually obscures other parts of the solution, we have split the solution into three parts. In this first part we develop those aspects of the theory which have nothing to do with the minimax nature of the problem, and postpone a more complete treatment of the H^∞ problem to Staffans [1998c] (the stable case) and Staffans [1999] (the unstable case). This makes it possible to treat a slightly more general case in this paper. In the full information H^∞ problem it is usually assumed that the negative parts of the input and output spaces have the same dimension; no such assumption is made here. Thus, the results presented here apply equally well to the H^∞ case, and to the standard and nonstandard quadratic cost minimization cases. In addition, we expect the theory to be applicable in some intermediate cases where only partial stabilization is achieved.

The key observation on which this paper is based is that if the system is stable and if the second derivative of $Q(x_0, u)$ with respect to u is nonsingular, then $Q(x_0, u)$ has a unique saddle point u^{crit} with respect to u for each $x_0 \in H$. This critical point is characterized by the fact that the derivative of $Q(x_0, u)$ with respect to u vanishes at u^{crit}, which is exactly the same property that was used to develop the theory presented in Staffans [1997 1998ab]. Thus, that theory can be extended to the case where Q has a saddle point (instead of being strictly convex) with minimal changes. One obvious complication is that we have to be careful with the definition of a "saddle point" in the unstable case where Q is unbounded. However, the main difference is that we have to introduce a nonstandard inner product $\langle \cdot, S\cdot \rangle_U$ in the input space U as well; recall that the difference between the standard and the nonstandard quadratic cost minimization problems is that in the nonstandard problem we use a nonstandard inner product $\langle \cdot, J\cdot \rangle_Y$ in Y.

The following are our main results. In Theorem 5.1 we show that in the stable case the problem of finding a critical control of state feedback type is equivalent to the problem of finding a (J, S)-inner-outer factorization of the input/output map $\mathcal{D}$ of the system. The corresponding result for a (unstable) jointly stabilizable and detectable system is given in Theorem 7.6. As Remarks 5.2 and 7.7 say, virtually all the additional conclusions about the factorization that could be drawn in the settings of Staffans [1997 1998b] remain valid. We show in Theorems 6.5 and 7.8 that the factorizations are (J, S)-lossless if and only is the Riccati operator is nonnegative on the reachable subspace.

We use the following set of notations.

$\mathcal{L}(U;Y)$, $\mathcal{L}(U)$: The set of bounded linear operators from U into Y or from U into itself, respectively.

I: The identity operator.

A^*: The (Hilbert space) adjoint of the operator A.

$\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{R}^-$: $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}^+ = [0, \infty)$, and $\mathbf{R}^- = (-\infty, 0]$.

$L^2(J;U)$: The set of U-valued L^2-functions on the interval J.

$L^2_\omega(J;U)$: $L^2_\omega(J;U) = \{ u \in L^2_{\mathrm{loc}}(J;U) \mid (t \mapsto \mathrm{e}^{-\omega t}u(t)) \in L^2(J;U) \}$.

$H^\infty_\omega(U;Y)$: The set of bounded analytic $\mathcal{L}(U;Y)$-valued functions over the half-plane $\Re z > \omega$, with the sup-norm.

$TI_\omega(U;Y)$, $TI_\omega(U)$: The set of bounded linear time-invariant operators from $L^2_\omega(\mathbf{R};U)$ into $L^2_\omega(\mathbf{R};Y)$, or from $L^2_\omega(\mathbf{R};U)$ into itself.

$TIC_\omega(U;Y)$, $TIC_\omega(U)$: The set of causal operators in $TI_\omega(U;Y)$ or $TI_\omega(U)$.

$\langle\cdot,\cdot\rangle_H$: The inner product in the Hilbert space H.

$\tau(t)$: The time shift group $\tau(t)u(s) = u(t+s)$ (this is a left-shift when $t > 0$ and a right-shift when $t < 0$).

π_J: $(\pi_J u)(s) = u(s)$ if $s \in J$ and $(\pi_J u)(s) = 0$ if $s \notin J$. Here J is a subset of $\mathbf{R}$.

π_+, π_-: $\pi_+ = \pi_{\mathbf{R}^+}$ and $\pi_- = \pi_{\mathbf{R}^-}$.

We extend an L^2_ω-function u defined on a subinterval J of $\mathbf{R}$ to the whole real line by requiring u to be zero outside of J, and we denote the extended function by $\pi_J u$. Thus, we use the same symbol π_J both for the embedding operator $L^2_\omega(J) \to L^2_\omega(\mathbf{R})$ and for the corresponding projection operator $L^2_\omega(\mathbf{R}) \to L^2_\omega(J)$. With this interpretation, $\pi_J L^2_\omega(\mathbf{R};U) = L^2_\omega(J;U) \subset L^2_\omega(\mathbf{R};U)$ for each interval $J \subset \mathbf{R}$.

Square brackets [ ] are used to denote optional parts of a statement. Such a statementis valid if all the text within square brackets is omitted, and also if the appropriate parts of the statement are replaced by the text in the brackets.

2 A Short Review of Well-Posed Linear Systems

The basic notions that we use are the same as in Staffans [1998a], and we refer the reader to that paper and to Staffans [1997] for more details. (Our notion of a well-posed linear system is the same as the notions in Salamon [1987 1989] and Weiss [1994ab], but we use a slightly modified set of axioms that has been adapted to the needs of the H^∞ theory).

In order to formulate the axioms satisfied by a well-posed linear system we introduce exponentially weighted L^2-spaces. For each Hilbert space U and each $\omega \in \mathbf{R}$ we let $L^2_\omega(\mathbf{R};U)$ be the weighted L^2-space

$$L^2_\omega(\mathbf{R};U) = \{ u \in L^2_{\mathrm{loc}}(\mathbf{R};U) \mid \big(t \mapsto \mathrm{e}^{-\omega t}u(t)\big) \in L^2(\mathbf{R};U) \}.$$

This is a Hilbert space with the natural norm $\|e^{-\omega\cdot}u(\cdot)\|_{L^2(\mathbf{R};U)}$. We also need the "past time" projection π_-, the "future time" projection π_+, and the "time shift" group $\tau(t)$ that operate on functions $u \in L^2_\omega(\mathbf{R};U)$ in the following way:

$$(\pi_- u)(s) = \begin{cases} u(s) & \text{if } s \in \mathbf{R}^-, \\ 0 & \text{if } s \in \mathbf{R}^+, \end{cases}$$
$$(\pi_+ u)(s) = \begin{cases} u(s) & \text{if } s \in \mathbf{R}^+, \\ 0 & \text{if } s \in \mathbf{R}^-, \end{cases}$$
$$(\tau(t)u)(s) = u(t+s), \quad t, s \in \mathbf{R}.$$

A bounded linear operator from $L^2_\omega(\mathbf{R};U)$ into $L^2_\omega(\mathbf{R};Y)$ is time-invariant if it commutes with the time shift τ. We denote this class of operators by $TI_\omega(U;Y)$, and the class of causal operators in $TI_\omega(U;Y)$ by $TIC_\omega(U;Y)$. In the case where $\omega = 0$ we simply write $TIC(U;Y)$.

Definition 2.1 *Let U, H, and Y be Hilbert spaces, and let $\omega \in \mathbf{R}$. A (causal) ω-stable well-posed linear system on (U, H, Y) is a quadruple $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are bounded linear operators of the following type:*

(i) $\mathcal{A}(t)\colon H \to H$ is a strongly continuous semigroup of bounded linear operators on H satisfying $\sup_{t\in\mathbf{R}^+} \|e^{-\omega t}\mathcal{A}(t)\| < \infty$;

(ii) $\mathcal{B}\colon L^2_\omega(\mathbf{R};U) \to H$ satisfies $\mathcal{A}(t)\mathcal{B}u = \mathcal{B}\tau(t)\pi_- u$ for all $u \in L^2_\omega(\mathbf{R};U)$ and all $t \in \mathbf{R}^+$;

(iii) $\mathcal{C}\colon H \to L^2_\omega(\mathbf{R};Y)$ satisfies $\mathcal{C}\mathcal{A}(t)x = \pi_+\tau(t)\mathcal{C}x$ for all $x \in H$ and all $t \in \mathbf{R}^+$;

(iv) $\mathcal{D}\colon L^2_\omega(\mathbf{R};U) \to L^2_\omega(\mathbf{R};Y)$ satisfies $\tau(t)\mathcal{D}u = \mathcal{D}\tau(t)u$, $\pi_-\mathcal{D}\pi_+ u = 0$, and $\pi_+\mathcal{D}\pi_- u = \mathcal{C}\mathcal{B}u$ for all $u \in L^2_\omega(\mathbf{R};U)$ and all $t \in \mathbf{R}$.

If, in addition, $e^{-\omega t}\mathcal{A}(t)x \to 0$ as $t \to \infty$ for all $x \in H$, then Ψ is strongly ω-stable. The system Ψ is [strongly] stable if it is [strongly] ω-stable with $\omega = 0$, and it is exponentially stable if it is ω-stable for some $\omega < 0$.

The different components of Ψ are called as follows: U is the input space, H is the state space, Y is the output space, $\mathcal{A}$ is the semigroup, $\mathcal{B}$ is the

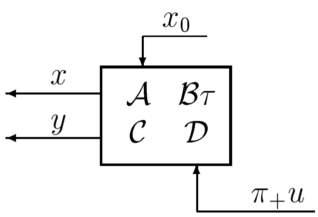

Figure 1: Input/state/output diagram of Ψ

controllability map, $\mathcal{C}$ is the observability map, and is $\mathcal{D}$ the input/output map of Ψ. In the initial value setting with initial time zero, initial value $x_0 \in H$, and control $u \in L^2_\omega(\mathbf{R}; U)$, the controlled state $x(t)$ at time $t \in \mathbf{R}^+$ and the observation y of Ψ are given by

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t) & \mathcal{B}_{\mathcal{T}}(t) \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x_0 \\ \pi_+ u \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t)x_0 + \mathcal{B}_{\mathcal{T}}(t)\pi_+ u \\ \mathcal{C}x_0 + \mathcal{D}\pi_+ u \end{bmatrix}. \tag{2.1}$$

We call Ψ a well-posed linear system on (U, H, Y) if it is an ω-stable well-posed linear system on (U, H, Y) for some $\omega \in \mathbf{R}$.

We remark that the condition imposed on $\mathcal{D}$ in Definition 2.1 requires that $\mathcal{D} \in TIC_\omega(U; Y)$ (i.e., $\mathcal{D}$ is time-invariant and causal with growth rate ω), and that the Hankel operator induced by $\mathcal{D}$ is equal to $\mathcal{C}\mathcal{B}$. Intuitively, the controllability map $\mathcal{B}$ maps past controls into the present state, the observability map $\mathcal{C}$ maps the present state into future observations, and the input/output map $\mathcal{D}$ maps inputs into outputs in a causal way.

The axioms listed above describe standard properties of the corresponding operators for systems with bounded control and observation operators B and

C. For such systems, we have

$$\begin{aligned}
\mathcal{B}u &= \int_{-\infty}^{0} \mathcal{A}(-s)Bu(s)\,ds, \\
\mathcal{C}x &= (t \mapsto C\mathcal{A}(t)x)\,, \\
\mathcal{D}u &= \left(t \mapsto \int_{-\infty}^{t} C\mathcal{A}(t-s)Bu(s)\,ds + Du(t)\right), \\
x(t) &= \mathcal{A}(t)x_0 + \int_0^t \mathcal{A}(t-s)Bu(s)\,ds, \quad t \in \mathbf{R}^+, \\
y(t) &= C\mathcal{A}(t)x_0 + \int_0^t C\mathcal{A}(t-s)Bu(s)\,ds + Du(t), \quad t \in \mathbf{R}^+,
\end{aligned}$$

where D is a given bounded linear feedthrough operator.

We use diagrams of the type drawn in Figure 1 to represent the relation between the state $x(t)$, the observation y, the initial value x_0, and the control u defined in (2.1). Throughout in our diagrams we use the following conventions:

(i) Initial states and controls enter at the top and the bottom, and they are fed into all the operators located in the column to which they are attached. In particular, note that x_0 is attached to the first column and u to the second.

(ii) Final states and observations leave to the left and the right, and they are the sums of all the elements in the row to which they are attached. In particular, note that $x(t)$ is attached to the top row, and y to the bottom row.

3 Critical Controls for Stable Systems

This work is built around the notion of a *critical control* for a stable well-posed linear system:

Definition 3.1 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a stable well-posed linear system on (U, H, Y), and let $J = J^* \in \mathcal{L}(Y)$. Define*

$$Q(x_0, u) = \int_{\mathbf{R}^+} \langle y(s), Jy(s)\rangle_Y \; ds,$$

where

$$y = \mathcal{C}x_0 + \mathcal{D}\pi_+ u$$

is the observation of Ψ *with initial value* $x_0 \in H$ *and control* $u \in L^2(\mathbf{R}^+; U)$. *The control* $u^{\mathrm{crit}}(x_0)$ *is* J*-critical if the (real) Fréchet derivative of* Q *with respect to* u *vanishes at* $(x_0, u^{\mathrm{crit}}(x_0))$.

Lemma 3.2 *The control* $u^{\mathrm{crit}}(x_0)$ *is* J*-critical if and only if the critical observation* $y^{\mathrm{crit}}(x_0) = \mathcal{C}x_0 + \mathcal{D}\pi_+ u^{\mathrm{crit}}(x_0)$ *satisfies*

$$\pi_+\mathcal{D}^* J y^{\mathrm{crit}}(x_0) = \pi_+\mathcal{D}^* J \left(\mathcal{C}x_0 + \mathcal{D}\pi_+ u^{\mathrm{crit}}(x_0)\right) = 0. \tag{3.1}$$

Proof. Without loss of generality, let us suppose that U is a real Hilbert space (if not, then we replace the inner product in U by the real inner product $\Re\langle\cdot,\cdot\rangle$). For each variation $\eta \in L^2(\mathbf{R}^+; U)$, the Fréchet derivative of $Q(x_0, u)$ with respect to u is given by

$$\begin{aligned} dQ(x_0,u)\eta &= 2\left\langle \mathcal{C}x_0 + \mathcal{D}\pi_+ u, J\mathcal{D}\pi_+\eta\right\rangle_{L^2(\mathbf{R}^+;U)} \\ &= 2\left\langle y, J\mathcal{D}\pi_+\eta\right\rangle_{L^2(\mathbf{R}^+;U)} \\ &= 2\left\langle \mathcal{D}^* J y, \eta\right\rangle_{L^2(\mathbf{R}^+;U)}. \end{aligned}$$

This is zero for all $\eta \in L^2(\mathbf{R}^+; U)$ if the critical observation $y^{\mathrm{crit}}(x_0) = \mathcal{C}x_0 + \mathcal{D}\pi_+ u^{\mathrm{crit}}(x_0)$ satisfies (3.1). $\square$

We shall throughout require the input/output map $\mathcal{D}$ of Ψ to be coercive in the following sense:

Definition 3.3 *Let* $J = J^* \in \mathcal{L}(Y)$. *An operator* $\mathcal{D} \in TIC(U;Y)$ *is* J*-coercive if the Toeplitz operator* $\pi_+\mathcal{D}^* J\mathcal{D}\pi_+$ *is invertible in* $\mathcal{L}(L^2(\mathbf{R}^+;U))$. *A stable well-posed linear system* $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ *on* (U,H,Y) *is* J*-coercive if its input/output map* $\mathcal{D}$ *is* J*-coercive.*

We remark that the symbol of the Toeplitz operator $\pi_+\mathcal{D}^* J\mathcal{D}\pi_+$ is known under the name "Popov function" (see, e.g., Weiss [1997]).

Lemma 3.4 *Let* $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ *be a stable* J*-coercive well-posed linear system on* (U,H,Y) *where* $J = J^* \in \mathcal{L}(Y)$. *Then, for every* $x_0 \in H$, *there is a unique* J*-critical control* $u^{\mathrm{crit}}(x_0)$, *namely*

$$u^{\mathrm{crit}}(x_0) = -(\pi_+\mathcal{D}^* J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^* J\mathcal{C}x_0. \tag{3.2}$$

The corresponding critical state $x^{\rm crit}(x_0)$, the critical observation $y^{\rm crit}(x_0)$, and the critical value of Q are given by

$$x^{\rm crit}(x_0) = \mathcal{A}x_0 - \mathcal{B}\tau\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\mathcal{C}x_0, \tag{3.3}$$

$$y^{\rm crit}(x_0) = \big(I - \mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\big)\,\mathcal{C}x_0, \tag{3.4}$$

$$Q(x_0, u^{\rm crit}(x_0)) = \big\langle x_0, \mathcal{C}^*\left(J - J\mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\right)\mathcal{C}x_0\big\rangle_H. \tag{3.5}$$

Furthermore, for every $\eta \in L^2(\mathbf{R}^+;U)$, we have

$$Q(x_0, u^{\rm crit}(x_0)+\eta) = Q(x_0, u^{\rm crit}(x_0)) + \langle \mathcal{D}\pi_+\eta, J\mathcal{D}\pi_+\eta\rangle_{L^2(\mathbf{R}^+;Y)}. \tag{3.6}$$

Proof. If Ψ is J-coercive, then $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ can be inverted, and we get (3.2) from (3.1). By substituting this value for $u^{\rm crit}(x_0)$ into $x^{\rm crit}(t,x_0) = \mathcal{A}(t)x_0 + \mathcal{B}\tau(t)\pi_+u^{\rm crit}(x_0)$, $y^{\rm crit}(x_0) = \mathcal{C}x_0 + \mathcal{D}\pi_+u^{\rm crit}(x_0)$, and $Q(x_0, u^{\rm crit}(x_0))$ (and making a straightforward computation) we get (3.3), (3.4), and (3.5). To derive (3.6) it suffices to expand $Q(x_0, u^{\rm crit}(x_0)+\eta)$ and use (3.1). $\square$

Definition 3.5 *Under the hypotheses of Lemma 3.4, define*

$$\begin{aligned}
\mathcal{A}_\circlearrowleft &= \mathcal{A} - \mathcal{B}\tau\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\mathcal{C},\\
\mathcal{C}_\circlearrowleft &= \big(I - \mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\big)\,\mathcal{C},\\
\mathcal{K}_\circlearrowleft &= -(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\mathcal{C},\\
\Pi &= \mathcal{C}^*\left(J - J\mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\right)\mathcal{C}.
\end{aligned}$$

The operator Π is called the Riccati operator of Ψ (with cost operator J).

Thus (according to Lemma 3.4), $x^{\rm crit}(x_0) = \mathcal{A}_\circlearrowleft x_0$, $y^{\rm crit}(x_0) = \mathcal{C}_\circlearrowleft x_0$, $u^{\rm crit}(x_0) = \mathcal{K}_\circlearrowleft x_0$, and $Q(x_0, u^{\rm crit}(x_0)) = \langle x_0, \Pi x_0\rangle_H$. Moreover, by (3.1)

$$\pi_+\mathcal{D}^*J\mathcal{C}_\circlearrowleft = 0. \tag{3.7}$$

Remark 3.6 *Observe that the critical operators $\mathcal{A}_\circlearrowleft$, $\mathcal{C}_\circlearrowleft$, $\mathcal{K}_\circlearrowleft$, and Π depend only on Ψ and on J. In particular, they do not depend on the operator S, which will be formally introduced later, in spite of the fact that S appears in many of the subsequent formulas.*

Lemma 3.7 *Make the same assumption as in Lemma 3.4, and introduce the same notations as in Definition 3.5. Then the following claims are true:*

(i) The operators $\mathcal{A}_{\circlearrowright}$, $\mathcal{C}_{\circlearrowright}$ and Π satisfy

$$\begin{aligned}\mathcal{A}_{\circlearrowright} &= \mathcal{A} + \mathcal{B}\tau\mathcal{K}_{\circlearrowright},\\ \mathcal{C}_{\circlearrowright} &= \mathcal{C} + \mathcal{D}\mathcal{K}_{\circlearrowright},\\ \Pi &= \mathcal{C}_{\circlearrowright}^{*}J\mathcal{C}_{\circlearrowright} = \mathcal{C}^{*}J\mathcal{C}_{\circlearrowright} = \mathcal{C}_{\circlearrowright}^{*}J\mathcal{C}.\end{aligned}$$

(ii) $\mathcal{A}_{\circlearrowright}$ is a strongly continuous, bounded semigroup on H, and $\mathcal{C}_{\circlearrowright}$ and $\mathcal{K}_{\circlearrowright}$ are admissible stable observability maps for $\mathcal{A}_{\circlearrowright}$ in the sense that $\mathcal{C}_{\circlearrowright} \in \mathcal{L}(H; L^2(\mathbf{R}; Y))$, $\mathcal{K}_{\circlearrowright} \in \mathcal{L}(H; L^2(\mathbf{R}; U))$, and

$$\begin{aligned}\mathcal{C}_{\circlearrowright}\mathcal{A}_{\circlearrowright}(t) &= \pi_{+}\tau(t)\mathcal{C}_{\circlearrowright},\\ \mathcal{K}_{\circlearrowright}\mathcal{A}_{\circlearrowright}(t) &= \pi_{+}\tau(t)\mathcal{K}_{\circlearrowright},\end{aligned}$$

for all $t \in \mathbf{R}^{+}$.

(iii) For all $t \in \mathbf{R}^{+}$,

$$\Pi = \mathcal{C}_{\circlearrowright}^{*}J\pi_{[0,t]}\mathcal{C}_{\circlearrowright} + \mathcal{A}_{\circlearrowright}^{*}(t)\Pi\mathcal{A}_{\circlearrowright}(t) \tag{3.8}$$

and

$$\pi_{[0,t]}\left(\mathcal{D}^{*}J\pi_{[0,t]}\mathcal{C}_{\circlearrowright} + \tau(-t)\mathcal{B}^{*}\Pi\mathcal{A}_{\circlearrowright}(t)\right) = 0. \tag{3.9}$$

Proof. (i) This follows from Lemma 3.4 and Definition 3.5.

(ii) We fix some $t \in \mathbf{R}^{+}$, and write u in the form $u = u^{\mathrm{crit}}(x_0) + \eta_1 + \eta_2$, where $\eta_1 = \pi_{[0,t]}\eta$, $\eta_2 = \pi_{[t,\infty)}\eta$, and $\eta \in L^2(\mathbf{R}^{+}; U)$ is arbitrary. Then

$$\begin{aligned}x(t) &= \mathcal{A}(t)x_0 + \mathcal{B}\tau(t)\pi_{+}u = x^{\mathrm{crit}}(t, x_0) + \mathcal{B}\tau(t)\eta_1,\\ \pi_{[0,t]}y &= \pi_{[0,t]}\left(\mathcal{C}x_0 + \mathcal{D}\pi_{+}u\right) = \pi_{[0,t]}\left(y^{\mathrm{crit}}(x_0) + \mathcal{D}\eta_1\right),\\ \pi_{+}\tau(t)y &= \mathcal{C}x(t) + \mathcal{D}\pi_{+}\tau(t)u = \pi_{+}\tau(t)y^{\mathrm{crit}}(x_0) + \mathcal{C}\mathcal{B}\tau(t)\eta_1 + \mathcal{D}\tau(t)\eta_2.\end{aligned}$$

We furthermore write $Q(x_0, u)$ in the form

$$\begin{aligned}Q(x_0, u) &= \int_0^t \langle y(s), Jy(s)\rangle_Y \, ds + \int_t^\infty \langle y(s), Jy(s)\rangle_Y \, ds\\ &= \int_0^t \langle y(s), Jy(s)\rangle_Y \, ds + \int_0^\infty \langle (\tau(t)y)(s), J(\tau(t)y)(s)\rangle_Y \, .\end{aligned}$$

Let us for a moment choose η_1 to be zero. Then $x(t) = x^{\text{crit}}(t, x_0)$ and $\pi_{[0,t]} y = \pi_{[0,t]} y^{\text{crit}}(x_0)$. Since $u^{\text{crit}}(x_0)$ is J-critical, the derivative of $Q(x_0, u)$ with respect to η_2 must vanish at the point $\eta_2 = 0$, and this implies that (cf. Lemma 3.2)

$$\pi_+ \mathcal{D}^* J \pi_+ \tau(t) y^{\text{crit}}(x_0) = \pi_+ \mathcal{D}^* J \left(\mathcal{C} x^{\text{crit}}(t, x_0) + \mathcal{D} \pi_+ \tau(t) u^{\text{crit}}(x_0) \right) = 0,$$

i.e., (3.1) holds with x_0, $y^{\text{crit}}(x_0)$, and $u^{\text{crit}}(x_0)$ replaced by $x^{\text{crit}}(t, x_0) = \mathcal{A}_\circlearrowleft(t) x_0$, $\pi_+ \tau(t) y^{\text{crit}}(x_0) = \pi_+ \tau(t) \mathcal{C}_\circlearrowleft x_0$, and $\pi_+ \tau(t) u^{\text{crit}}(x_0) = \pi_+ \tau(t) \mathcal{K}_\circlearrowleft x_0$, respectively. By Lemma 3.4 and Definition 3.5, this implies that

$$\begin{aligned}
\pi_+ \tau(t) \mathcal{K}_\circlearrowleft x_0 &= \mathcal{K}_\circlearrowleft x^{\text{crit}}(t, x_0) = \mathcal{K}_\circlearrowleft \mathcal{A}_\circlearrowleft(t) x_0, \\
\pi_+ \tau(t) \mathcal{C}_\circlearrowleft x_0 &= \mathcal{C}_\circlearrowleft x^{\text{crit}}(t, x_0) = \mathcal{C}_\circlearrowleft \mathcal{A}_\circlearrowleft(t) x_0, \\
\mathcal{A}_\circlearrowleft(s + t) x_0 &= \mathcal{A}_\circlearrowleft(s) x^{\text{crit}}(t, x_0) = \mathcal{A}_\circlearrowleft(s) \mathcal{A}_\circlearrowleft(t) x_0.
\end{aligned}$$

Thus, $\mathcal{A}_\circlearrowleft$ is a semigroup, and $\mathcal{K}_\circlearrowleft$ and $\mathcal{C}_\circlearrowleft$ are admissible observability maps for $\mathcal{A}_\circlearrowleft$. The strong continuity and boundedness of $\mathcal{A}_\circlearrowleft$ are immediate.

(iii) Formula (3.8) follows from the fact that

$$\begin{aligned}
\langle x_0, \Pi x_0 \rangle_H &= Q(x_0, u^{\text{crit}}(x_0)) \\
&= \int_0^t \left\langle y^{\text{crit}}(x_0, s), J y^{\text{crit}}(x_0, s) \right\rangle_Y \, ds + Q(x^{\text{crit}}(t, x_0), \pi_+ \tau(t) u^{\text{crit}}(x_0)) \\
&= \int_0^t \langle (\mathcal{C}_\circlearrowleft x_0)(s), J(\mathcal{C}_\circlearrowleft x_0)(s) \rangle_Y \, ds + \langle \mathcal{A}_\circlearrowleft(t) x_0, \Pi \mathcal{A}_\circlearrowleft(t) x_0 \rangle_H .
\end{aligned}$$

To prove (3.9) we instead take $\eta_2 = 0$, differentiate $Q(x_0, u)$ with respect to η_1, and set the result equal to zero. In this way we arrive at the equation

$$\pi_{[0,t]} \left(\mathcal{D}^* J \pi_{[0,t]} \mathcal{C}_\circlearrowleft + \tau(-t) \mathcal{B}^* \mathcal{C}^* J \tau(t) \mathcal{C}_\circlearrowleft \right) = 0.$$

By parts (i) and (ii),

$$\mathcal{C}^* J \tau(t) \mathcal{C}_\circlearrowleft = \mathcal{C}^* J \pi_+ \tau(t) \mathcal{C}_\circlearrowleft = \mathcal{C}^* J \mathcal{C}_\circlearrowleft \mathcal{A}_\circlearrowleft(t) = \Pi \mathcal{A}_\circlearrowleft(t),$$

which substituted into the preceding formula gives (3.9). □

4 Spectral and Inner-Outer Factorizations

In the stable case our feedback/feedforward representation of the critical control $u^{\text{crit}}(x_0)$ depends on the use of a (J, S)-*inner-outer factorization* of the input/output map $\mathcal{D}$, or equivalently on a *S-spectral factorization* of $\mathcal{D}^*J\mathcal{D}$. In this section we therefore present and discuss these factorizations. Most of the results of this section can be found in [Staffans 1998b, Section 2] in the case where S is positive, but there the reader is asked to supply the proofs along the lines of the proofs given in Staffans [1997]. For the convenience of the reader we this time include short proofs.

Definition 4.1 *The operator $A = A^* \in \mathcal{L}(H)$ is positive [uniformly positive] if $\langle x, Ax\rangle \geq 0$ [$\langle x, Ax\rangle \geq \epsilon\|x\|^2$ for some $\epsilon > 0$] for all $x \in H$. It is [uniformly] negative if $-A$ is [uniformly] positive. The notations $A \geq B$ and $B \leq A$ [$A >> B$ and $B << A$] mean that $A - B$ is [uniformly] positive.*

Definition 4.2 *We say that $\mathcal{E} \in TI(U)$ has the (canonical) spectral factorization $\mathcal{Y}^*\mathcal{X}$ if $\mathcal{E}$ can be written in the form $\mathcal{E} = \mathcal{Y}^*\mathcal{X}$ where $\mathcal{Y}$ and $\mathcal{X}$ are invertible in $TIC(U)$ (i.e., both these operators and their inverses are causal, time-invariant, and bounded on $L^2(\mathbf{R}; U)$).*

Lemma 4.3 *Let $\mathcal{E} \in TI(U)$.*

(i) If $\mathcal{E}$ has the spectral factorization $\mathcal{E} = \mathcal{Y}^*\mathcal{X}$, and if $E \in \mathcal{L}(U)$ is invertible, then $((E^{-1})^*\mathcal{Y})^*(E\mathcal{X})$ is another spectral factorization of $\mathcal{E}$. Moreover, every spectral factorization $\mathcal{E} = \widetilde{\mathcal{Y}}^*\widetilde{\mathcal{X}}$ of $\mathcal{E}$ is of this form, i.e., $\widetilde{\mathcal{Y}} = (E^{-1})^*\mathcal{Y}$ and $\widetilde{\mathcal{X}} = E\mathcal{X}$ for some invertible $E \in \mathcal{L}(U)$.

(ii) If $\mathcal{E} = \mathcal{E}^*$ has a spectral factorization $\mathcal{E} = \mathcal{Y}^*\mathcal{X}$, then there is an invertible operator $S = S^* \in \mathcal{L}(U)$ such that $\mathcal{Y} = S\mathcal{X}$. Thus, $\mathcal{E} = \mathcal{X}^*S\mathcal{X}$. (See also Definition 4.5.)

(iii) If $\mathcal{E}$ has a spectral factorization $\mathcal{E} = \mathcal{Y}^*\mathcal{X}$, then the Toeplitz operator $\pi_+\mathcal{E}\pi_+$ is invertible in $\mathcal{L}(L^2(\mathbf{R}^+; U))$, and its inverse is given by $\pi_+(\pi_+\mathcal{E}\pi_+)^{-1}\pi_+ = \mathcal{X}^{-1}\pi_+(\mathcal{Y}^*)^{-1}$.

(iv) $\mathcal{E}$ has a spectral factorization of the form $\mathcal{E} = \mathcal{X}^*\mathcal{X}$ iff $\mathcal{E} >> 0$.

Proof. (i) Obviously, if E is invertible, then $((E^{-1})^*\mathcal{Y})^*(E\mathcal{X})$ is a spectral factorization of $\mathcal{E}$ whenever $\mathcal{Y}^*\mathcal{X}$ is so. Conversely, suppose that we have two different factorization $\mathcal{E} = \mathcal{Y}^*\mathcal{X} = \widetilde{\mathcal{Y}}^*\widetilde{\mathcal{X}}$. Then

$$\widetilde{\mathcal{X}}\mathcal{X}^{-1} = \left(\widetilde{\mathcal{Y}}^*\right)^{-1}\mathcal{Y}^*.$$

The left-hand side is causal and the right-hand side is anti-causal, so these operators are static, and, by [Staffans 1997, Lemma 6],

$$\widetilde{\mathcal{X}}\mathcal{X}^{-1} = \left(\widetilde{\mathcal{Y}}^*\right)^{-1}\mathcal{Y}^* = E,$$

for some operator $E \in \mathcal{L}(U)$. Thus, $\widetilde{\mathcal{X}} = E\mathcal{X}$ and $\widetilde{\mathcal{Y}} = (E^{-1})^*\mathcal{Y}$. The invertibility of E follows from the invertibility of $\mathcal{X}$ and $\widetilde{\mathcal{X}}$.

(ii) If $\mathcal{E} = \mathcal{E}^*$, then both $\mathcal{Y}^*\mathcal{X}$ and $\mathcal{X}^*\mathcal{Y}$ are spectral factorizations of $\mathcal{E}$, and by part (i), $\mathcal{Y} = S\mathcal{X}$ for some invertible $S \in \mathcal{L}(U)$. That $S = S^*$ follows from the fact that $S = (\mathcal{X}^*)^{-1}\mathcal{E}\mathcal{X}^{-1}$.

(iii) Use the causality of $\mathcal{X}$ and anti-causality of $(\mathcal{Y}^*)^{-1}$ to compute

$$\begin{aligned}
\mathcal{X}^{-1}\pi_+(\mathcal{Y}^*)^{-1}\pi_+\mathcal{E}\pi_+ &= \mathcal{X}^{-1}\pi_+(\mathcal{Y}^*)^{-1}\pi_+\mathcal{Y}^*\mathcal{X}\pi_+ \\
&= \mathcal{X}^{-1}\pi_+(\mathcal{Y}^*)^{-1}\mathcal{Y}^*\mathcal{X}\pi_+ \\
&= \mathcal{X}^{-1}\pi_+\mathcal{X}\pi_+ \\
&= \mathcal{X}^{-1}\mathcal{X}\pi_+ \\
&= \pi_+.
\end{aligned}$$

Thus, $\mathcal{X}^{-1}\pi_+(\mathcal{Y}^*)^{-1}$ is a left inverse of $\pi_+\mathcal{E}\pi_+$ on $L^2(\mathbf{R}^+;U)$. A similar computation shows that it is also a right inverse.

(iv) Assume $\mathcal{E} >> 0$. By [Rosenblum and Rovnyak 1985, Theorem 3.4, p. 50] and [Rosenblum and Rovnyak 1985, Theorem 3.7, p. 54], $\mathcal{E}$ has a factorization $\mathcal{E} = \mathcal{X}^*\mathcal{X}$ for some outer $\mathcal{X}$. The invertibility of $\mathcal{E}$ implies that $\mathcal{X}$ is one-to-one and has closed range, hence $\mathcal{X}$ is invertible (for details, see the proof of Lemma 4.6(ii)).[2] The opposite claim is obvious. $\square$

Lemma 4.4 *Let $\mathcal{E} \in TI(U)$.*

(i) The operator $\tau(-t)\pi_+\mathcal{E}\pi_+\tau(t)$ tends strongly to $\mathcal{E}$ as $t \to -\infty$.

(ii) The Toeplitz operator $\pi_+\mathcal{E}\pi_+$ is uniformly positive on $L^2(\mathbf{R}^+;U)$ if and only if the time-invariant operator $\mathcal{E}$ is uniformly positive on $L^2(\mathbf{R};U)$. Moreover, in this case $\mathcal{E}^{-1} \geq \pi_+(\pi_+\mathcal{E}\pi_+)^{-1}\pi_+$.

[2] The definition of an outer operator in Rosenblum and Rovnyak [1985] differs slightly from our definition: there the range space of $\mathcal{X}$ need not be the same as the initial space. However, these two spaces are unitarily equivalent whenever $\mathcal{X}$ is invertible, so the range space can be identified with U in this case.

(iii) *If the Toeplitz operator* $\pi_+\mathcal{E}\pi_+$ *is invertible in* $\mathcal{L}(L^2(\mathbf{R}^+;U))$*, then the time-invariant operator* $\mathcal{E}$ *is invertible in* $TI(L^2(\mathbf{R};U))$ *(but the converse is not true).*

(iv) *If* $\mathcal{E}$ *has a spectral factorization, then* $\tau(-t)\pi_+(\pi_+\mathcal{E}\pi_+)^{-1}\pi_+\tau(t)$ *tends strongly to* $\mathcal{E}^{-1}$ *as* $t \to -\infty$*. Here* $(\pi_+\mathcal{E}\pi_+)^{-1}$ *stands for the inverse of the Toeplitz operator* $\pi_+\mathcal{E}\pi_+$ *in* $\mathcal{L}(L^2((\mathbf{R}^+;U))$*. In particular, this is true if* $\mathcal{E} >> 0$*.*

Proof. (i) Use the time-invariance of $\mathcal{E}$ to write

$$\tau(-t)\pi_+\mathcal{E}\pi_+\tau(t) = \tau(-t)\pi_+\tau(t)\mathcal{E}\tau(-t)\pi_+\tau(t),$$

and let $t \to -\infty$. The operator above tends strongly to $\mathcal{E}$ since $\tau(-t)\pi_+\tau(t) = \pi_{[t,\infty)}$ tends strongly to the identity in $L^2(\mathbf{R};U)$.

(ii) By definition, if $\mathcal{E}$ is uniformly positive on $L^2(\mathbf{R};U)$, then $\langle u, \mathcal{E}u\rangle \geq \epsilon\|u\|^2$ for some $\epsilon > 0$ and all $u \in L^2(\mathbf{R};U)$. In particular, this implies that $\langle \pi_+u, \mathcal{E}\pi_+u\rangle \geq \epsilon\|\pi_+u\|^2$, and thus $\pi_+\mathcal{E}\pi_+$ is uniformly positive on $L^2(\mathbf{R}^+;U)$. Conversely, if $\pi_+\mathcal{E}\pi_+$ is uniformly positive on $L^2(\mathbf{R}^+;U)$, then

$$\langle u, \tau(-t)\pi_+\mathcal{E}\pi_+\tau(t)u\rangle = \langle \pi_+\tau(t)u, \mathcal{E}\pi_+\tau(t)u\rangle \geq \epsilon\|\pi_+\tau(t)u\|^2$$

for some $\epsilon > 0$ and all $u \in L^2(\mathbf{R};U)$. Let $t \to -\infty$, and use (i) to conclude that $\mathcal{E}$ is uniformly positive on $L^2(\mathbf{R};U)$. To prove the last claim we use parts (iii) and (iv) of Lemma 4.3 to get

$$\pi_+(\pi_+\mathcal{E}\pi_+)^{-1}\pi_+ = (\mathcal{X}^*)^{-1}\pi_+\mathcal{X}^{-1} \leq (\mathcal{X}^*)^{-1}\mathcal{X}^{-1} = \mathcal{E}^{-1}.$$

(iii) If the Toeplitz operator $\pi_+\mathcal{E}\pi_+$ is invertible on $L^2(\mathbf{R}^+;U)$, then so is its adjoint, and both $\pi_+\mathcal{E}^*\pi_+\mathcal{E}\pi_+$ and $\pi_+\mathcal{E}\pi_+\mathcal{E}^*\pi_+$ are uniformly positive on $L^2(\mathbf{R}^+;U)$. Trivially, this implies that $\pi_+\mathcal{E}^*\mathcal{E}\pi_+$ and $\pi_+\mathcal{E}\mathcal{E}^*\pi_+$ are uniformly positive. By part (i), both $\mathcal{E}^*\mathcal{E}$ and $\mathcal{E}\mathcal{E}^*$ is uniformly positive on $L^2(\mathbf{R};U)$, and this implies that $\mathcal{E}$ must be invertible. The time-invariance of $\mathcal{E}^{-1}$ follows from the time-invariance of $\mathcal{E}$. As a counterexample to the converse claim we can take $\mathcal{E}$ to be the time shift operator $\tau(t)$ on $L^2(\mathbf{R};\mathbf{C})$, with either $t > 0$ or $t < 0$, or we can use the self-adjoint counterexample $\mathcal{E} = \begin{bmatrix} 0 & \tau(t) \\ \tau(-t) & 0 \end{bmatrix}$ on $L^2(\mathbf{R};\mathbf{C}^2)$.

(iv) Use part (iii) of Lemma 4.3 to write

$$\begin{aligned}\tau(-t)\pi_+(\pi_+\mathcal{E}\pi_+)^{-1}\pi_+\tau(t) &= \tau(-t)\mathcal{X}^{-1}\pi_+(\mathcal{Y}^*)^{-1}\tau(t) \\ &= \mathcal{X}^{-1}\tau(-t)\pi_+\tau(t)(\mathcal{Y}^*)^{-1} \\ &= \mathcal{X}^{-1}\pi_{[t,\infty)}(\mathcal{Y}^*)^{-1},\end{aligned}$$

which tends strongly to $\mathcal{X}^{-1}(\mathcal{Y}^*)^{-1} = \mathcal{E}^{-1}$ as $t \to -\infty$. $\square$

In the sequel when we apply these results we primarily take $\mathcal{E}$ to be given by $\mathcal{E} = \mathcal{D}^* J \mathcal{D}$, and we extend Definition 4.2 as follows:

Definition 4.5 *Let $J = J^* \in \mathcal{L}(Y)$ and $S = S^* \in \mathcal{L}(U)$.*

(i) The operator $\mathcal{N} \in TIC(U;Y)$ is (J,S)-inner if $\mathcal{N}^* J \mathcal{N} = S$.

(ii) An operator $\mathcal{X} \in TIC(U;Y)$ is outer if the image of $L^2(\mathbf{R}^+;U)$ under $\mathcal{X}\pi_+$ is dense in $L^2(\mathbf{R}^+;Y)$.

(iii) An operator $\mathcal{X} \in TIC(U)$ is an (invertible) S-spectral factor of $\mathcal{D}^* J \mathcal{D} \in TI(U)$ if $\mathcal{X}$ is invertible in $TIC(U)$ and $\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* S \mathcal{X}$.

(iv) The factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J,S)-inner-outer factorization of $\mathcal{D} \in TIC(U;Y)$ if $\mathcal{N} \in TIC(U;Y)$ is (J,S)-inner and $\mathcal{X} \in TIC(U)$ is outer.

(v) In each case S is called the sensitivity operator of $\mathcal{N}$ or of the factorization.

By specializing Lemma 4.3 to the case where $\mathcal{E} = \mathcal{D}^* J \mathcal{D}$ we get most of the following result:

Lemma 4.6 *Let $\mathcal{D} \in TIC(U;Y)$, $J = J^* \in \mathcal{L}(Y)$, $S = S^* \in \mathcal{L}(U)$, and $\widetilde{S} = \widetilde{S}^* \in \mathcal{L}(U)$. Then*

(i) If $\mathcal{X}$ is a S-spectral factor of $\mathcal{D}^* J \mathcal{D}$ then $\mathcal{N}\mathcal{X} = (\mathcal{D}\mathcal{X}^{-1})\,\mathcal{X}$ is a (J,S)-inner-outer factorization of $\mathcal{D}$. Conversely, if $\mathcal{N}\mathcal{X}$ is a (J,S)-inner-outer factorization of $\mathcal{D}$ and $\mathcal{X}$ is invertible in $TIC(U)$, then $\mathcal{X}$ is a S-spectral factor of $\mathcal{D}^* J \mathcal{D}$.

(ii) Suppose that $\mathcal{D}$ has a (J,S)-inner-outer factorization $\mathcal{N}\mathcal{X}$. Then $\mathcal{D}$ is J-coercive (i.e., the Toeplitz operator $\pi_+\mathcal{D}^* J \mathcal{D}\pi_+$ is invertible in $\mathcal{L}(L^2(\mathbf{R}^+;U))$) if and only if S is invertible in $\mathcal{L}(U)$ and $\mathcal{X}$ is invertible in $TIC(U)$. In this case $\mathcal{X}$ is a S-spectral factor of $\mathcal{D}^* J \mathcal{D}$, and the inverse of the Toeplitz operator $\pi_+\mathcal{D}^* J \mathcal{D}\pi_+$ can be written in the form $(\pi_+\mathcal{D}^* J \mathcal{D}\pi_+)^{-1} = \mathcal{X}^{-1} S^{-1} \pi_+ (\mathcal{X}^*)^{-1}$. In particular, $\mathcal{X}^{-1} S^{-1} \pi_+ (\mathcal{X}^*)^{-1}$ does not depend on the particular factorization, only on $\mathcal{D}$ and J.

(iii) Suppose that $\mathcal{D}$ is J-coercive and that $\mathcal{D}$ has a $(J, \widetilde{S})$-inner-outer factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$. Then the set of all possible (J, S)-inner-outer factorizations $\mathcal{N}\mathcal{X}$ of $\mathcal{D}$ can be parameterized as $\mathcal{X} = E^{-1}\widetilde{\mathcal{X}}$, $\mathcal{N} = \widetilde{\mathcal{N}}E$, and $S = E^*\widetilde{S}E$, where $E \in \mathcal{L}(U)$ is an arbitrary invertible operator.

Proof. (i) This follows directly from Definition 4.5.

(ii) If both S and $\mathcal{X}$ are invertible, then the claim about the J-coercivity of $\mathcal{D}$ and the formula for the inverse of $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ follow from Lemma 4.3. Thus, the only part of (ii) that requires a new proof is that S and $\mathcal{X}$ are invertible whenever $\mathcal{D}$ is J-coercive.

Suppose that $\mathcal{D}$ is J-coercive. By part (i), $\mathcal{D}^*J\mathcal{D} = \mathcal{X}^*S\mathcal{X}$, so $\pi_+\mathcal{X}^*S\mathcal{X}\pi_+$ is invertible in $\mathcal{L}(L^2(\mathbf{R}^+; U))$. We claim that $\mathcal{X}\pi_+$ is one-to-one on $L^2(\mathbf{R}^+; U)$ and has a closed range. To prove this we argue as follows. That $\mathcal{X}\pi_+$ must be one-to-one on $L^2(\mathbf{R}^+; U)$ follows from the invertibility of $\pi_+\mathcal{X}^*S\mathcal{X}\pi_+$. If the range of $\mathcal{X}\pi_+$ is not closed, then we can find a sequence $u_n \in L^2(\mathbf{R}^+; U)$ with $\|u_n\|_{L^2(\mathbf{R}^+;U)} = 1$ for each n such that $\mathcal{X}\pi_+u_n \to 0$ in $L^2(\mathbf{R}^+; U)$ as $n \to \infty$. Then also $\pi_+\mathcal{X}^*S\mathcal{X}\pi_+u_n \to 0$ in $L^2(\mathbf{R}^+; U)$ as $n \to \infty$, but this contradicts the fact that $\pi_+\mathcal{X}^*S\mathcal{X}\pi_+$ has a continuous inverse. Thus, the range of $\mathcal{X}\pi_+$ is closed in $L^2(\mathbf{R}^+; U)$. Since $\mathcal{X}$ is required to be outer, the range of $\mathcal{X}\pi_+$ is dense in $L^2(\mathbf{R}^+; U)$. Thus, the range of $\mathcal{X}\pi_+$ is all of $L^2(\mathbf{R}^+; U)$, hence $\pi_+\mathcal{X}\pi_+ = \mathcal{X}\pi_+$ is invertible in $\mathcal{L}(L^2(\mathbf{R}^+; U)$. By Lemma 4.4(iii), $\mathcal{X}$ is invertible in $TI(U)$. To show that the inverse is causal it suffices to observe that $\mathcal{X}\pi_+$ maps $L^2(\mathbf{R}^+; U)$ onto itself, hence the inverse must have the same property. The invertibility of S is a consequence of the fact that $S = (\mathcal{X}^*)^{-1}(\mathcal{D}^*J\mathcal{D})\mathcal{X}^{-1}$ is the product of three invertible operators (see Lemma 4.4(iii)).

(iii) This follows from (ii) and Lemma 4.3. $\square$

By using a spectral factorization we can simplify the formulas in Definition 3.5 as follows.

Lemma 4.7 *Let $J = J^* \in \mathcal{L}(Y)$, $S = S^* \in \mathcal{L}(U)$, and let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a stable J-coercive well-posed linear system on (U, H, Y). In addition, suppose that $\mathcal{D}$ has a (J, S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$. Define $\mathcal{M} = \mathcal{X}^{-1}$. Then the critical operators $\mathcal{A}_{\circlearrowleft}$, $\mathcal{C}_{\circlearrowleft}$, $\mathcal{K}_{\circlearrowleft}$, and Π in Definition 3.5 can be* *written in the form*

$$\begin{aligned} \mathcal{K}_{\circlearrowleft} &= -\mathcal{M}S^{-1}\pi_+\mathcal{N}^*J\mathcal{C}, && (4.1)\\ \mathcal{A}_{\circlearrowleft} &= \mathcal{A} - \mathcal{B}\mathcal{M}\tau S^{-1}\pi_+\mathcal{N}^*J\mathcal{C}, && (4.2)\\ \mathcal{C}_{\circlearrowleft} &= \left(I - \mathcal{N}S^{-1}\pi_+\mathcal{N}^*J\right)\mathcal{C}, && (4.3)\\ \Pi &= \mathcal{C}^*\left(J - J\mathcal{N}S^{-1}\pi_+\mathcal{N}^*J\right)\mathcal{C}. && (4.4) \end{aligned}$$

Proof. This follows from Lemma 4.6. $\square$

Remark 4.8 *We warn the reader that there do exist input-output maps $\mathcal{D}$ with invertible Toeplitz operator $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ which do not have a (J,S)-inner-outer factorization in the sense of Definition 4.5. It is well known that the factors in a "generalized factorization" need not always be bounded; see [Clancey and Gohberg 1981, Section VII]. It is less obvious that it is possible to produce a counterexample which satisfies all the extra conditions imposed here and in Staffans [1998c] and still does not have a (bounded) factorization, but it is indeed possible (Ball and Spitkovsky [1996]). The conclusion of part (iii) of Lemma 4.6 says that if a factorization does exist, then the set of all possible factorizations can be parameterized as described above.*

However, in some cases (J,S)-inner-outer factorizations are known to exist, for example when $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+ >> 0$ on $L^2(\mathbf{R}^+;U\times W)$ (see Lemma 4.3(iv)), and for systems of Pritchard-Salamon type (see van Keulen [1993] and Weiss [1997]). Still another example is the following one:

Lemma 4.9 *Let U be finite-dimensional, and suppose that $\mathcal{E}\in TI(U)$ is a convolution operator of the form (for all $u\in L^2(\mathbf{R};U)$ and almost all $t\in\mathbf{R}$)*

$$(\mathcal{E}u)(t) = Eu(t) + \int_{-\infty}^{\infty} A(t-s)u(s)\,ds,$$

where $E\in\mathcal{L}(U)$ and $A\in L^1(\mathbf{R};\mathcal{L}(U))$. Then $\mathcal{E}$ has a spectral factorization iff the Toeplitz operator $\pi_+\mathcal{E}\pi_+$ is invertible in $\mathcal{L}(L^2(\mathbf{R}^+;U))$.

It is an open question to what extent this result is true when U is infinite-dimensional.

Proof. By Lemma 4.3(iii), the Toeplitz operator $\pi_+\mathcal{E}\pi_+$ is invertible in $\mathcal{L}(L^2(\mathbf{R}^+;U))$ whenever $\mathcal{E}$ has a spectral factorization.

Conversely, suppose that the Toeplitz operator $\pi_+\mathcal{E}\pi_+$ is invertible in $\mathcal{L}(L^2(\mathbf{R}^+;U))$. Then, by Lemma 4.4(iii), the time-invariant operator $\mathcal{E}$ is invertible in $\mathcal{L}(L^2(\mathbf{R};U))$. By [Clancey and Gohberg 1981, Theorem 6.3, p. 63], this implies that $\mathcal{E}$ has a, possibly non-canonical, spectral factorization. According to [Clancey and Gohberg 1981, Corollary 1.1, p. 75], the invertibility of the Toeplitz operator $\pi_+\mathcal{E}\pi_+$ implies that the factorization is canonical. $\square$

Corollary 4.10 *Let U be finite-dimensional, and suppose that $\mathcal{D} \in TIC(U;Y)$ is a convolution operator of the form (for all $u \in L^2(\mathbf{R};U)$ and almost all $t \in \mathbf{R}$)*

$$(\mathcal{D}u)(t) = Du(t) + \int_{-\infty}^{t} A(t-s)u(s)\,ds,$$

where $D \in \mathcal{L}(U;Y)$ and $A \in L^1(\mathbf{R}^+;\mathcal{L}(U;Y))$. Then $\mathcal{D}^*J\mathcal{D}$ has a spectral factorization iff the Toeplitz operator $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ is invertible in $\mathcal{L}(L^2(\mathbf{R}^+;U))$.

Proof. This follows from Lemma 4.9 with $\mathcal{E} = \mathcal{D}^*J\mathcal{D}$. $\square$

Let us end this section by giving some necessary and sufficient conditions for the *causal* invertibility of an operator $\mathcal{X} \in TIC(U)$.

Lemma 4.11 *Let $\mathcal{X} \in TIC(U)$.*

(i) If $\mathcal{X}$ has an inverse in $TIC(U)$, then

- *(a) $\mathcal{X}\mathcal{X}^* >> 0$ on $L^2(\mathbf{R},U)$,*
- *(b) $\mathcal{X}^*\mathcal{X} >> 0$ on $L^2(\mathbf{R},U)$,*
- *(c) $\mathcal{X}\pi_+\mathcal{X}^* >> 0$ on $L^2(\mathbf{R}^+,U)$,*
- *(d) $\mathcal{X}^*\pi_-\mathcal{X} >> 0$ on $L^2(\mathbf{R}^-,U)$.*

(ii) If (a) and (d) hold, then $\mathcal{X}$ has an inverse in $TIC(U)$.

(iii) If (b) and (c) hold, then $\mathcal{X}$ has an inverse in $TIC(U)$.

Proof. (i) Claims (a) and (b) are obvious (both $\mathcal{X}$ and $\mathcal{X}^*$ are invertible in $TI(U)$). Clearly, $\mathcal{X}\pi_+\mathcal{X}^* \geq 0$ on $L^2(\mathbf{R}^+,U)$, so to show that $\mathcal{X}\pi_+\mathcal{X}^* >> 0$ it suffices to show that $\mathcal{X}\pi_+\mathcal{X}^*$ is invertible on $L^2(\mathbf{R}^+,U)$. But this follows from Lemma 4.3(iii); it is the inverse of the Toeplitz operator $\pi_+\mathcal{Y}^*\mathcal{Y}\pi_+ >> 0$

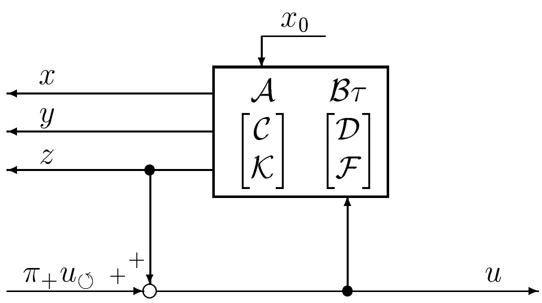

Figure 2: Optimal state feedback connection

where $\mathcal{Y} = \mathcal{X}^{-1}$. This proves claim (c). Claim (d) is proved in a similar way (e.g., through a reflection the time direction).

(ii) Obviously, (d) implies that $\pi_-\mathcal{X}^*\mathcal{X}\pi_- \geq \pi_-\mathcal{X}^*\pi_-\mathcal{X}\pi_- = \mathcal{X}^*\pi_-\mathcal{X} >> 0$ on $L^2(\mathbf{R}^-, U)$. From here we conclude that the time invariant operator $\mathcal{X}^*\mathcal{X}$ is uniformly positive on $L^2(\mathbf{R}; U)$ (cf. the proof of Lemma 4.4(ii)). Thus, combining this with (a) we find that $\mathcal{X}$ is invertible in $\mathcal{L}(L^2(\mathbf{R}; U))$. Denote the inverse by $\mathcal{Y}$. This inverse is necessarily time invariant. To show that it is causal we use condition (d). Let $u \in L^2(\mathbf{R}; U)$ be arbitrary, and denote $v = \mathcal{Y}u$. Then $u = \mathcal{X}v = \mathcal{X}\mathcal{Y}u$, so by (d), there is an $\epsilon > 0$ such that

$$\|u\|^2_{L^2(\mathbf{R}^-;U)} \geq \epsilon\|\mathcal{Y}u\|^2_{L^2(\mathbf{R}^-;U)}.$$

Thus, $\mathcal{Y}u$ vanishes on $\mathbf{R}^-$ whenever u does, and this means that $\mathcal{Y}$ is causal.

(iii) This proof is similar to the proof of (ii), and it is left to the reader (for example, one can reflect the time direction, and apply (ii)). $\square$

5 Closed Loop Formula for the Critical Control

With the preliminary results in Sections 3 and 4 at our disposal it is now easy to extend [Staffans 1997, Theorem 27] and [Staffans 1998b, Theorem 2.6] as follows:

Theorem 5.1 ([Staffans 1998b, Theorem 2.6]) *Let $J = J^* \in \mathcal{L}(Y)$ and let $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be a stable J-coercive well-posed linear system on (U, H, Y). Let $x_0 \in H$, and let $u^{\text{crit}}(x_0)$, $x^{\text{crit}}(x_0)$, and $y^{\text{crit}}(x_0)$ be the critical control,* *state, and observation (see Lemma 3.4), and let* Π *be the Riccati operator (see Definition 3.5).*

(i) Suppose that $\mathcal{D}$ *has a* (J,S)*-inner-outer factorization* $\mathcal{D}=\mathcal{N}\mathcal{X}$*. Then* S *is invertible in* $\mathcal{L}(U)$*,* $\mathcal{X}$ *is invertible in* $TIC(U)$*, and* $\mathcal{X}$ *is a* S*-spectral factor of* $\mathcal{D}^*J\mathcal{D}$*. Define* $\mathcal{M}=\mathcal{X}^{-1}$ *and*

$$\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix} = \begin{bmatrix}-S^{-1}\pi_+\mathcal{N}^*J\mathcal{C} & (I-\mathcal{X})\end{bmatrix}.$$

Then $\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix}$ *is a stable and stabilizing state feedback pair for* Ψ *[Staffans 1997, Definition 22] and*

$$\begin{aligned}\begin{bmatrix} x^{\mathrm{crit}}(t,x_0) \\ y^{\mathrm{crit}}(x_0) \\ u^{\mathrm{crit}}(x_0)\end{bmatrix} &= \begin{bmatrix}\mathcal{A}_\circlearrowleft(t) \\ \mathcal{C}_\circlearrowleft \\ \mathcal{K}_\circlearrowleft\end{bmatrix} x_0 = \begin{bmatrix}\mathcal{A}(t)+\mathcal{B}\mathcal{M}\tau(t)\mathcal{K} \\ \mathcal{C}+\mathcal{N}\mathcal{K} \\ \mathcal{M}\mathcal{K}\end{bmatrix} x_0 \\ &= \begin{bmatrix}\mathcal{A}(t) \\ \mathcal{C} \\ 0\end{bmatrix} x_0 - \begin{bmatrix}\mathcal{B}\mathcal{M}\tau(t) \\ \mathcal{N} \\ \mathcal{M}\end{bmatrix} S^{-1}\pi_+\mathcal{N}^*J\mathcal{C}x_0\end{aligned}$$

is equal to the state and output of the closed loop system $\Psi_\circlearrowleft$ *defined by*

$$\Psi_\circlearrowleft = \begin{bmatrix}\mathcal{A}_\circlearrowleft & \mathcal{B}_\circlearrowleft \\ \begin{bmatrix}\mathcal{C}_\circlearrowleft \\ \mathcal{K}_\circlearrowleft\end{bmatrix} & \begin{bmatrix}\mathcal{D}_\circlearrowleft \\ \mathcal{F}_\circlearrowleft\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\mathcal{A}+\mathcal{B}\mathcal{M}\tau\mathcal{K} & \mathcal{B}\mathcal{M} \\ \begin{bmatrix}\mathcal{C}+\mathcal{N}\mathcal{K} \\ \mathcal{M}\mathcal{K}\end{bmatrix} & \begin{bmatrix}\mathcal{N} \\ \mathcal{M}-I\end{bmatrix}\end{bmatrix}$$

with initial value x_0*, initial time zero, and zero control* $u_\circlearrowleft$ *(see Figure 2). The Riccati operator* Π *of* Ψ *can be written in the following alternative forms:*

$$\Pi = \mathcal{C}^*J\mathcal{C} - \mathcal{K}^*S\mathcal{K} = \mathcal{C}^*J\mathcal{C}_\circlearrowleft = \mathcal{C}_\circlearrowleft^*J\mathcal{C}_\circlearrowleft = \mathcal{C}_\circlearrowleft^*J\mathcal{C}.$$

(ii) Conversely, suppose that $\begin{bmatrix} y^{\mathrm{crit}}(x_0) \\ u^{\mathrm{crit}}(x_0)\end{bmatrix}$ *is equal to the observation of some stable state feedback perturbation* $\Psi_\circlearrowleft$ *of* Ψ *with initial value* x_0*, initial time* 0*, zero control* $u_\circlearrowleft$*, and some admissible stable state feedback pair* $\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix}$*. Then there exists an (invertible) operator* $S = S^* \in \mathcal{L}(U)$ *such that* $\mathcal{N}\mathcal{X}$ *is a* (J,S)*-inner-outer factorization of* $\mathcal{D}$*, where* $\mathcal{N} = \mathcal{D}(I-\mathcal{F})^{-1}$ *and* $\mathcal{X} = (I-\mathcal{F})$*. Moreover,* $\mathcal{K}$ *is given by* $\mathcal{K} = -S^{-1}\pi_+\mathcal{N}^*J\mathcal{C}$*.*

(iii) *Let the two equivalent conditions (i) and (ii) hold. If* $y = \mathcal{C}_{\circlearrowleft} x_0 + \mathcal{D}_{\circlearrowleft} \pi_+ u_{\circlearrowleft}$ *is the first output of the critical closed loop system* $\Psi_{\circlearrowleft}$ *with initial state* $x_0 \in H$ *and control* $u_{\circlearrowleft} \in L^2(\mathbf{R}^+; U)$ *(see Figure 2), then the closed loop cost* $Q_{\circlearrowleft}(x_0, u_{\circlearrowleft})$ *is given by*

$$Q_{\circlearrowleft}(x_0, u_{\circlearrowleft}) = \langle y, Jy \rangle_{L^2(\mathbf{R}^+;Y)} = \langle x_0, \Pi x_0 \rangle_H + \langle u_{\circlearrowleft}, S u_{\circlearrowleft} \rangle_{L^2(\mathbf{R}^+;U)} . \tag{5.1}$$

The proof of this theorem is very similar to the proof of [Staffans 1995, Theorem 5.1], so we only outline it as follows.

Proof. (i) By Lemma 4.6, S and $\mathcal{X}$ are invertible in $\mathcal{L}(U)$ and $TIC(U)$, respectively. Moreover, since $\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* S \mathcal{X}$ we have $\mathcal{X} = S^{-1} \mathcal{N}^* J \mathcal{D}$. Thus $\mathcal{K} = -S^{-1} \pi_+ \mathcal{N}^* J \mathcal{C}$ and $\mathcal{F} = I - S^{-1} \mathcal{N}^* J \mathcal{D}$. It follows from [Staffans 1998b, Lemma 4.10] that we can use the pair $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix}$ to extend the system Ψ into a larger well-posed linear system

$$\Psi_{\text{ext}} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{F} \end{bmatrix} \end{bmatrix} .$$

Clearly this system is stable (all its components are bounded operators). The given formula for the closed loop system $\Psi_{\circlearrowleft}$ is copied from [Staffans 1998a, Lemma 3.13] (with $(I - \mathcal{F})^{-1}$ replaced by $\mathcal{M}$), and Lemma 4.7 shows that the state and observation of the closed loop system $\Psi_{\circlearrowleft}$ is identical to the critical state, observation and control for Ψ. The given formulas for the Riccati operator Π are the same as in Lemmas 3.7 and 4.7.

(ii) Define $\mathcal{X} = I - \mathcal{F}$ and $\mathcal{N} = \mathcal{D} \mathcal{X}^{-1}$. The operator $\mathcal{X}$ is invertible in $TIC(U)$ since both Ψ and $\Psi_{\circlearrowleft}$ are supposed to be stable. We want to show that $\mathcal{X}$ is an S-spectral factor of $\mathcal{D}^* J \mathcal{D}$ for some invertible $S = S^* \in \mathcal{L}(U)$, i.e., that $\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* S \mathcal{X}$. The crucial step in the proof is to show that the operator $\mathcal{Y}^* = \mathcal{D}^* J \mathcal{D} \mathcal{X}^{-1} = \mathcal{D}^* J \mathcal{N}$ is anti-causal, and this is done in exactly the same way as in the proof of [Staffans 1995, Theorem 5.1]. We leave this part of the proof to the reader.

(iii) The proof of (iii) is identical to the proof of [Staffans 1998b, Theorem 2.6(iii)]. $\square$

In [Staffans 1998b, Sections 5–7] we proved a number of results concerning the behavior of the critical closed loop system in Theorem 5.1 on a finite time interval $[0, t]$, and we also established the validity of certain algebraic

Riccati equations. In those results the sensitivity operator S was throughout assumed to be positive, but the positivity of S did not play any significant role in the proofs.[3] The only place where the positivity was used was in the proofs of [Staffans 1998b, Formulae (5.1) and (5.3)], which are identical to (3.8) and (3.9). This means that the following result holds:

Remark 5.2 *Suppose that the two equivalent conditions (i) and (ii) in Theorem 5.1 hold, i.e., let* $J = J^* \in \mathcal{L}(Y)$, *let* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a stable* J*-coercive well-posed linear system on* (U, H, Y), *and suppose that* $\mathcal{D}$ *has a* (J, S)*-inner-outer factorization* $\mathcal{N}\mathcal{X}$. *Then* S *is invertible in* $\mathcal{L}(U)$, $\mathcal{X}$ *is invertible in* $TIC(U)$, *and all the claims in [Staffans 1998b, Sections 5–7] are valid if we throughout require all the systems to be stable, drop the positivity requirement on* S, *replace all references to [Staffans 1998b, Lemma 2.4 and Theorem 4.4] by references to Lemma 4.6 and Theorem 5.1, respectively, and throughout replace* $x^{\mathrm{opt}}(x_0)$, $y^{\mathrm{opt}}(x_0)$ *and* $u^{\mathrm{opt}}(x_0)$ *by* $x^{\mathrm{crit}}(x_0)$, $y^{\mathrm{crit}}(x_0)$ *and* $u^{\mathrm{crit}}(x_0)$, *respectively. The parameterization results [Staffans 1998b, Propositions 4.7 and 4.8] remain valid in the same sense.*

In particular, let us cite the following result, that will be important in the next section:

Lemma 5.3 ([Staffans 1998b, Lemma 5.4]) *Let* $J = J^* \in \mathcal{L}(Y)$, *let* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a stable* J*-coercive well-posed linear system on* (U, H, Y), *and suppose that* $\mathcal{D}$ *has a* (J, S)*-inner-outer factorization* $\mathcal{N}\mathcal{X}$. *Then, with the notations of Theorem 5.1, for all* $t \in \mathbf{R}^+$,

$$\pi_+\tau(-t)\mathcal{B}_{\circlearrowleft}^*\Pi\mathcal{B}_{\circlearrowleft}\tau(t)\pi_+ + \pi_{[0,t]}\mathcal{N}^*\pi_{[0,t]}J\mathcal{N}\pi_{[0,t]} = S\pi_{[0,t]}.$$

6 (J, S)-Lossless Factorizations

In the H^∞ theory it is not sufficient with a (J, S)-inner-outer factorization; what one really needs is a (J, S)-lossless-outer factorization. This has been emphasized by, e.g., Ball and Helton [1988], Green [1992], and Curtain and Green [1997]. We refer the reader to these papers and to Staffans [1998c] for further discussions of the importance of this notion.

[3] The positivity was needed in order to make the problem a *minimization* problem instead of a minimax problem.

Usually the notion of (J,S)-losslessness is defined in the frequency domain, but in our case it makes mores sense to work in the time domain. We refer the reader to Ball and Helton [1988] for an extensive discussion of how the different losslessness notions are related to each other. The time domain version has natural extensions to time-variant systems (see Gohberg [1992]) and to nonlinear systems (see Ball and Helton [1992] and Ball and van der Schaft [1996]).

We define the time domain version of (J,S)-losslessness as follows:

Definition 6.1 *Let $J = J^* \in \mathcal{L}(Y)$ and $S = S^* \in \mathcal{L}(U)$. Let $\mathcal{D} \in TIC(U;Y)$, $\mathcal{N} \in TIC(U;Y)$, and $\mathcal{X} \in TIC(U)$.*

(i) The operator $\mathcal{D}$ is (J,S)-dissipative with respect to $\mathcal{X}$ if

$$\mathcal{D}^*\pi_- J\mathcal{D} \le \mathcal{X}^*\pi_- S\mathcal{X},$$

that is, if

$$\int_{-\infty}^{0} \langle (\mathcal{D}u)(s), J(\mathcal{D}u)(s)\rangle_Y \, ds \le \int_{-\infty}^{0} \langle (\mathcal{X}u)(s), S(\mathcal{X}u)(s)\rangle_U \, ds$$

for all $u \in L^2(\mathbf{R}^-;U)$. In the case where $\mathcal{X}$ is the identity operator we call $\mathcal{D}$ (J,S)-dissipative. Thus, $\mathcal{N}$ is (J,S)-dissipative if

$$\int_{-\infty}^{0} \langle (\mathcal{N}u)(s), J\mathcal{N}u)(s)\rangle_Y \, ds \le \int_{-\infty}^{0} \langle u(s), Su(s)\rangle_U \, ds$$

for all $u \in L^2(\mathbf{R}^-;U)$.

(ii) A S-dissipative spectral factor $\mathcal{X}$ of $\mathcal{D}^*J\mathcal{D}$ is a S-spectral factor of $\mathcal{D}^*J\mathcal{D}$ with the property that $\mathcal{D}$ is (J,S)-dissipative with respect to $\mathcal{X}$.

(iii) The operator $\mathcal{N}$ is (J,S)-lossless if $\mathcal{N}$ is both (J,S)-inner and (J,S)-dissipative.

(iv) A (J,S)-lossless-outer factorization of $\mathcal{D}$ is a (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ with a (J,S)-lossless inner factor $\mathcal{N}$.

There are some another equivalent characterizations of (J,S)-dissipativity:

Lemma 6.2 *Let $J = J^* \in \mathcal{L}(Y)$ and $S = S^* \in \mathcal{L}(U)$. Let $\mathcal{D} \in TIC(U;Y)$, $\mathcal{N} \in TIC(U;Y)$, and $\mathcal{X} \in TIC(U)$.*

(i) The following conditions are equivalent:

(a) $\mathcal{D}$ *is* (J,S)*-dissipative with respect to* $\mathcal{X}$*;*

(b) $\mathcal{D}^*\pi_{(-\infty,t]}J\mathcal{D} \le \mathcal{X}^*\pi_{(-\infty,t]}S\mathcal{X}$ *for all* $t \in \mathbf{R}$*, i.e.,*

$$\int_{-\infty}^{t} \langle (\mathcal{D}u)(s), J(\mathcal{D}u)(s)\rangle_Y \, ds \le \int_{-\infty}^{t} \langle (\mathcal{X}u)(s), S(\mathcal{X}u)(s)\rangle_U \, ds$$

for all $u \in L^2(\mathbf{R};U)$ *and all* $t \in \mathbf{R}$*;*

(c) $\pi_{[0,t]}\mathcal{D}^*\pi_{[0,t]}J\mathcal{D}\pi_{[0,t]} \le \pi_{[0,t]}\mathcal{X}^*\pi_{[0,t]}S\mathcal{X}\pi_{[0,t]}$ *for all* $t \in \mathbf{R}^+$*, i.e.,*

$$\int_0^t \langle (\mathcal{D}\pi_+u)(s), J(\mathcal{D}\pi_+u)(s)\rangle_Y \, ds \le \int_0^t \langle (\mathcal{X}\pi_+u)(s), S(\mathcal{X}\pi_+u)(s)\rangle_U \, ds$$

for all $u \in L^2(\mathbf{R}^+;U)$ *and all* $t \in \mathbf{R}^+$*.*

(ii) The following conditions are equivalent:

(a) $\mathcal{N}$ *is* (J,S)*-dissipative;*

(b) $\mathcal{N}^*\pi_{(-\infty,t]}J\mathcal{N} \le \pi_{(-\infty,t]}S$ *for all* $t \in \mathbf{R}$*, i.e.,*

$$\int_{-\infty}^{t} \langle (\mathcal{N}u)(s), J\mathcal{N}u)(s)\rangle_Y \, ds \le \int_{-\infty}^{t} \langle u(s), Su(s)\rangle_U \, ds$$

for all $u \in L^2(\mathbf{R};U)$ *and all* $t \in \mathbf{R}$*;*

(c) $\pi_{[0,t]}\mathcal{N}^*\pi_{[0,t]}J\mathcal{N}\pi_{[0,t]} \le \pi_{[0,t]}S$ *for all* $t \in \mathbf{R}^+$*, i.e.,*

$$\int_0^t \langle (\mathcal{N}\pi_+u)(s), J\mathcal{N}\pi_+u)(s)\rangle_Y \, ds \le \int_0^t \langle u(s), Su(s)\rangle_U \, ds$$

for all $u \in L^2(\mathbf{R}^+;U)$ *and all* $t \in \mathbf{R}^+$*.*

(iii) If $\mathcal{N}$ *is* (J,S)*-inner, then the following conditions are equivalent:*

(a) $\mathcal{N}$ *is* (J,S)*-lossless.*

(b) $\pi_-\mathcal{N}^*\pi_+J\mathcal{N}\pi_- \geq 0$, *i.e.*,

$$\int_0^\infty \langle(\mathcal{N}\pi_-u)(s), J\mathcal{N}\pi_-u)(s)\rangle_Y \, ds \geq 0$$

for all $u \in L^2(\mathbf{R}; U)$.

(c) $\pi_{(-\infty,t]}\mathcal{N}^*\pi_{[t,\infty)}J\mathcal{N}\pi_{(-\infty,t]} \geq 0$ *for all* $t \in \mathbf{R}$, *i.e.*,

$$\int_t^\infty \big\langle(\mathcal{N}\pi_{(-\infty,t]}u)(s), J\mathcal{N}\pi_{(-\infty,t]}u)(s)\big\rangle_Y \, ds \geq 0$$

for all $u \in L^2(\mathbf{R}; U)$ *and all* $t \in \mathbf{R}$.

Proof. (i) To show that the (a) and (b) are equivalent it suffices to replace u by $\tau(t)u$ in Definition 6.1(i) and use the time-invariance of $\mathcal{D}$ and $\mathcal{X}$. The same argument shows that (c) is equivalent to the requirement that

$$\begin{aligned}\int_t^0 & \big\langle(\mathcal{D}\pi_{[t,\infty)}u)(s), J(\mathcal{D}\pi_{[t,\infty)}u)(s)\big\rangle_Y \, ds \\ & \leq \int_t^0 \big\langle(\mathcal{X}\pi_{[t,\infty)}u)(s), S(\mathcal{X}\pi_{[t,\infty)}u)(s)\big\rangle_U \, ds\end{aligned}$$

for all $u \in L^2(\mathbf{R}; U)$ and $t < 0$. If (a) holds, then this inequality is true (take $u(s) = 0$ for $s < t$), and conversely, if this inequality is true, then we can let $t \to -\infty$ to show that (a) holds.

(ii) Clearly, (ii) follows from (i) and Definition 6.1.

(iii) Since $\mathcal{N}$ is causal and (J, S)-inner, we have for all $u \in L^2(\mathbf{R}; U)$,

$$\begin{aligned}\int_{-\infty}^0 & \langle(\mathcal{N}u)(s), J(\mathcal{N}u)(s)\rangle_Y \, ds \\ & = \int_{-\infty}^0 \langle(\mathcal{N}\pi_-u)(s), J(\mathcal{N}\pi_-u)(s)\rangle_Y \, ds \\ & = \int_{-\infty}^0 \langle(u(s), Su(s)\rangle_U \, ds - \int_0^\infty \langle(\mathcal{N}\pi_-u)(s), J(\mathcal{N}\pi_-u)(s)\rangle_Y \, ds.\end{aligned}$$

Thus $\int_0^\infty \langle(\mathcal{N}\pi_-u)(s), J(\mathcal{N}\pi_-u)(s)\rangle_Y \, ds \geq 0$ if and only if

$$\int_{-\infty}^0 \langle(\mathcal{N}u)(s), J(\mathcal{N}u)(s)\rangle_Y \, ds \leq \int_{-\infty}^0 \langle(u(s), Su(s)\rangle_U \, ds.$$

This proves the equivalence of (a) and (b). The equivalence of (b) and (c) follows from the time invariance of $\mathcal{N}$. $\square$

In one case the (J, S)-losslessness property is trivial:

Lemma 6.3 *If $J \geq 0$, then every (J, S)-inner operator $\mathcal{N}$ is (J, S)-lossless.*

Proof of Lemma 6.3. This follows from Lemma 6.2(iii) (the positivity of J implies (b)). $\square$

Observe that the assumption of Lemma 6.3 implies that $S \geq 0$, too.

Definition 6.4 *The reachable subspace of the well-posed linear system $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ on (U, H, Y) is the closure of the range of its controllability map $\mathcal{B}$, i.e., the closure of the set*

$$\left\{ \mathcal{B}\tau(t)\pi_+ u \,\middle|\, t \in \mathbf{R}^+,\ u \in L^2(\mathbf{R}^+; U) \right\}.$$

Theorem 6.5 *Let $J = J^* \in \mathcal{L}(Y)$, let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a stable J-coercive well-posed linear system on (U, H, Y), and suppose that $\mathcal{D}$ has a (J, S)-inner-outer factorization $\mathcal{N}\mathcal{X}$. Then the following conditions are equivalent:*

(i) $\mathcal{D}^* J \mathcal{D}$ has a S-dissipative spectral factor $\mathcal{X}$.

(ii) Every S-spectral factor $\mathcal{X}$ of $\mathcal{D}^* J \mathcal{D}$ is S-dissipative,

(iii) $\mathcal{D}$ has a (J, S)-lossless-outer factorization $\mathcal{N}\mathcal{X}$.

(iv) Every (J, S)-inner-outer factorization of $\mathcal{D}$ is (J, S)-lossless.

(v) The Riccati operator Π is nonnegative on the reachable subspace of Ψ.

Proof. We begin the proof by observing that the equivalence of (i), (ii), (iii) and (iv) follows from Lemma 4.6. It remains to show that (v) is equivalent to the other conditions. Clearly, Lemmas 5.3 and 6.2 implies that (i) holds iff Π is nonnegative on the reachable subspace of the closed loop system $\Psi_\circlearrowleft$. But the closed loop controllability map $\mathcal{B}_\circlearrowleft$ is given by $\mathcal{B}_\circlearrowleft = \mathcal{B}\mathcal{X}^{-1}$, and both $\mathcal{X}$ and $\mathcal{X}^{-1}$ are causal, so the reachability subspace of the closed loop system $\Psi_\circlearrowleft$ is the same as the reachability subspace of the open loop system Ψ. Thus (i) and (v) are equivalent. $\square$

Corollary 6.6 *If the cost function $Q(x_0, u)$ is nonnegative, i.e., if*

$$\int_{\mathbf{R}^+} \langle (\mathcal{C}x_0)(s) + (\mathcal{D}\pi_+u)(s), J\left((\mathcal{C}x_0)(s) + (\mathcal{D}\pi_+u)(s)\right)\rangle_Y \, ds \geq 0$$

for all $x_0 \in H$ and $u \in L^2(\mathbf{R}^+; U)$, then the (J, S)-inner-outer factorization $\mathcal{N}\mathcal{X}$ in Theorem 5.1 is (J, S)-lossless.

Proof. To prove this it suffices to observe that the Riccati operator Π is nonnegative on the whole space H in this case. $\square$

Remark 6.7 *Note that there is no essential loss of generality if we assume that the reachable subspace of Ψ is the whole state space H, because otherwise we may simply factor out and discard the orthogonal complement to the reachable subspace. This will give us a new stable well-posed linear system which is approximately controllable, i.e., the range of the controllability map $\mathcal{B}$ is dense in the new state space. Then the system is lossless if and only if the Riccati operator is nonnegative.*

Our time domain notion of losslessness is related to the corresponding frequency domain notion, which is built around the notion of *a transfer function.* As is well-known, there is a one-to-one correspondence between $TIC_\omega(U; Y)$ and the set of $\mathcal{L}(U; Y)$-valued H^∞ functions over the half-plane $\Re z > \omega$. We denote this set of functions by $H^\infty_\omega(U; Y)$. The norm in this space is the usual H^∞-norm.

Lemma 6.8 *The two spaces $TIC_\omega(U; Y)$ and $H^\infty_\omega(U; Y)$ are isometrically isomorphic. More precisely, to each operator $\mathcal{D} \in TIC_\omega(U; Y)$ there corresponds a unique function $\widehat{\mathcal{D}} \in H^\infty_\omega(U; Y)$, with $\|\mathcal{D}\|_{TIC_\omega(U;Y)} = \|\widehat{\mathcal{D}}\|_{H^\infty(U;Y)}$, such that for each $u \in L^2_\omega(\mathbf{R}^+; U)$, the Laplace transform of $\mathcal{D}u$ is given by $\widehat{\mathcal{D}}(s)\widehat{u}(s)$, $\Re s > \omega$, where $\widehat{u}$ is the Laplace transform of u.*

Proof. This result is classic; see, for example, Fourès and Segal [1955], [Thomas 1997, Theorem 9.1], or [Weiss 1991, Theorem 2.3]. $\square$

Definition 6.9 *The function $\widehat{\mathcal{D}}$ in Lemma 6.8 is called the transfer function of $\mathcal{D}$.*

Lemma 6.10 *Let $\mathcal{D} \in TIC_\omega(U; Y)$, $s \in \mathbf{C}$ with $\Re s > \omega$, and $u \in U$. Define $\mathrm{e}_s(t) = \mathrm{e}^{st}$, $t \in \mathbf{R}$. Then $\mathcal{D}(\mathrm{e}_s u) = \mathrm{e}_s \widehat{\mathcal{D}}(s)u$.*

Proof. This follows from [Weiss 1991, p. 198]. $\square$

Lemma 6.11 *Let $J \in \mathcal{L}(Y)$ and $S = S^* \in \mathcal{L}(U)$. Let $\mathcal{D} \in TIC(U;Y)$, $\mathcal{N} \in TIC(U;Y)$, and $\mathcal{X} \in TIC(U)$.*

(i) If $\mathcal{D}$ is (J,S)-dissipative with respect to $\mathcal{X}$, then the corresponding transfer functions $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{X}}$ satisfy

$$\widehat{\mathcal{D}}^*(s)J\widehat{\mathcal{D}}(s) \le \widehat{\mathcal{X}}^*(s)S\widehat{\mathcal{X}}(s), \qquad \Re s > 0.$$

(ii) If $\mathcal{N}$ is (J,S)-lossless, then the transfer function $\widehat{\mathcal{N}}$ of $\mathcal{N}$ satisfies

$$\widehat{\mathcal{N}}^*(s)J\widehat{\mathcal{N}}(s) \le S, \qquad \Re s > 0.$$

Note the absence of all invertibility assumptions on S and $\mathcal{X}$ in this lemma. In the finite-dimensional case this result is closely related to the first part of [Ball and Helton 1988, Proposition 1.1].

Proof. Clearly (i) implies (ii), so it suffices to prove (i).

Fix some $s \in \mathbf{C}$ with $\Re s > 0$ and $u \in U$. Define $v(t) = \mathrm{e}^{st}u$, $t \le 0$, and extend v to a function in $L^2(\mathbf{R};U)$ in an arbitrary way. By Lemma 6.10 and the causality of $\mathcal{D}$ and $\mathcal{X}$,

$$(\mathcal{D}v)(t) = \mathrm{e}^{st}\widehat{\mathcal{D}}(s)u, \qquad (\mathcal{X}v)(t) = \mathrm{e}^{st}\widehat{\mathcal{X}}(s)u, \qquad t \le 0.$$

The (J,S)-dissipativity of $\mathcal{D}$ with respect to $\mathcal{X}$, applied to the function v, then gives

$$\langle \widehat{\mathcal{D}}(s)u, J\widehat{\mathcal{D}}(s)u\rangle \int_{-\infty}^0 |\mathrm{e}^{2st}|\, ds \le \langle \widehat{\mathcal{X}}(s)u, S\widehat{\mathcal{D}}(s)u\rangle \int_{-\infty}^0 |\mathrm{e}^{2st}|\, ds.$$

Divide by $\int_{-\infty}^0 |\mathrm{e}^{2st}|\, ds$ to conclude that $\widehat{\mathcal{D}}^*(s)J\widehat{\mathcal{D}}(s) \le \widehat{\mathcal{X}}^*(s)S\widehat{\mathcal{X}}(s)$ for all $s \in \mathbf{C}$ with $\Re s > 0$. $\square$

With the help of Theorem 6.5 and Lemma 6.11 we can improve one of the results that is a part of Remark 5.2. In this result we need the following regularity notion:

Definition 6.12 *The operator $\mathcal{D} \in TIC_\omega(U;Y)$ is regular if, for every $u \in U$, the strong Abel mean $Du = \lim_{\lambda \to +\infty} \widehat{\mathcal{D}}(\lambda)u$ exists for every $u \in U$; here λ tends to infinity along the real axis. We call the operator $D\colon U \to Y$ the feedthrough operator of $\mathcal{D}$.*

Theorem 6.13 *Let* $J = J^* \in \mathcal{L}(Y)$, *let* $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ *be a stable* J*-coercive well-posed linear system on* (U, H, Y)*, and suppose that* $\mathcal{D}$ *is regular and that* $\mathcal{D}^*J\mathcal{D}$ *has a regular* S*-spectral factor* $\mathcal{X}$*. Denote the feedthrough operators of* $\mathcal{D}$ *and* $\mathcal{X}$ *by* D *and* X*, respectively. Then the difference* $X^*SX - D^*JD$ *is positive [negative] definite whenever* Π *is positive [negative] definite on the reachable subspace.*

Proof. Suppose that $\Pi \geq 0$ on the reachable subspace. Then, by Theorem 6.5 and Lemma 6.11(ii), $\langle \widehat{\mathcal{D}}(s)u, J\widehat{\mathcal{D}}(s)u\rangle \leq \langle \widehat{\mathcal{X}}(s)u, S\widehat{\mathcal{X}}(s)u\rangle$ for all $u \in U$ and $s \in \mathbf{C}$ with $\Re s > 0$. Let $s = \lambda \to +\infty$ to conclude that $D^*JD \leq X^*SX$. If $\Pi \leq 0$ on the reachable subspace, then we replace J, S, and Π by $-J$, $-S$, and $-\Pi$, and apply the preceding result. $\square$

7 Critical Controls for Unstable Systems

It is a well-known property of the finite-dimensional full information H^∞ problem that its solution is not affected by certain types of preliminary feedback. More precisely, the critical output y^{crit} and state x^{crit} stay the same and so does the Riccati operator Π, but the critical control u^{crit} adjusts to compensate for the preliminary feedback. In particular, it is standard practice to first use a preliminary static feedback in order to simplify the feedthrough terms before applying the Riccati equation theory; see [Green and Limebeer 1995, pp. 160–170]. In Lemma 7.2 below we prove that a similar statement is true in our case. This makes it possible to study the unstable H^∞ problem by a method which in principle is quite simple: we first stabilize the system using a preliminary feedback, and then we apply the theory for stable systems.

The assumptions that we impose on an unstable system Ψ imply that the input/output map $\mathcal{D}$ of Ψ has a right coprime factorization $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$, defined as follows:

Definition 7.1 *Let* $J = J^* \in \mathcal{L}(Y)$ *and let* $S = S^* \in \mathcal{L}(U)$*. Let* $\mathcal{D} \in TIC_\alpha(U; Y)$ *for some* $\alpha \in \mathbf{R}$.

(i) The operators $\mathcal{N} \in TIC(U; Y)$ *and* $\mathcal{M} \in TIC(U; Z)$ *are right coprime if there exist operators* $\widetilde{\mathcal{Y}} \in TIC(U)$ *and* $\widetilde{\mathcal{X}} \in TIC(Z; U)$ *that together with* $\mathcal{N}$ *and* $\mathcal{M}$ *satisfy the Bezout identity* $\widetilde{\mathcal{Y}}\mathcal{N} + \widetilde{\mathcal{X}}\mathcal{M} = I$ *in* $TIC(U)$.

(ii) The pair $(\mathcal{N}, \mathcal{M})$ is a right coprime factorization of $\mathcal{D}$ if $\mathcal{N} \in TIC(U;Y)$ and $\mathcal{M} \in TIC(U)$ are right coprime, $\mathcal{M}$ has an inverse in $TIC_\alpha(U)$, and $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$.

(iii) The pair $(\mathcal{N}, \mathcal{M})$ is an (J,S)-inner right coprime factorization of $\mathcal{D}$ if $\mathcal{N}$ is (J,S)-inner and $(\mathcal{N}, \mathcal{M})$ is a right coprime factorization of $\mathcal{D}$.

(iv) The pair $(\mathcal{N}, \mathcal{M})$ is an (J,S)-lossless right coprime factorization of $\mathcal{D}$ if $\mathcal{N}$ is (J,S)-lossless and $(\mathcal{N}, \mathcal{M})$ is a right coprime factorization of $\mathcal{D}$.

Our treatment of the unstable case rests on the following basic result:

Lemma 7.2 *Let $J = J^* \in \mathcal{L}(Y)$, let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a well-posed linear system on (U,H,Y) with jointly stabilizing feedback and output injection pairs $\begin{bmatrix}\mathcal{K}^1 & \mathcal{F}^1\end{bmatrix}$ and $\left[\begin{smallmatrix} \mathcal{H} \\ \mathcal{G} \end{smallmatrix}\right]$ [Staffans 1998a, Definition 3.15]. Let*

$$\begin{aligned}
\Psi_\flat &= \begin{bmatrix} \mathcal{A}_\flat & \mathcal{B}_\flat \\ \begin{bmatrix} \mathcal{C}_\flat \\ \mathcal{K}^1_\flat \end{bmatrix} & \begin{bmatrix} \mathcal{D}_\flat \\ \mathcal{F}^1_\flat \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A} + \mathcal{B}\tau \left(I - \mathcal{F}^1\right)^{-1} \mathcal{K}^1 & \mathcal{B}\left(I - \mathcal{F}^1\right)^{-1} \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}\left(I - \mathcal{F}^1\right)^{-1}\mathcal{K}^1 \\ \left(I - \mathcal{F}^1\right)^{-1}\mathcal{K}^1 \end{bmatrix} & \begin{bmatrix} \mathcal{D}\left(I - \mathcal{F}^1\right)^{-1} \\ \left(I - \mathcal{F}^1\right)^{-1} - I \end{bmatrix} \end{bmatrix}
\end{aligned}$$

be the state feedback perturbed version of Ψ [Staffans 1998a, Lemma 3.13], with feedback pair $\begin{bmatrix}\mathcal{K}^1 & \mathcal{F}^1\end{bmatrix}$.

(i) $(\mathcal{D}_\flat, I + \mathcal{F}^1_\flat) = \left(\mathcal{D}\left(I - \mathcal{F}^1\right)^{-1}, \left(I - \mathcal{F}^1\right)^{-1}\right)$ is a right coprime factorization of $\mathcal{D}$.

(ii) The output $y = \mathcal{C}x_0 + \mathcal{D}\pi_+ u$ of Ψ with initial value $x_0 \in H$ and control $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$ is equal to the first output $y = \mathcal{C}_\flat x_0 + \mathcal{D}_\flat \pi_+ u_\flat$ of $\Psi^1_\flat$ with the same initial value $x_0 \in H$ and control $u_\flat \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$ if we choose u and $u_\flat$ to satisfy

$$u = \left(I - \mathcal{F}^1\right)^{-1}\left(\mathcal{K}^1 x_0 + \pi_+ u_\flat\right) = \mathcal{K}^1_\flat x_0 + \left(I + \mathcal{F}^1_\flat\right)\pi_+ u_\flat, \tag{7.1}$$

or equivalently (see [Staffans 1998a, Figures 3.4 and 3.7]),

$$u_\flat = -\mathcal{K}^1 x_0 + \left(I - \mathcal{F}^1\right)\pi_+ u.$$

With this choice of u and $u_\flat$, also the states $x(t) = \mathcal{A}(t)x_0 + \mathcal{B}\tau(t)\pi_+ u$ and $x(t) = \mathcal{A}_\flat(t)x_0 + \mathcal{B}_\flat\tau(t)\pi_+ u_\flat$ of the two systems are equal for all $t \in \mathbf{R}^+$. Moreover, $u_\flat \in L^2(\mathbf{R}^+; U)$ iff both $y \in L^2(\mathbf{R}^+; Y)$ and $u \in L^2(\mathbf{R}^+; U)$.

(iii) If $\Psi_\flat$ is J-coercive, then the controls $u \in L^2(\mathbf{R}^+; U)$ of Ψ and $u_\flat \in L^2(\mathbf{R}^+; U)$ of $\Psi_\flat$ are uniquely determined by the initial state x_0 and the (first) output y. In particular, if the output $y = \mathcal{C}x_0 + \mathcal{D}\pi_+ u$ of Ψ with initial value x_0 and control $u \in L^2(\mathbf{R}^+; U)$ is equal to the first output $\mathcal{C}_\flat x_0 + \mathcal{D}_\flat \pi_+ u_\flat$ of $\Psi_\flat$ with initial value x_0 and control $u_\flat \in L^2(\mathbf{R}^+; U)$, then u and $u_\flat$ must satisfy (7.1).

(iv) Let $\begin{bmatrix} \widetilde{\mathcal{K}} & \widetilde{\mathcal{F}} \end{bmatrix}$ and $\begin{bmatrix} \widetilde{\mathcal{H}} \\ \widetilde{\mathcal{G}} \end{bmatrix}$ be another set of jointly stabilizing feedback and output injection pairs for Ψ, and define $\widetilde{\Psi}_\flat$ in the same way as $\Psi_\flat$ was defined, but with $\begin{bmatrix} \mathcal{K}^1 & \mathcal{F}^1 \end{bmatrix}$ replaced by $\begin{bmatrix} \widetilde{\mathcal{K}} & \widetilde{\mathcal{F}} \end{bmatrix}$. (In particular, if Ψ is stable, then we can choose $\begin{bmatrix} \widetilde{\mathcal{K}} & \widetilde{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} \widetilde{\mathcal{H}} \\ \widetilde{\mathcal{G}} \end{bmatrix} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$.) Then $\mathcal{D}_\flat$ is J-coercive if and only if $\widetilde{\mathcal{D}}_\flat$ is J-coercive, $\mathcal{D}_\flat$ has a (J,S)-inner-outer factorization if and only if $\widetilde{\mathcal{D}}_\flat$ has a (J,S)-inner-outer factorization, and $\mathcal{D}_\flat$ has a (J,S)-lossless-outer factorization if and only if $\widetilde{\mathcal{D}}_\flat$ has a (J,S)-lossless-outer factorization.

(v) Suppose that one (hence both) of the systems $\Psi_\flat$ and $\widetilde{\Psi}_\flat$ in (iv) is J-coercive. Then both systems have the same critical observation $y^{\mathrm{crit}}(x_0)$, critical state $x^{\mathrm{crit}}(x_0)$, and Riccati operator Π. The corresponding control u^{crit} for Ψ is the same in both cases, and it is given by (7.1) with u and $u_\flat$ replaced by u^{crit} and $u_\flat^{\mathrm{crit}}$, respectively.

(vi) The input/output map $\mathcal{D}$ of Ψ has a (J,S)-inner coprime factorization iff $\mathcal{D}_\flat^* J \mathcal{D}_\flat$ has a S-spectral factor, and $\mathcal{D}$ has a (J,S)-lossless coprime factorization iff $\mathcal{D}_\flat^* J \mathcal{D}_\flat$ has a S-dissipative spectral factor. In these cases $\Psi_\flat$ is J-dissipative iff S is invertible.

(vii) Suppose that $\Psi_\flat$ is J-coercive. Then the input/output map $\mathcal{D}$ of Ψ has a (J,S)-inner coprime factorization iff $\mathcal{D}_\flat$ has a (J,S)-inner-outer factorization, and $\mathcal{D}$ has a (J,S)-lossless coprime factorization iff $\mathcal{D}_\flat$ has a (J,S)-lossless-outer factorization.

Proof. (i) This follows from [Staffans 1998a, Theorem 4.4].
(ii) This is [Staffans 1998b, Lemma 3.9(i)].

(iii) This proof is identical to the proof of [Staffans 1998b, Lemma 3.9(iv)].

(iv) By [Staffans 1998a, Lemma 4.3], we have that the set of all possible right ω-coprime factorizations of $\mathcal{D}$ can be parameterized in the form $(\mathcal{N}\mathcal{U}, \mathcal{M}\mathcal{U})$, where $(\mathcal{N}, \mathcal{M})$ is a fixed right coprime factorization of Ψ. This combined with part (i) implies that $\mathcal{D}_\flat = \widetilde{\mathcal{D}}_\flat\mathcal{U}$, where $\mathcal{U} = (I - \widetilde{\mathcal{F}})(I - \mathcal{F}^1)^{-1}$ belongs to $TIC(U)$ and is invertible in $TIC(U)$. Since $\mathcal{U}$ is causal and $\mathcal{U}^*$ is anti-causal, we have

$$\pi_+\mathcal{D}_\flat^* J\mathcal{D}_\flat\pi_+ = \pi_+\mathcal{U}^*\mathcal{D}^* J\mathcal{D}\mathcal{U}\pi_+ = \pi_+\mathcal{U}^*\pi_+\mathcal{D}^* J\mathcal{D}\pi_+\mathcal{U}\pi_+.$$

Both the operator $\pi_+\mathcal{U}\pi_+$ and its adjoint $\pi_+\mathcal{U}^*\pi_+$ are bounded invertible operators on $L^2(\mathbf{R}^+; U)$ (for example, the inverse of $\pi_+\mathcal{U}\pi_+$ is $\pi_+\mathcal{U}^{-1}\pi_+$), so $\pi_+\mathcal{D}_\flat^* J\mathcal{D}_\flat\pi_+$ is invertible if and only if $\pi_+\mathcal{D}^* J\mathcal{D}\pi_+$ is invertible, i.e., $\Psi_\flat$ is J coercive if and only if Ψ is so. The proofs of the claims concerning the existence of (dissipative) S-spectral factors are similar but simpler.

(v) Define

$$\tilde{u}_\flat = -\widetilde{\mathcal{K}}x_0 + \mathcal{U}\left(\mathcal{K}^1 x_0 + \pi_+ u_\flat\right),$$

where $\mathcal{U} = (I - \widetilde{\mathcal{F}})(I - \mathcal{F}^1)^{-1}$. By part (i), the input $\tilde{u}_\flat$ to $\widetilde{\Psi}_\flat$ produces the same first output y as the input $u_\flat$ to $\Psi_\flat$ does. By [Staffans 1998b, Lemmas 4.3 and 4.5], the term $(\mathcal{U}\mathcal{K}^1 - \widetilde{\mathcal{K}})x_0$ belongs to $L^2(\mathbf{R}^+; Y)$. Thus, for fixed x_0, the preceding formula defines a continuous invertible transformation between $\tilde{u}_\flat \in L^2(\mathbf{R}^+; U)$ and $u_\flat \in L^2(\mathbf{R}^+; U)$. Make the change of variable from $\tilde{u}_\flat$ to $u_\flat$ in the cost function for $\widetilde{\Psi}_\flat$, differentiate with respect to $u_\flat$, and set the result equal to zero to show that the two problems have the same critical output and state and the same Riccati operator (and that their critical controls are related as in the formula above). The last claim follows from (ii).

(vi)–(vii) These claims follow from the definitions, from Lemma 4.6(ii), and from the parameterization of all coprime factorizations given in [Staffans 1998a, Lemma 4.3] (cf. the proof of (iv) and observe that the invertible outer factor can be absorbed into the denominator $\mathcal{M}$). $\square$

Motivated by Lemma 7.2, we introduce the following definition.

Definition 7.3 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a jointly stabilizable and detectable well-posed linear system on (U, H, Y) [Staffans 1998a, Definition 3.16]. Let $\left[\mathcal{K}^1 \quad \mathcal{F}^1\right]$ and $\left[\begin{smallmatrix} \mathcal{H} \\ \mathcal{G} \end{smallmatrix}\right]$ be jointly stabilizing feedback and output injection pairs for Ψ, and define $\Psi_\flat$ as in Lemma 7.2. Let $J = J^* \in \mathcal{L}(Y)$.*

(i) *The system* Ψ *is* J*-coercive if* $\Psi_\flat$ *is* J*-coercive.*

(ii) *In the* J*-coercive case the control* $u^{\text{crit}}(x_0) \in L^2(\mathbf{R}^+;U)$ *is* J*-critical for* Ψ *if* $u_\flat^{\text{crit}} = -\mathcal{K}^1 x_0 + (I - \mathcal{F}^1)\, u^{\text{crit}}(x_0)$ *is* J*-critical for* $\Psi_\flat$*.*

(iii) *In the* J*-coercive case the Riccati operator of* Ψ *is equal to the Riccati operator of* $\Psi_\flat$*.*

With this definition, we get the following analogue of Lemma 3.4:

Lemma 7.4 *In addition to the assumptions of Lemma 7.2, suppose that* Ψ *is* J*-coercive. Then, for every* $x_0 \in H$*, there is a unique* J*-critical control* $u^{\text{crit}}(x_0)$*, given by*

$$u^{\text{crit}}(x_0) = \mathcal{K}_\flat^1 x_0 - \left(I + \mathcal{F}_\flat^1\right) (\pi_+ \mathcal{D}_\flat^* J \mathcal{D}_\flat \pi_+)^{-1} \pi_+ \mathcal{D}_\flat^* J \mathcal{C}_\flat x_0. \tag{7.2}$$

In particular, $u^{\text{crit}}(x_0)$ *does not depend on the particular state feedback pair* $\begin{bmatrix} \mathcal{K}^1 & \mathcal{F}^1 \end{bmatrix}$*. The corresponding critical state* $x^{\text{crit}}(x_0)$*, the critical observation* $y^{\text{crit}}(x_0)$*, and the critical value of* Q *are given by*

$$x^{\text{crit}}(t, x_0) = \mathcal{A}_\flat(t) x_0 - \mathcal{B}_\flat \tau(t) \pi_+ (\pi_+ \mathcal{D}_\flat^* J \mathcal{D}_\flat \pi_+)^{-1} \pi_+ \mathcal{D}_\flat^* J \mathcal{C}_\flat x_0, \tag{7.3}$$

$$y^{\text{crit}}(x_0) = \left(I - \mathcal{D}_\flat \pi_+ (\pi_+ \mathcal{D}_\flat^* J \mathcal{D}_\flat \pi_+)^{-1} \pi_+ \mathcal{D}_\flat^* J\right) \mathcal{C}_\flat x_0, \tag{7.4}$$

$$Q(x_0, u^{\text{crit}}(x_0)) = \left\langle x_0, \mathcal{C}_\flat^* \left(J - J\mathcal{D}_\flat \pi_+ (\pi_+ \mathcal{D}_\flat^* J \mathcal{D}_\flat \pi_+)^{-1} \pi_+ \mathcal{D}_\flat^* J\right) \mathcal{C}_\flat x_0 \right\rangle_H . \tag{7.5}$$

Proof. This follows from Lemma 3.4, Lemma 7.2, and Definition 7.3. $\square$

If $\mathcal{D}$ has a (J,S)-inner right coprime factorization, then we can add the following conclusions to Lemma 7.4:

Lemma 7.5 *In addition to the assumptions of Lemma 7.4, suppose that* $\mathcal{D}$ *has a* (J,S)*-inner right coprime factorization* $(\mathcal{N}, \mathcal{M})$*. Then* S *is invertible in* $\mathcal{L}(U)$*,*

(i) *the inverse of* $\pi_+ \mathcal{D}_\flat^* J \mathcal{D}_\flat \pi_+$ *can be written in the form*

$$(\pi_+ \mathcal{D}_\flat^* J \mathcal{D}_\flat \pi_+)^{-1} = \mathcal{Z} S^{-1} \pi_+ \mathcal{Z}^*, \tag{7.6}$$

where $\mathcal{Z}$ *is invertible in* $TIC(U)$ *and satisfies* $\mathcal{M} = (I + \mathcal{F}_\flat^1)\, \mathcal{Z}$*, and* *(ii) the critical control $u^{\mathrm{crit}}(x_0)$, the critical state $x^{\mathrm{crit}}(x_0)$, the critical observation $y^{\mathrm{crit}}(x_0)$, and the Riccati operator can be written in the form*

$$u^{\mathrm{crit}}(x_0) = \mathcal{K}_\flat^1 x_0 - \mathcal{M}S^{-1}\pi_+\mathcal{N}^* J\mathcal{C}_\flat x_0, \tag{7.7}$$

$$x^{\mathrm{crit}}(t,x_0) = \mathcal{A}_\flat(t)x_0 - \mathcal{B}\mathcal{M}\tau(t)S^{-1}\pi_+\mathcal{N}^* J\mathcal{C}_\flat x_0, \tag{7.8}$$

$$y^{\mathrm{crit}}(x_0) = \left(I - \mathcal{N}S^{-1}\pi_+\mathcal{N}^* J\right)\mathcal{C}_\flat x_0, \tag{7.9}$$

$$\Pi = \mathcal{C}_\flat^* \left(J - J\mathcal{N}S^{-1}\pi_+\mathcal{N}^* J\right)\mathcal{C}_\flat. \tag{7.10}$$

We are now ready to extend Theorem 5.1 to the unstable case:

Theorem 7.6 *Let $J = J^* \in \mathcal{L}(Y)$, and let $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be a jointly stabilizable and detectable J-coercive well-posed linear system on (U,H,Y). Let $x_0 \in H$, and let $u^{\mathrm{crit}}(x_0)$, $x^{\mathrm{crit}}(x_0)$, and $y^{\mathrm{crit}}(x_0)$ be the critical control, state, and observation (see Lemma 7.5), and let Π be the Riccati operator of Ψ.*

(i) Suppose that $\mathcal{D}$ has a (J,S)-inner right coprime factorization $(\mathcal{N},\mathcal{M})$. Then S is invertible in $\mathcal{L}(U)$, and there is a unique feedback map $\mathcal{K}$ such that $\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix} = \begin{bmatrix}\mathcal{K} & (I - \mathcal{M}^{-1})\end{bmatrix}$ is an admissible stabilizing state feedback pair for Ψ and

$$\begin{bmatrix} x^{\mathrm{crit}}(t,x_0) \\ y^{\mathrm{crit}}(x_0) \\ u^{\mathrm{crit}}(x_0) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_\circlearrowleft(t) \\ \mathcal{C}_\circlearrowleft \\ \mathcal{K}_\circlearrowleft \end{bmatrix} x_0 = \begin{bmatrix} \mathcal{A}(t) + \mathcal{B}\mathcal{M}\tau(t)\mathcal{K} \\ \mathcal{C} + \mathcal{N}\mathcal{K} \\ \mathcal{M}\mathcal{K} \end{bmatrix} x_0$$

is equal to the state and output of the closed loop system $\Psi_\circlearrowleft$ defined by

$$\Psi_\circlearrowleft = \begin{bmatrix} \mathcal{A}_\circlearrowleft & \mathcal{B}_\circlearrowleft \\ \begin{bmatrix}\mathcal{C}_\circlearrowleft \\ \mathcal{K}_\circlearrowleft\end{bmatrix} & \begin{bmatrix}\mathcal{D}_\circlearrowleft \\ \mathcal{F}_\circlearrowleft\end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}\tau\mathcal{M}\mathcal{K} & \mathcal{B}\mathcal{M} \\ \begin{bmatrix}\mathcal{C} + \mathcal{N}\mathcal{K} \\ \mathcal{M}\mathcal{K}\end{bmatrix} & \begin{bmatrix}\mathcal{N} \\ \mathcal{M} - I\end{bmatrix} \end{bmatrix}$$

with initial value x_0, initial time zero, and zero control $u_\circlearrowleft$ (see Figure 2). The feedback map $\mathcal{K}$ is uniquely determined by the fact that $\mathcal{C}_\circlearrowleft = \mathcal{C} + \mathcal{N}\mathcal{K} \in \mathcal{L}(H; L^2(\mathbf{R}^+;Y))$, $\mathcal{K}_\circlearrowleft = \mathcal{M}\mathcal{K} \in \mathcal{L}(H; L^2(\mathbf{R}^+;U))$, and $\pi_+\mathcal{N}^* J\mathcal{C}_\circlearrowleft = 0$. Moreover, the Riccati operator of Ψ is given by

$$\Pi = \mathcal{C}_\circlearrowleft^* J\mathcal{C}_\circlearrowleft = (\mathcal{C} + \mathcal{N}\mathcal{K})^* J(\mathcal{C} + \mathcal{N}\mathcal{K}).$$

(ii) If $y = \mathcal{C}_\circlearrowleft x_0 + \mathcal{D}_\circlearrowleft \pi_+ u_\circlearrowleft$ is the first output of the optimal closed loop system $\Psi_\circlearrowleft$ in (i) with initial state $x_0 \in H$ and control $u_\circlearrowleft \in L^2(\mathbf{R}^+; U)$ (see Figure 2), then the closed loop cost $Q_\circlearrowleft(x_0, u_\circlearrowleft)$ is given by

$$Q_\circlearrowleft(x_0, u_\circlearrowleft) = \langle y, Jy \rangle_{L^2(\mathbf{R}^+;Y)} = \langle x_0, \Pi x_0 \rangle_H + \langle u_\circlearrowleft, S u_\circlearrowleft \rangle_{L^2(\mathbf{R}^+;Y)} . \tag{7.11}$$

(iii) If Ψ is jointly ω-stabilizable and detectable for some $\omega < 0$ [Staffans 1998a, Definition 3.16], and if $\mathcal{N}$ and $\mathcal{M}$ in (i) are right ω-coprime [Staffans 1998a, Definition 4.1], then the closed loop system $\Psi_\circlearrowleft$ is ω-stable.

(iv) If $(\mathcal{N}, \mathcal{M})$ are given, then the feedback map $\mathcal{K}$, the Riccati operator Π, the closed loop semigroup $\mathcal{A}_\circlearrowleft$, and the closed loop controllability and feedback maps $\mathcal{C}_\circlearrowleft$ and $\mathcal{K}_\circlearrowleft$ can be computed as follows: Choose some arbitrary jointly stabilizing feedback and output injection pairs $\begin{bmatrix} \mathcal{K}^1 & \mathcal{F}^1 \end{bmatrix}$ and $\left[\begin{smallmatrix} \mathcal{H} \\ \mathcal{G} \end{smallmatrix}\right]$. Then

$$\begin{aligned}
\mathcal{K} &= \mathcal{M}^{-1}\mathcal{K}^1_\flat - S^{-1}\pi_+\mathcal{N}^* J \mathcal{C}_\flat, \\
\begin{bmatrix} \mathcal{A}_\circlearrowleft \\ \mathcal{C}_\circlearrowleft \\ \mathcal{K}_\circlearrowleft \end{bmatrix} &= \begin{bmatrix} \mathcal{A}_\flat \\ \mathcal{C}_\flat \\ \mathcal{K}^1_\flat \end{bmatrix} - \begin{bmatrix} \mathcal{B}\mathcal{M}\tau \\ \mathcal{N} \\ \mathcal{M} \end{bmatrix} S^{-1}\pi_+\mathcal{N}^* J \mathcal{C}_\flat, \\
\Pi &= \mathcal{C}^*_\flat J \mathcal{C}_\flat - \left(\mathcal{K} - \mathcal{M}^{-1}\mathcal{K}^1_\flat\right)^* S \left(\mathcal{K} - \mathcal{M}^{-1}\mathcal{K}^1_\flat\right) \\
&= \mathcal{C}^*_\flat \left(J - J\mathcal{N}S^{-1}\pi_+\mathcal{N}^* J\right) \mathcal{C}_\flat = \mathcal{C}^*_\flat J \mathcal{C}_\circlearrowleft = \mathcal{C}^*_\circlearrowleft J \mathcal{C}_\flat,
\end{aligned}$$

where $\mathcal{A}_\flat = \mathcal{A} + \mathcal{B}\tau\mathcal{K}^1_\flat$, $\mathcal{C}_\flat = \mathcal{C} + \mathcal{D}\mathcal{K}^1_\flat$ and $\mathcal{K}^1_\flat = (I - \mathcal{F}^1)^{-1}\mathcal{K}^1$. (If Ψ is stable, then we can can take $\mathcal{K}^1_\flat = 0$, $\mathcal{A}_\flat = \mathcal{A}$, and $\mathcal{C}_\flat = \mathcal{C}$, and get the same formulas as in Theorem 5.1).

Both the statement and the proof of this theorem is virtually identical to the statement and proof of [Staffans 1998b, Theorem 4.4], and we leave its proof to the reader.

The following analogue of Remark 5.2 is true as well:

Remark 7.7 *Let $J = J^* \in \mathcal{L}(Y)$, and let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a jointly stabilizable and detectable J-coercive well-posed linear system on (U, H, Y), and suppose that $\mathcal{D}$ has a (J, S)-inner right coprime factorization $(\mathcal{N}, \mathcal{M})$ for some $S = S^* \in \mathcal{L}(U)$. Then S is invertible in $\mathcal{L}(U)$ and all the claims in*

[Staffans 1998b, Sections 5–7] are valid if we drop the positivity requirement on S, replace all references to [Staffans 1998b, Theorem 4.4] by references to Theorem 7.6, and throughout replace $x^{\mathrm{opt}}(x_0)$, $y^{\mathrm{opt}}(x_0)$ and $u^{\mathrm{opt}}(x_0)$ by $x^{\mathrm{crit}}(x_0)$, $y^{\mathrm{crit}}(x_0)$ and $u^{\mathrm{crit}}(x_0)$, respectively. Also the parameterization results [Staffans 1998b, Propositions 4.7 and 4.8] remain valid in the same sense.

Proof. By Theorem 5.2, these claims are true if we replace Ψ by the stabilized system $\Psi_\flat$ in Lemma 7.2. The crucial step is to show that (3.8) and (3.9) hold, and this follows from the fact that (3.8) and (3.9) hold if we replace Ψ by $\Psi_\flat$. $\square$

By the same method Theorem 6.5 can be extended as follows:

Theorem 7.8 *Let $J = J^* \in \mathcal{L}(Y)$, and let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a jointly stabilizable and detectable J-coercive well-posed linear system on (U, H, Y), and suppose that $\mathcal{D}$ has a (J,S)-inner right coprime factorization $(\mathcal{N}, \mathcal{M})$ for some $S = S^* \in \mathcal{L}(U)$. Then the following conditions are equivalent:*

(i) $\mathcal{D}$ has a (J,S)-lossless coprime factorization.

(ii) Every (J,S)-inner coprime factorization of $\mathcal{D}$ is (J,S)-lossless.

(iii) The Riccati operator Π is nonnegative on the reachable subspace of Ψ.

We leave the easy proof to the reader.

We have now developed that part of our solution to the stable full information model matching problem which does not distinguish between the control and the disturbance as far as we need. In the sequel we shall make a clear distinction between the control and the disturbance, use a minimax setting, and study several additional properties of the closed loop system. See Staffans [1998c] for the stable case and Staffans [1999] for the unstable case.

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