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State/Signal Linear Time-Invariant Systems Theory, Part I: Discrete Time Systems

Damir Z. Arov and Olof J. Staffans

Abstract. This is the first paper in a series of several papers in which we develop a state/signal linear time-invariant systems theory. In this first part we shall present the general state/signal setting in discrete time. Our following papers will deal with conservative and passive state/signal systems in discrete time, the general state/signal setting in continuous time, and conservative and passive state/signal systems in continuous time, respectively. The state/signal theory that we develop differs from the standard input/state/output theory in the sense that we do not distinguish between input signals and output signals, only between the "internal" states x and the "external" signals w. In the development of the general state/signal systems theory we take both the state space $\mathcal{X}$ and the signal space $\mathcal{W}$ to be Hilbert spaces. In later papers where we discuss conservative and passive systems we assume that the signal space $\mathcal{W}$ has an additional Kreĭn space structure. The definition of a state/signal system has been designed in such a way that to any state/signal system there exists at least one decomposition of the signal space $\mathcal{W}$ as the direct sum $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ such that the evolution of the system can be described by the standard input/state/output system of equations with input space $\mathcal{U}$ and output space $\mathcal{Y}$. (In a passive state/signal system we may take $\mathcal{U}$ and $\mathcal{Y}$ to be the positive and negative parts, respectively, of a fundamental decomposition of the Kreĭn space $\mathcal{W}$.) Thus, to each state/signal system corresponds infinitely many input/state/output systems constructed in the way described above. A state/signal system consists of a state/signal node and the set of trajectories generated by this node. A state/signal node is a triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$, where V is a subspace with appropriate properties of the product space $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$. In this first paper we extend standard input/state/output notions, such as existence and uniqueness of solutions, continuous dependence on initial data, observability, controllability, stabilizability, detectability, and minimality to the state/signal setting. Three classes of representations of state/systems are presented (one of which is the class of input/state/output representations), and the families of all the transfer functions of these representations are studied. We also discuss realizations of signal behaviors by state/signal systems, as well as dilations and compressions of these systems. (Duality will be discussed later in connection with passivity and conservativity.)

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Contents

1. Introduction

The main motivation for this work comes from the notion of a multi-port network. Such a network consists of *internal branches*, where the evolution of the data is described by, e.g., systems of ordinary or partial differential equations involving *state variables* (lumped or distributed), and *external branches* (*ports*), where the evolution of the *port variables* is only partially restricted by the network equations. Typically one part of the port variables can be prescribed in an arbitrary way (this is the "input" part), after which the remaining "output" part of the port variables can be computed from the network equations. However, the splitting of the port variables into an input part and output part is not specified, and many different choices are possible.

To be a little more concrete, let us consider a two-port Kirchhoff network, i.e., a Kirchhoff network with two external branches. To each of these branches we associate at each time instant t a normalized voltage/current pair $(v_1(t), i_1(t))$, respectively, $(v_2(t), i_2(t))$ (normalization means that we divide each voltage by $\sqrt{R}$ and multiply each current by $\sqrt{R}$, where R is a fixed resistance). Thus, the complete set of port variables is the four-dimensional vector $w(t) = (v_1(t), i_1(t), v_2(t), i_2(t))$.

Sometimes we may use $u(t) = (v_1(t), i_1(t))$ as the input data, and regard $(v_2(t), i_2(t))$ as the output data (or the other way around). This case is called the *transmission* case, and it is used, e.g., in the cascade synthesis of two-ports. However, this choice of input and output data is not always possible or reasonable. Another possibility is to choose $u(t) = (i_1(t), i_2(t))$ as the input data and $y(t) = (v_1(t), v_2(t))$ as the output data (or the other way around). These cases are referred to as the *impedance* and *admittance* cases, and they are used, e.g., in series and parallel connections of networks. Neither is this choice of input and output data always possible or reasonable. In his development of the theory of passive Kirchhoff networks V. Belevitch [Bel68] proposed the use of the incoming wave data $u(t) = (\frac{1}{\sqrt{2}}(v_1(t) + i_1(t)), \frac{1}{\sqrt{2}}(v_2(t) + i_2(t)))$ as input data and the outgoing wave data $u(t) = (\frac{1}{\sqrt{2}}(v_1(t) - i_1(t)), \frac{1}{\sqrt{2}}(v_2(t) - i_2(t)))$ as output. This case is called the *scattering* case, and this particular decomposition is always possible and meaningful for passive Kirchhoff networks. In all these cases the physical network is the same, but depending on the decomposition of $w(t) = (v_1(t), i_1(t), v_2(t), i_2(t))$ into an input part and an output part we get very different input/state/output characteristics.

The idea of considering the evolution of external signals $w(t)$ without an explicit decomposition into an input part $u(t)$ and an output part $y(t)$ is the most fundamental ingredient in the behavioral theory initiated by J. Willems (see, e.g., [PW98] for a recent presentation of behavioral theory). Our approach differs from the standard behavioral approach in the sense that we always include a state variable in the equations describing the evolution of the system, and we more or less ignore polynomial descriptions as well as dynamics generated by ordinary differential equations. It is genuinely infinite-dimensional, and it appears to be applicable to a large class of infinite-dimensional problems. A first step in this direction was taken by J. Ball and O. Staffans [BS05], where the main notion of a state/signal node and its trajectories are found in an implicit way.

A state/signal system consists of a state/signal node and the set of trajectories generated by this node. A state/signal node is a triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$, where $\mathcal{X}$ (the state space) and $\mathcal{W}$ (the signal space) are Hilbert spaces, and V is a subspace of the product space $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ with appropriate properties. In this paper we shall only discuss systems with discrete time. The list of properties that the subspace V should satisfy in this case is given in Definition 2.1. By a *trajectory* $(x(\cdot), w(\cdot))$ of Σ on $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ we mean a pair of sequences $\{x(n)\}_{n=0}^\infty$ and $\{w(n)\}_{n=0}^\infty$ satisfying

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^+. \tag{1.1}$$

The properties of the subspace V have been chosen in such a way that there exists at least one *admissible* decomposition (actually infinitely many decompositions) of the signal space $\mathcal{W}$ as the direct sum $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of an input space $\mathcal{U}$ and an output space $\mathcal{Y}$ such that trajectories are defined by a usual input/state/output system of equations

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \qquad n \in \mathbb{Z}^+, \\ x(0) &= x_0, \end{aligned} \tag{1.2}$$

where the coefficients A, B, C, and D are bounded linear operators between the respective Hilbert spaces, i.e., $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right])$. The set of all trajectories $(x(\cdot), w(\cdot))$ of the state/signal system (1.1) can be obtained from the set of trajectories of (1.2) by taking the state sequence $x(\cdot)$ to be the same and taking $w(\cdot) = y(\cdot) + u(\cdot)$. The latter equation we write alternatively in the form $w(\cdot) = \left[\begin{smallmatrix} y(\cdot) \\ u(\cdot) \end{smallmatrix}\right]$, and likewise, instead of $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ we write alternatively $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$.

In addition to these input/state/output representations, there are two other useful types of representations, namely *driving variable* and *output nulling* representations. In a *driving variable representation* we parameterize the trajectories by using an extra driving variable ℓ with values in an auxiliary *driving variable* Hilbert space $\mathcal{L}$. The trajectories of the system are described by a system of equations

$$\begin{aligned} x(n+1) &= A'x(n) + B'\ell(n), \\ w(n) &= C'x(n) + D'\ell(n), \qquad n \in \mathbb{Z}^+, \\ x(0) &= x_0, \end{aligned} \tag{1.3}$$

where the coefficients (A', B', C', D') are bounded linear operators between the respective Hilbert spaces, i.e., $\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right])$, and D' is injective and has closed range. The set of all trajectories $(x(\cdot), w(\cdot))$ of the state/signal system Σ can be obtained from the set of trajectories $(x(\cdot), \ell(\cdot), w(\cdot))$ of (1.3) by simply dropping the driving variable $\ell(\cdot)$. In an *output nulling representation* we formally consider the signal component w as an input which is restricted by an additional equation posed in an auxiliary *error space* $\mathcal{K}$. The trajectories of this new input/state/output system are described by a system of equations

$$\begin{aligned} x(n+1) &= A''x(n) + B''w(n), \\ e(n) &= C''x(n) + D''w(n), \qquad n \in \mathbb{Z}^+, \\ x(0) &= x_0, \end{aligned} \tag{1.4}$$

where the coefficients (A'', B'', C'', D'') are bounded linear operators between the respective Hilbert spaces, i.e., $\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{K} \end{smallmatrix}\right])$, and D'' is surjective. The reason for the name "output nulling" for this representation is that $(x(\cdot), w(\cdot))$ is a trajectory of Σ if and only if $(x(\cdot), w(\cdot), e(\cdot))$ with $e(n) = 0$ for all n is a trajectory of the input/state/output system described by (1.4).

To each state/signal system there corresponds infinitely many representations of each of the three types described above. We prove the existence of these three types of representations, discuss their properties, and also discuss the relationships between different representations of the same type or of different types.

Each input/state/output representation (1.2) of a given state/signal system has a $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued transfer function given by

$$\mathfrak{D}(z) = D + zC(1_{\mathcal{X}} - zA)^{-1}B, \quad z \in \Lambda_A, \tag{1.5}$$

where Λ_A is the set of points $z \in \mathbb{C}$ for which $(1_{\mathcal{X}} - zA)$ has a bounded inverse, plus the point at infinity if A is boundedly invertible. Thus, each state/signal system has infinitely many such transfer functions, one corresponding to each input/state/output representation. All of these transfer functions can be obtained from one fixed input/state/output representation through the use of a linear fractional transformation. More precisely, let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ and $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ be two admissible input/output decompositions of the signal space $\mathcal{W}$ of a given state/signal system Σ, and denote the corresponding transfer functions by $\mathfrak{D}$ and $\mathfrak{D}_1$, respectively. Let

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{Y}_1}^{\mathcal{U}_1}|_{\mathcal{Y}} & P_{\mathcal{Y}_1}^{\mathcal{U}_1}|_{\mathcal{U}} \\ P_{\mathcal{U}_1}^{\mathcal{Y}_1}|_{\mathcal{Y}} & P_{\mathcal{U}_1}^{\mathcal{Y}_1}|_{\mathcal{U}} \end{bmatrix}, \tag{1.6}$$

where $P_{\mathcal{Y}_1}^{\mathcal{U}_1}|_{\mathcal{Y}}$ is the restriction to $\mathcal{Y}$ of the projection of $\mathcal{W}$ onto $\mathcal{Y}_1$ along $\mathcal{U}_1$, etc. (Note that Θ can be interpreted as a decomposition of the identity in $\mathcal{W}$ with respect to the two sum decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$.) Then $\mathfrak{D}_1$ is the value of the linear fractional transform of $\mathfrak{D}$ with coefficient matrix Θ, i.e.,

$$\mathfrak{D}_1(z) = [\Theta_{11}\mathfrak{D}(z) + \Theta_{12}][\Theta_{21}\mathfrak{D}(z) + \Theta_{22}]^{-1}, \quad z \in \Lambda_A \cap \Lambda_{A_1}. \tag{1.7}$$

We also introduce notions of controllability, observability, and minimality of state/signal systems. These notions are defined in terms of the properties of its trajectories, without any reference to the various representations described above, but it is possible to give equivalent conditions for controllability and observability in terms of the different types of representations described above. In particular, we prove that a state/signal system is controllable (or observable, or minimal) if and only if at least one corresponding input/state/output system (1.2) (hence all of them) has the same property.

In Section 2 we discuss the main notions of the theory: state/signal nodes, the corresponding trajectories, and their basic properties. In Sections 3 and 4 we study driving variable and output nulling representations, respectively. Here we also define the notions of controllability and observability and develop tests for controllability and observability in terms of driving variable and output nulling representations. Input/state/output representations are studied in Section 5. Here we also give criteria for the admissibility of a decomposition of the signal space $\mathcal{W}$ into an input space $\mathcal{U}$ and an output space $\mathcal{Y}$ and describe the connections between different representations. Different kinds of transfer functions related to different representations of state/signal systems and their connections are studied in Section 6. In Section 7 we introduce and study signal behaviors and their realizations by means of state/signal systems. Dilations of state/signal systems are studied in depth in Section 8. In particular, we show that a dilation of a state/signal system has the same signal behavior and also the same set of input/output transfer functions (restricted to a neighborhood of zero) as the original system.

The main result of this section characterizes dilations in terms of the existence of a decomposition of the state space into parts with certain invariance properties. All the proofs are given in the state/signal setting, and we obtain standard input/state/output results as corollaries of our main results. Finally, Section 9 is devoted to a study of different stabilizability properties of state/signal systems in terms of the existence of stable representations of driving variable, output nulling, or input/state/output type. Not only power stability, but also strong stability is studied.

Notation. The space of bounded linear operators from one normed space $\mathcal{X}$ to another normed space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X};\mathcal{Y})$, and we abbreviate $\mathcal{B}(\mathcal{X};\mathcal{X})$ to $\mathcal{B}(\mathcal{X})$. The domain of a linear operator A is denoted by $\mathcal{D}(A)$, its range by $\mathcal{R}(A)$, and its kernel by $\mathcal{N}(A)$. The restriction of A to some subspace $\mathcal{Z} \subset \mathcal{D}(A)$ is denoted by $A|_{\mathcal{Z}}$. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$. For each $A \in \mathcal{B}(\mathcal{X})$ we let Λ_A be the set of points $z \in \mathbb{C}$ for which $(1_{\mathcal{X}} - zA)$ has a bounded inverse, plus the point at infinity if A is boundedly invertible.

$\mathbb{C}$ is the complex plane, $\mathbb{D}$ is the open unit disk in $\mathbb{C}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, and $\mathbb{Z}^- = \{-1, -2, \ldots\}$. The space $H^2(\mathbb{D};\mathcal{U})$, where $\mathcal{U}$ is a Hilbert space, consists of all analytic $\mathcal{U}$-valued functions ϕ on $\mathbb{D}$ which satisfy $\|\phi\|^2 := \sup_{0\le r<1} \frac{1}{2\pi} \oint_{|z|=r} \|\phi(z)\|^2 |dz| < \infty$. The space $H^\infty(\mathbb{D};\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, consists of all bounded analytic $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued functions on $\mathbb{D}$. The sequence spaces $\ell^1(\mathbb{Z}^+;\mathcal{U})$ and $\ell^2(\mathbb{Z}^+;\mathcal{U})$ contain those $\mathcal{U}$-valued sequences $u(\cdot)$ on $\mathbb{Z}^+$ which satisfy $\sum_{n\in\mathbb{Z}^+} \|u(n)\| < \infty$, respectively, $\sum_{n\in\mathbb{Z}^+} \|u(n)\|^2 < \infty$, and $\ell^\infty(\mathbb{Z}^+;\mathcal{U})$ consists of all bounded $\mathcal{U}$-valued sequences on $\mathbb{Z}^+$.

We denote the projection onto a closed subspace $\mathcal{Y}$ of a space $\mathcal{X}$ along some complementary subspace $\mathcal{U}$ by $P_{\mathcal{Y}}^{\mathcal{U}}$. The closed linear span or linear span of a sequence of subsets $\mathfrak{R}_n \subset \mathcal{X}$ where n runs over some index set Λ is denoted by $\vee_{n\in\Lambda} \mathfrak{R}_n$ and $\operatorname{span}_{n\in\Lambda} \mathfrak{R}_n$, respectively.

We denote the product of the two locally convex topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$ by $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$. In particular, although $\mathcal{X}$ and $\mathcal{Y}$ may be Hilbert spaces (in which case the product topology in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ is induced by an inner product), we shall not require that $\left[\begin{smallmatrix}\mathcal{X}\\ 0\end{smallmatrix}\right] \perp \left[\begin{smallmatrix}0\\ \mathcal{Y}\end{smallmatrix}\right]$ in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$. Furthermore, in this case we identify a vector $\left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right] \in \left[\begin{smallmatrix}\mathcal{X}\\ 0\end{smallmatrix}\right]$ with $x \in \mathcal{X}$ and a vector $\left[\begin{smallmatrix}0\\ y\end{smallmatrix}\right] \in \left[\begin{smallmatrix}0\\ \mathcal{Y}\end{smallmatrix}\right]$ with $y \in \mathcal{Y}$. (Thus, we also denote the ordered direct sum $\mathcal{X} \dotplus \mathcal{Y}$ by $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$.)

2. State/signal nodes and trajectories

In this section we shall study time-invariant linear systems induced by something that we call a *state/signal node*.

Definition 2.1. A triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$, where the (*internal*) *state space* $\mathcal{X}$ and the (*external*) *signal space* $\mathcal{W}$ are Hilbert spaces and V is a subspace of the product space $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ is called a *state/signal node* if it has the following properties:[1]

(i) V is closed in $\mathfrak{K}$;

(ii) For every $x \in \mathcal{X}$ there is some $\begin{bmatrix} z \\ w \end{bmatrix} \in \begin{bmatrix} X \\ \mathcal{W} \end{bmatrix}$ such that $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$;

(iii) If $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$, then $z = 0$;

(iv) The set $\left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \text{ for some } z \in \mathcal{X} \right\}$ is closed in $\begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}$.

We call $\mathfrak{K}$ the *node space* and V the *generating subspace*.

As we shall see in a moment (in Proposition 2.2, Lemmas 2.3–2.4 and Theorem 2.5), all of these conditions have a clear meaning related to the fact that we shall use the generating subspace V as the main tool in our definition of a *trajectory*. To *define* such a trajectory it is not important that (i)–(iv) hold.

We define a *trajectory* $(x(\cdot), w(\cdot))$ *along an arbitrary subspace* V of $\mathfrak{K}$ on the time interval $[n_1, n_2]$, where n_1, $n_2 \in \mathbb{Z}$, $n_1 \le n_2$, to be a pair of sequences $\{x(k)\}_{k=n_1}^{n_2+1}$ and $\{w(k)\}_{k=n_1}^{n_2}$ satisfying

$$\begin{bmatrix} x(k+1) \\ x(k) \\ w(k) \end{bmatrix} \in V, \qquad n_1 \le k \le n_2. \tag{2.1}$$

We shall also allow $n_1 = -\infty$ or $n_2 = \infty$, in which case we replace $\le$ by $<$ in the formula above. Most of our trajectories will be considered on $\mathbb{Z}^+$. We shall refer to the sequence $x(\cdot)$ as the *state component* and to the sequence $w(\cdot)$ as the *signal component* of the trajectory $(x(\cdot), w(\cdot))$. In the case where n_1 is finite we shall call $x(n_1)$ the *initial state* of this trajectory.

It follows immediately from Definition 2.1 that the set of trajectories along a given subspace V of $\mathfrak{K}$ has the following two properties:

1) *if* $(x(\cdot), w(\cdot))$ *is a trajectory along* V *on* $[n_1, n_2]$*, then for each* $k \in \mathbb{Z}$*, the shifted pair of sequences* $(x(\cdot + k), w(\cdot + k))$ *is a trajectory along* V *on* $[n_1 - k, n_2 - k]$.

2) *if* $(x_1(\cdot), w_1(\cdot))$ *is a trajectory along* V *on* $[n_1, n_2]$*, if* $(x_2(\cdot), w_2(\cdot))$ *is a trajectory along* V *on* $[n_2+1, n_3]$*, and if* $x_1(n_2+1) = x_2(n_2+1)$*, then the concatenation* $(x(\cdot), w(\cdot))$ *defined by* $(x(k), w(k)) = (x_1(k), w_1(k))$ *for* $k \in [n_1, n_2]$*,* $(x(k), w(k)) = (x_2(k), w_2(k))$ *for* $k \in [n_2+1, n_3]$*, and* $x(n_3+1) = x_2(n_3+1)$*, is a trajectory along* V *on* $[n_1, n_3]$.

Property 1) means that the set of trajectories along V is *time-invariant*, and property 2) says that x has the *state property*; cf. [PW98, p. 119].

[1] Recall that we denote the direct product $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$ by $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$. Later when we introduce *passive* nodes we shall require $\mathcal{X}$ to be a Hilbert space, $\mathcal{W}$ to be a Kreĭn space, and equip $\mathfrak{K}$ with a particular Kreĭn space structure rather than the Hilbert space structure that it inherits from $\mathcal{X}$ and $\mathcal{W}$. This is the reason why we throughout ignore the Hilbert space inner product in $\mathfrak{K}$ induced by the inner products in $\mathcal{X}$ and $\mathcal{W}$. The only way in which we use the fact that $\mathcal{X}$ and $\mathcal{W}$ are Hilbert spaces is in the assertion that every closed subspace of $\mathfrak{K}$ has a complementary subspace. The same comments applies to all other Hilbert spaces and their products that appear in this paper.

Properties (ii) and (iii) in Definition 2.1 are reflected in the properties of the set of all trajectories along V as follows:

Proposition 2.2. *Let V be a subspace of the product space $\mathfrak{K} := \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$.*

1) *The following three statements are equivalent:*
   (a) *V has property (ii) in Definition 2.1;*
   (b) *for every $x_0 \in \mathcal{X}$ there is a trajectory $(x(\cdot), w(\cdot))$ along V on $\mathbb{Z}^+$ with $x(0) = x_0$;*
   (c) *every trajectory $(x(\cdot), w(\cdot))$ along V defined on some interval $[0, n_2]$ can be extended to a trajectory on $\mathbb{Z}^+$.*
2) *The following four statements are equivalent:*
   (a) *V has property (iii) in Definition 2.1;*
   (b) *if $(x(\cdot), w(\cdot))$ is a trajectory on $[n_1, n_2]$ along V, then for every $k \in [n_1, n_2]$, the value of $x(k+1)$ is determined uniquely by $\left[\begin{smallmatrix}x(k)\\ w(k)\end{smallmatrix}\right]$;*
   (c) *if $(x(\cdot), w(\cdot))$ is a trajectory on $[n_1, n_2]$ along V, then the value of $x(n_2+1)$ is determined uniquely by $x(n_1)$ and $w(k)$, $n_1 \le k \le n_2$.*
   (d) *if $(x(\cdot), w(\cdot))$ is a trajectory on $[n_1, n_2]$ along V with $x(n_1) = 0$, then the value of $x(n_2+1)$ is determined uniquely by $w(k)$, $n_1 \le k \le n_2$.*

Proof. Proof of 1): The implications (b) ⇒ (a) and (c) ⇒ (a) are obvious.

We next prove that (a) ⇒ (b). Suppose that (a) holds. Let $x_0 \in \mathcal{X}$, and define $x(0) = x_0$. It follows from property (ii) in Definition 2.1 that there exist $x(1)$ and $w(0)$ such that $\left[\begin{smallmatrix}x(1)\\ x(0)\\ w(0)\end{smallmatrix}\right] \in V$. By the same argument with $x(0)$ replaced by $x(1)$, there exist $x(2)$ and $w(1)$ such that $\left[\begin{smallmatrix}x(2)\\ x(1)\\ w(1)\end{smallmatrix}\right] \in V$. By induction, we will obtain (b).

The proof of the fact that (a) ⇒ (c) is the same as the proof of the implication (a) ⇒ (b) given above, except that we start from time $n+1$ and the initial value $x(n+1)$ (instead of time zero and initial value x_0).

The proof of 2) is left to the reader. □

By the *state/signal system generated by the state/signal node* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ we mean this node itself together with the set of all trajectories along V. For simplicity we use the same notation Σ for the system as we used for the original node. We shall also refer to the trajectories along V as *the trajectories of* Σ.

We shall next develop certain representations of the subspace V in Definition 2.1, and begin with the following lemmas.

Lemma 2.3. *Let V be a subspace of the product space $\mathfrak{K} := \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$. Let $G_{2,3}\colon V \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ be the bounded linear operator that maps the vector $\left[\begin{smallmatrix}z\\ x\\ w\end{smallmatrix}\right] \in V$ into $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$. Then the following conditions are equivalent:*

1) *V has property (iii);*
2) *$G_{2,3}$ is injective;*

3) *V has a graph representation over the last two components $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ of $\mathfrak{K}$, i.e., there exists a linear operator F, mapping $\mathcal{D}(F) \subset \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ into $\mathcal{X}$ such that $\left[\begin{smallmatrix}z\\ x\\ w\end{smallmatrix}\right] \in V$ if and only if $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \mathcal{D}(F)$ and $z = F\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$.*

Assuming 1)*, with $G_{2,3}$ and F defined as in* 2) *and* 3)*, the operator F is uniquely determined by V (hence so is $\mathcal{D}(F)$), $\mathcal{R}(G_{2,3}) = \mathcal{D}(F)$, $G_{2,3}^{-1}\colon \mathcal{D}(F) \to V$ is given by $G_{2,3}^{-1} = \left[\begin{smallmatrix} & F & \\ 1_{\mathcal{X}} & & 0\\ 0 & & 1_{\mathcal{W}}\end{smallmatrix}\right]$, and*

$$V = G_{2,3}^{-1}\mathcal{D}(F) = \left\{ \begin{bmatrix} z\\ x\\ w\end{bmatrix} \;\middle|\; z = F\left(\begin{bmatrix} x\\ w\end{bmatrix}\right),\ \begin{bmatrix} x\\ w\end{bmatrix} \in \mathcal{D}(F)\right\} \tag{2.2}$$

Lemma 2.4. *Let V be a subspace of the product space $\mathfrak{K} := \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$. Assume that V has property (iii), and let F be the operator defined in Lemma* 2.3*. Then*

1) *V has property (i) if and only if F is closed,*
2) *V has property (ii) if and only if the linear operator $\mathcal{D}(F) \to \mathcal{X}$ that maps $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \mathcal{D}(F)$ into $x \in \mathcal{X}$ is surjective,*
3) *V has property (iv) if and only if $\mathcal{D}(F)$ is closed,*
4) *V has properties (i) and (iv) if and only if F is bounded and $\mathcal{D}(F)$ is closed.*

We leave the straightforward proofs of Lemmas 2.3 and 2.4 to the reader.

By combining Lemmas 2.3 and 2.4 we get the following theorem:

Theorem 2.5. *Let V be a subspace of the product space $\mathfrak{K} := \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$. Then V has properties (i)–(iv) listed in Definition* 2.1*, i.e., $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a state/signal node, if and only if V has a graph representation over the last two components $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ of $\mathfrak{K}$ with a bounded linear operator $F\colon \mathcal{D}(F) \subset \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right] \to \mathcal{X}$ with closed domain, i.e.,*

$$V = \left\{ \begin{bmatrix} z\\ x\\ w\end{bmatrix} \;\middle|\; z = F\left(\begin{bmatrix} x\\ w\end{bmatrix}\right),\ \begin{bmatrix} x\\ w\end{bmatrix} \in \mathcal{D}(F)\right\}, \tag{2.3}$$

with the additional property that the linear operator $\mathcal{D}(F) \to \mathcal{X}$ that maps $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \mathcal{D}(F)$ into $x \in \mathcal{X}$ is surjective.

In the next three sections we shall develop three different types of representations of a state/signal system Σ: *driving variable* representations, *output nulling* representations, and *input/state/output* representations. They complement each other, and all of them are important in slightly different connections.

3. The driving variable representation

In our first representation of the generating subspace V we write V as the image of a bounded linear injective operator of the following type.

Lemma 3.1. *Let V be a subspace of the product space $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$, where $\mathcal{X}$ and $\mathcal{W}$ are Hilbert spaces. If there exists a Hilbert space $\mathcal{L}$ and four operators*

$$A' \in \mathcal{B}(\mathcal{X}),\ B' \in \mathcal{B}(\mathcal{L};\mathcal{X}),\ C' \in \mathcal{B}(\mathcal{X},\mathcal{W}),\ \text{and}\ D' \in \mathcal{B}(\mathcal{L};\mathcal{W}), \tag{3.1}$$

where

$$D' \text{ is injective and has a closed range} \tag{3.2}$$

such that

$$V = \mathcal{R}\left(\begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix}\right) = \left\{ \begin{bmatrix} A'x + B'\ell \\ x \\ C'x + D'\ell \end{bmatrix} \,\middle|\, x \in \mathcal{X},\ \ell \in \mathcal{L} \right\}, \tag{3.3}$$

then V has properties (i)–(iv) listed in Definition 2.1*, i.e., $(V;\mathcal{X},\mathcal{W})$ is a state/ signal node. Conversely, if V has properties (i)–(iv) listed in Definition* 2.1 *then V is given by* (3.3) *for some Hilbert space $\mathcal{L}$ and some operators A', B', C', and D' satisfying* (3.1) *and* (3.2).

Proof. We begin by proving that the representation (3.1)–(3.3) implies that V has properties (i)–(iv) in Definition 2.1. Trivially, (3.1) and (3.3) imply (ii). It is also clear that the injectivity of D' implies that the operator $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]$ is injective. Thus, by defining $\mathcal{D}(F) = \mathcal{R}\left(\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]\right)$ and $F = \begin{bmatrix} A' & B' \end{bmatrix} \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]^{-1}$ we get the graph representation (2.3) of V. According to Lemma 2.3, this implies that V has property (iii). The closedness of $\mathcal{R}(D')$ implies that also $\mathcal{R}\left(\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]\right)$ is closed, because $\mathcal{R}\left(\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & 1_{\mathcal{W}} \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{R}(D') \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & 1_{\mathcal{W}} \end{smallmatrix}\right]$ is boundedly invertible. Finally, the closed graph theorem implies that $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]^{-1}$ is bounded on $\mathcal{D}(F)$, hence so is F, and by part 4) of Lemma 2.4, V has properties (i) and (iv). We have now showed that V has all the properties (i)–(iv).

Conversely, suppose that V has properties (i)–(iv) in Definition 2.1. Let $G_2 \in \mathcal{B}(V;\mathcal{X})$ be the bounded linear operator that maps $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$ into $x \in \mathcal{X}$. We take $\mathcal{L} = \mathcal{N}(G_2)$, and define $B' \in \mathcal{B}(\mathcal{L};\mathcal{X})$ and $D' \in \mathcal{B}(\mathcal{X};\mathcal{W})$ by $B'\left[\begin{smallmatrix} z \\ 0 \\ w \end{smallmatrix}\right] = z$ and $D'\left[\begin{smallmatrix} z \\ 0 \\ w \end{smallmatrix}\right] = w$ for each $\left[\begin{smallmatrix} z \\ 0 \\ w \end{smallmatrix}\right] \in \mathcal{L}$. Clearly $\ell = \left[\begin{smallmatrix} B' \\ 0 \\ D' \end{smallmatrix}\right]\ell$ for all $\ell \in \mathcal{L}$, $\left[\begin{smallmatrix} B' \\ D' \end{smallmatrix}\right]$ is injective on $\mathcal{L}$, and the range of $\left[\begin{smallmatrix} B' \\ D' \end{smallmatrix}\right]$ is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$. By property (ii) in Definition 2.1, G_2 maps V *onto* $\mathcal{X}$. Let $G_{2,\text{right}}^{-1} \in \mathcal{B}(\mathcal{X};V)$ be an arbitrary right-inverse of G_2 (such a bounded right-inverse exists since V is closed). This right-inverse must be of the form $G_{2,\text{right}}^{-1} = \left[\begin{smallmatrix} A' \\ 1_{\mathcal{X}} \\ C' \end{smallmatrix}\right]$ (the middle component must be the identity operator since $G_2\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] = x$ for all $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$). By property (i), $V = \mathcal{R}\left(G_{2,\text{right}}^{-1}\right) \dotplus \mathcal{L}$, hence

$$V = \begin{bmatrix} A' \\ 1_{\mathcal{X}} \\ C' \end{bmatrix} \mathcal{X} \dotplus \mathcal{L} = \begin{bmatrix} A' \\ 1_{\mathcal{X}} \\ C' \end{bmatrix} \mathcal{X} \dotplus \begin{bmatrix} B' \\ 0 \\ D' \end{bmatrix} \mathcal{L}.$$

This implies (3.3).

We still have to show that D' is injective and has closed range, and for this we need properties (iii) and (iv) (which we have not used up to now). By construction the operator $\begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix}$ is injective. It then follows from Lemma 2.3 that the operator $G_{2,3}\begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix} = \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ also must be injective (since we now assume (iii)). This implies that D' is injective. That the range is closed follows from (iv), i.e., from the closedness of $\mathcal{D}(F) = \mathcal{R}\left(\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]\right)$, since (as we observed above) $\mathcal{R}\left(\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]\right) = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C' & 1_{\mathcal{W}} \end{bmatrix}\begin{bmatrix} \mathcal{X} \\ \mathcal{R}(D') \end{bmatrix}$, where $\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C' & 1_{\mathcal{W}} \end{bmatrix}$ is boundedly invertible. □

We shall call a colligation $\Sigma_{dv/s/s} := \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$, where $\mathcal{L}$ is a Hilbert space and A', B', C', and D' satisfy (3.1)–(3.3) a *driving variable representation of the state/signal node* $\Sigma = (V; \mathcal{X}, \mathcal{W})$. We shall also refer to $\Sigma_{dv/s/s}$ as a *driving-variable/state/signal node*. By the driving-variable/state/signal *system* $\Sigma_{dv/s/s}$ we mean the node $\Sigma_{dv/s/s}$ itself together with the set of all trajectories $(x(\cdot), \ell(\cdot), w(\cdot))$ generated by this node through the equations

$$\begin{aligned} x(k+1) &= A'x(k) + B'\ell(k), \\ w(k) &= C'x(k) + D'\ell(k), \qquad n_1 \le k \le n_2. \end{aligned} \tag{3.4}$$

The space $\mathcal{L}$ considered above is called a *driving variable space*, and the vector $\ell \in \mathcal{L}$ in (3.3) is called a *driving variable*. (The notion of a driving variable is known in the finite-dimensional setting from the theory of behaviors; see, e.g., [WT02].) From each trajectory $(x(\cdot), \ell(\cdot), w(\cdot))$ of the driving-variable/state/signal system $\Sigma_{dv/s/s}$ we get a trajectory $(x(\cdot), w(\cdot))$ of the state/signal system Σ by simply deleting the driving variable component ℓ. It follows from part 3) of Proposition 3.2 below that this correspondence between the trajectories of the two types of systems is one-to-one.

Let us next point out some important properties of driving variable representations.

Proposition 3.2. *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a state/signal node with the driving variable representation* $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$*, and let* $F \colon \mathcal{D}(F) \to \mathcal{X}$ *be the linear operator defined in Lemma 2.3. Then the following assertions are true.*

1) $\mathcal{R}\left(\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]\right) = \mathcal{D}(F)$, $\mathcal{R}(B') = \mathfrak{R}_0$, $\mathcal{R}(D') = \mathcal{U}_0$, *and the preimage of* $\mathcal{R}(D')$ *under* C' *is given by* $\mathfrak{U}_0$, *where*

$$\begin{aligned} \mathfrak{R}_0 &= \left\{F\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \in \mathcal{D}(F)\right\} \\ &= \left\{z \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix} z \\ 0 \\ w \end{smallmatrix}\right] \in V \textit{ for some } w \in \mathcal{W}\right\}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \mathcal{U}_0 &= \left\{w \in \mathcal{W} \;\middle|\; \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \in \mathcal{D}(F)\right\} \\ &= \left\{w \in \mathcal{W} \;\middle|\; \left[\begin{smallmatrix} z \\ 0 \\ w \end{smallmatrix}\right] \in V \textit{ for some } z \in \mathcal{X}\right\}, \end{aligned} \tag{3.6}$$

$$\mathfrak{U}_0 = \left\{x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{D}(F)\right\}$$
$$= \left\{x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix} z \\ x \\ 0 \end{smallmatrix}\right] \in V \text{ for some } z \in \mathcal{X}\right\}. \tag{3.7}$$

Consequently, the ranges of B'*,* D'*, and* $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]$ *do not depend on the particular choice of* $\Sigma_{dv/s/s}$*.*

2) *The space* $\mathcal{L}$ *is isomorphic to the space* $\mathcal{U}_0$ *defined in* (3.6)*.*
3) *The operator* $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]$ *has a bounded inverse mapping* $\mathcal{D}(F)$ *one-to-one onto* $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right]$*, and the vector* ℓ *in the representation* (3.3) *is uniquely determined by* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *via*

$$\begin{bmatrix} x \\ \ell \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix}^{-1} \begin{bmatrix} x \\ w \end{bmatrix}, \qquad \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{D}(F). \tag{3.8}$$

4) *The operator* $\begin{bmatrix} A' & B' \end{bmatrix}$ *is given by*

$$\begin{bmatrix} A' & B' \end{bmatrix} = F \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]. \tag{3.9}$$

Consequently, A' *is determined uniquely by* C' *and* B' *is determined uniquely by* D'*.*

Proof. Assertion 1) follows from (3.3) and the definition of F. To see that assertion 2) holds it suffices to note that the operator D' maps $\mathcal{L}$ one-to-one onto $\mathcal{U}_0$, and by the closed graph theorem, then inverse of this operator is also bounded. Assertions 3) and 4) were established as a part of the proof of Lemma 3.1. □

Theorem 3.3. *Let* $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ *be a driving variable representation of a state signal system* Σ*, and let*

$$\begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ K' & M' \end{bmatrix} \tag{3.10}$$

where

$$K' \in \mathcal{B}(\mathcal{X}; \mathcal{L}),\ M' \in \mathcal{B}(\mathcal{L}_1; \mathcal{L}),\ \text{and } M' \text{ has a bounded inverse,} \tag{3.11}$$

for some Hilbert space $\mathcal{L}_1$*. Then* $\Sigma^1_{dv/s/s} = \left(\left[\begin{smallmatrix} A_1' & B_1' \\ C_1' & D_1' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}_1, \mathcal{W}\right)$ *is a driving variable representation of* Σ*. Conversely, every driving variable representation* $\Sigma^1_{dv/s/s}$ *of* Σ *may be obtained from formula* (3.10) *for some operators* K' *and* M' *satisfying* (3.11)*. The operators* K' *and* M' *are uniquely defined by* $\Sigma_{dv/s/s}$ *and* $\Sigma^1_{dv/s/s}$ *via*

$$D'K' = C_1' - C' \text{ and } D'M' = D_1'. \tag{3.12}$$

Proof. Suppose that $\Sigma^1_{dv/s/s} = \left(\left[\begin{smallmatrix} A_1' & B_1' \\ C_1' & D_1' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}_1, \mathcal{W}\right)$ given by (3.10) for some operators K' and M' satisfying (3.11). It follows from (3.11) that $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ K' & M' \end{smallmatrix}\right]$ maps $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right]$ one-to-one onto $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L}_1 \end{smallmatrix}\right]$. By (3.3) and (3.10),

$$\begin{bmatrix} A_1' & B_1' \\ 1_{\mathcal{X}} & 0 \\ C_1' & D_1' \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{L}_1 \end{bmatrix} = \begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ K' & M' \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{L}_1 \end{bmatrix} = \begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{L} \end{bmatrix} = V.$$

Furthermore, $D'_1 = D'M'$ is injective and has closed range. Thus $\Sigma^1_{dv/s/s}$ is a driving variable representation of Σ.

We next turn to the converse part. By statements 1) and 3) of Proposition 3.2, the operator $\left[\begin{smallmatrix} G' & H' \\ K' & M' \end{smallmatrix}\right] := \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C'_1 & D'_1 \end{smallmatrix}\right]$ is a bounded linear operator mapping $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L}_1 \end{smallmatrix}\right]$ one-to-one onto $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right]$. It follows from the identity $\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C'_1 & D'_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right] \left[\begin{smallmatrix} G' & H' \\ K' & M' \end{smallmatrix}\right]$ that $G' = 1_{\mathcal{X}}$ and that $H' = 0$, and the invertibility of $\left[\begin{smallmatrix} G' & H' \\ K' & M' \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ K' & M' \end{smallmatrix}\right]$ implies that M' is invertible. Thus, (3.11) and (3.12) hold. By statement 4) of Proposition 3.2,

$$F = \begin{bmatrix} A' & B' \end{bmatrix} \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C' & D' \end{smallmatrix}\right]^{-1} = \begin{bmatrix} A'_1 & B'_1 \end{bmatrix} \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ C'_1 & D'_1 \end{smallmatrix}\right]^{-1},$$

hence $\begin{bmatrix} A'_1 & B'_1 \end{bmatrix} = \begin{bmatrix} A' & B' \end{bmatrix} \left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ K' & M' \end{smallmatrix}\right]$. Thus equation (3.10) holds.

Finally, we remark that (3.12) determines K' and M' uniquely since D' is injective. □

Definition 3.4. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal system.

1) By an *externally generated* trajectory of Σ on $[0, n]$ or on $\mathbb{Z}^+$ we mean a trajectory $(x(\cdot), w(\cdot))$ satisfying $x(0) = 0$.
2) The *reachable subspace* $\mathfrak{R}_n$ of Σ *in time* n is the subspace of all the final states $x(n+1)$ of all externally generated trajectories $(x(\cdot), w(\cdot))$ of the system Σ on the interval $[0, n]$.
3) The (approximately) *reachable subspace* $\mathfrak{R}$ of Σ (in infinite time) is the closure in $\mathcal{X}$ of all the possible values of the state components $x(\cdot)$ of all externally generated trajectories $(x(\cdot), w(\cdot))$ of the system Σ on $\mathbb{Z}^+$.
4) The system is (approximately) *controllable* if the reachable subspace is all of $\mathcal{X}$.

Thus,

$$\mathfrak{R}_n \subset \mathfrak{R}_{n+1}, \quad \mathfrak{R} = \vee_{n \in \mathbb{Z}^+} \mathfrak{R}_n$$

(we get the first inclusion by taking $x(0) = 0$ and $w(0) = 0$, so that also $x(1) = 0$; for the second inclusion we use part 1) of Proposition 2.2). Observe, in particular, that the subspace $\mathfrak{R}_0$ defined above coincides with the subspace $\mathfrak{R}_0$ defined in (3.5).

The subspaces $\mathfrak{R}_n$ and $\mathfrak{R}$ in Definition 3.4 have the following simple characterizations in terms of an arbitrary driving variable representation of Σ.

Proposition 3.5. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal system, with a driving variable representation $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$. Then the subspaces $\mathfrak{R}_n$ defined above and the reachable subspace $\mathfrak{R}$ are given by*

$$\mathfrak{R}_n = \operatorname{span}\{\mathcal{R}\left((A')^k B'\right) \mid 0 \le k \le n\}, \quad n \in \mathbb{Z}^+, \tag{3.13}$$

$$\mathfrak{R} = \vee_{k \in \mathbb{Z}^+} \mathcal{R}\left((A')^k B'\right). \tag{3.14}$$

In particular, Σ is controllable if and only if

$$\mathcal{X} = \vee_{k \in \mathbb{Z}^+} \mathcal{R}\left((A')^k B'\right). \tag{3.15}$$

Proof. Let $(x(\cdot), w(\cdot))$ be an externally generated trajectory of Σ on $[0, n]$. It follows from the representation (3.3) (by induction) that $x(n + 1)$ can be written in the form

$$x(n+1) = \sum_{k=0}^{n} (A')^k B' \ell(n-k)$$

for some sequence $\{\ell(k)\}_{k=0}^{n}$. Thus, $x(n + 1)$ belongs to the linear span of $\{\mathcal{R}\left((A')^k B'\right)\}_{k=0}^{n}$. Conversely, to each such sequence $\{\ell(k)\}_{k=0}^{n}$ corresponds a trajectory on $[0, n]$ for which $x(n+1)$ is given by the formula above. This proves (3.13). Letting $n \to \infty$ in (3.13) we get (3.14). The final statement follows from (3.14) and the definition of controllability. □

4. The output nulling representation

In our second representation of the generating subspace V we write V as the kernel of a surjective operator of the following type.

Lemma 4.1. *Let V be a subspace of the product space $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, where $\mathcal{X}$ and $\mathcal{W}$ are Hilbert spaces. If there exists a Hilbert space $\mathcal{K}$ and four operators*

$$A'' \in \mathcal{B}(\mathcal{X}),\ B'' \in \mathcal{B}(\mathcal{W}; \mathcal{X}),\ C'' \in \mathcal{B}(\mathcal{X}, \mathcal{K}),\ \text{and}\ D'' \in \mathcal{B}(\mathcal{W}; \mathcal{K}) \tag{4.1}$$

where

$$D'' \text{ is surjective} \tag{4.2}$$

such that

$$V = \mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & A'' & B'' \\ 0 & C'' & D'' \end{bmatrix}\right) = \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in \mathfrak{K} \;\middle|\; \begin{matrix} z = A''x + B''w \\ 0 = C''x + D''w \end{matrix} \right\}, \tag{4.3}$$

then V has properties (i)–(iv) *listed in Definition* 2.1*, i.e., $(V; \mathcal{X}, \mathcal{W})$ is a state/signal node. Conversely, if V has properties* (i)–(iv) *listed in Definition* 2.1 *then V is given by* (4.3) *for some Hilbert space $\mathcal{K}$ and some operators A'', B'', C'', and D'' satisfying* (4.1) *and* (4.2)*.*

Proof. Trivially, if V is given by (4.3), then V has property (iii). That (i) holds follows from the fact that V is the kernel of the bounded linear operator $\left[\begin{smallmatrix} -1_{\mathcal{X}} & A'' & B'' \\ 0 & C'' & D'' \end{smallmatrix}\right]$. Define F as in Lemma 2.3. That (iv) holds follows from the fact that $\mathcal{D}(F)$ is the kernel of the bounded linear operator $\begin{bmatrix} C'' & D'' \end{bmatrix}$. Finally, (ii) holds since the surjectivity of D'' guarantees that for every $x \in \mathcal{X}$ it is possible to find some $w \in \mathcal{W}$ such that $C''x + D''w = 0$, i.e., $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F)$.

Conversely, suppose that V has properties (i)–(iv). Then the operator F in Lemma 2.3 is bounded and $\mathcal{D}(F)$ is closed. Let $\begin{bmatrix} C'' & D'' \end{bmatrix} \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]; \mathcal{K})$ be an arbitrary surjective operator with $\mathcal{N}\left(\begin{bmatrix} C'' & D'' \end{bmatrix}\right) = \mathcal{D}(F)$ (e.g., let $\mathcal{K}$ be a complementary subspace to $\mathcal{D}(F)$ in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ and let $\begin{bmatrix}C'' & D''\end{bmatrix} = P_{\mathcal{K}}^{\mathcal{D}(F)}$). Let $\begin{bmatrix}A'' & B''\end{bmatrix}$ be an arbitrary extension of F to an operator in $\mathcal{B}(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right];\mathcal{X})$ (e.g., take $\begin{bmatrix}A'' & B''\end{bmatrix} = FP_{\mathcal{D}(F)}^{\mathcal{K}}$ with $\mathcal{K}$ chosen as above). Then $\begin{bmatrix}C'' & D''\end{bmatrix}$ is surjective and (4.1) and (4.3) hold.

It remains to show that D'' is surjective, and for this we need property (ii) (which has not yet been used). It follows from (4.3) that (ii) holds if and only if $\mathcal{R}(C'') \subset \mathcal{R}(D'')$. Because of the surjectivity of $\begin{bmatrix}C'' & D''\end{bmatrix}$, this is equivalent to (4.2). □

We shall call a colligation $\Sigma_{s/s/on} := \left(\left[\begin{smallmatrix}A'' & B''\\ C'' & D''\end{smallmatrix}\right];\mathcal{X},\mathcal{W},\mathcal{K}\right)$, where $\mathcal{K}$ is a Hilbert space and A'', B'', C'', and D'' satisfy (4.1)–(4.3) an *output nulling representation of the state/signal node* $\Sigma = (V;\mathcal{X},\mathcal{W})$. (Output nulling representations are known in the finite-dimensional case from the theory of behaviors; see, e.g., [WT02].) We shall also refer to $\Sigma_{s/s/on}$ as a *signal/state/output nulling node*. By the signal/state/output nulling *system* $\Sigma_{s/s/on}$ we mean the node $\Sigma_{s/s/on}$ itself together with the set of all trajectories generated by this node. However, the notion of a trajectory of such a node differs slightly from the corresponding notions for a state/signal node or a driving-variable/state/signal node. By a trajectory of $\Sigma_{s/s/on}$ on $[n_1, n_2]$ we mean a triple of sequences $(x(\cdot), w(\cdot), e(\cdot))$ which satisfy

$$\begin{aligned} x(k+1) &= A''x(k) + B''w(k),\\ e(k) &= C''x(k) + D''w(k), \qquad n_1 \le k \le n_2. \end{aligned} \tag{4.4}$$

Here we interpret w as input data and e as output data. Thus, *not every trajectory of* (4.4) *corresponds to a trajectory of the corresponding state/signal system* Σ; this is true exactly for those trajectories whose output $e(\cdot)$ is null (i.e., it vanishes identically). We shall refer to e as the *error variable*, and to the space $\mathcal{K}$ as the *error space*.

Output nulling representations have a number of important properties listed below.

Proposition 4.2. *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a state/signal node with the output nulling representation $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix}A'' & B''\\ C'' & D''\end{smallmatrix}\right];\mathcal{X},\mathcal{W},\mathcal{K}\right)$, and let $F\colon \mathcal{D}(F) \to \mathcal{X}$ be the linear operator defined in Lemma 2.3. Then the following assertions are true.*

1) *The operator F is given by*

$$F = \begin{bmatrix}A'' & B''\end{bmatrix}|_{\mathcal{D}(F)} \ \text{with}\ \mathcal{D}(F) = \mathcal{N}\left(\begin{bmatrix}C'' & D''\end{bmatrix}\right). \tag{4.5}$$

2) *We have*

$$\mathcal{N}(D'') = \mathcal{U}_0,\ \mathcal{N}(C'') = \mathfrak{U}_0,\ \mathcal{R}(B''|_{\mathcal{U}_0}) = \mathfrak{R}_0, \tag{4.6}$$

where $\mathfrak{R}_0$, $\mathcal{U}_0$, and $\mathfrak{U}_0$ are defined in (3.5)–(3.7)*. Consequently, the ranges and kernels listed above do not depend on the particular choice of $\Sigma_{s/s/on}$.*

3) *Let $\mathcal{Y}_0$ be a direct complement in $\mathcal{W}$ to the space $\mathcal{U}_0$ defined in* (3.6)*, i.e., $\mathcal{W} = \mathcal{Y}_0 \dot{+} \mathcal{U}_0$. Then $D''|_{\mathcal{Y}_0}$ maps $\mathcal{Y}_0$ one-to-one onto $\mathcal{K}$ and $\left[\begin{smallmatrix}1_{\mathcal{X}} & B''|_{\mathcal{Y}_0}\\ 0 & D''|_{\mathcal{Y}_0}\end{smallmatrix}\right]$ maps*

$\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}_0\end{smallmatrix}\right]$ *one-to-one onto* $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{K}\end{smallmatrix}\right]$*, and consequently, these operators are boundedly invertible. Moreover,*

$$\begin{bmatrix}F\\ 0\end{bmatrix}\begin{bmatrix}1_{\mathcal{X}}\\ H_{\mathcal{Y}_0}\end{bmatrix} = \begin{bmatrix}A'' & B''\\ C'' & D''\end{bmatrix}\begin{bmatrix}1_{\mathcal{X}}\\ H_{\mathcal{Y}_0}\end{bmatrix}, \tag{4.7}$$

or equivalently,

$$\begin{bmatrix}A''\\ C''\end{bmatrix} = \begin{bmatrix}1_{\mathcal{X}} & B''|_{\mathcal{Y}_0}\\ 0 & D''|_{\mathcal{Y}_0}\end{bmatrix}\left(\begin{bmatrix}F\\ 0\end{bmatrix}\begin{bmatrix}1_{\mathcal{X}}\\ H_{\mathcal{Y}_0}\end{bmatrix} - \begin{bmatrix}0\\ H_{\mathcal{Y}_0}\end{bmatrix}\right), \tag{4.8}$$

where $H_{\mathcal{Y}_0}\colon \mathcal{X} \to \mathcal{W}$ *is the operator defined by* $H_{\mathcal{Y}_0}x = w$*, where* w *is the unique element in* $\mathcal{Y}_0$ *such that* $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \mathcal{D}(F)$*. Consequently,* A'' *is determined uniquely by* B'' *and* C'' *is determined uniquely by* D''*.*

4) *The space* $\mathcal{K}$ *is isomorphic to every direct complement in* $\mathcal{W}$ *to the space* $\mathcal{U}_0$ *defined in* (3.6)*.*

Proof. We leave the straightforward proofs of 1) and 2) to the reader. That the restriction of D'' to any complement $\mathcal{Y}_0$ of $\mathcal{U}_0$ is invertible with a bounded inverse follows from the fact that $\mathcal{N}(D'') = \mathcal{U}_0$. This implies that the restriction of $\left[\begin{smallmatrix}1_{\mathcal{X}} & B''\\ 0 & D''\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}_0\end{smallmatrix}\right]$ is invertible with a bounded inverse. Formula (4.7) follows from (4.3) and (4.5). Clearly (4.8) is equivalent to (4.7). Finally, 4) follows from the invertibility of $D''|_{\mathcal{Y}_0}$ established in 3). □

Theorem 4.3. *Let* $\Sigma_{s/s/on} = (\left[\begin{smallmatrix}A'' & B''\\ C'' & D''\end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K})$ *be an output nulling representation of a state/signal system* Σ*, and let*

$$\begin{bmatrix}A_1'' & B_1''\\ C_1'' & D_1''\end{bmatrix} = \begin{bmatrix}1_{\mathcal{X}} & K''\\ 0 & M''\end{bmatrix}\begin{bmatrix}A'' & B''\\ C'' & D''\end{bmatrix}, \tag{4.9}$$

where

$$K'' \in \mathcal{B}(\mathcal{K}, \mathcal{X}),\ M'' \in \mathcal{B}(\mathcal{K}, \mathcal{K}_1),\ \text{and } M'' \text{ has a bounded inverse,} \tag{4.10}$$

for some Hilbert space $\mathcal{K}_1$*. Then*

$$\Sigma^1_{s/s/on} = \left(\left[\begin{smallmatrix}A_1'' & B_1''\\ C_1'' & D_1''\end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}_1\right)$$

is an output nulling representation of Σ*. Conversely, every output nulling representation* $\Sigma^1_{s/s/on}$ *of* Σ *may be obtained from the formula* (4.9) *for some operators* M'' *and* K'' *satisfying* (4.10)*. The operators* M'' *and* K'' *are uniquely defined by* $\Sigma_{s/s/on}$ *and* $\Sigma^1_{s/s/on}$ *via*

$$M''D'' = D_1'' \text{ and } K''D'' = B_1'' - B''. \tag{4.11}$$

Proof. Suppose that $\Sigma^1_{s/s/on} = \left(\left[\begin{smallmatrix}A_1'' & B_1''\\ C_1'' & D_1''\end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}_1\right)$ is given by (4.9) for some operators K'' and M'' satisfying (4.10). It follows from (4.9) and (4.10) that

$D_1'' = M''D''$ is surjective, that $\begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & K'' \\ 0 & 0 & M'' \end{bmatrix}$ is invertible, and that

$$\mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & A_1'' & B_1'' \\ 0 & C_1'' & D_1'' \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & K'' \\ 0 & 0 & M'' \end{bmatrix}\begin{bmatrix} -1_{\mathcal{X}} & A'' & B'' \\ 0 & C'' & D'' \end{bmatrix}\right)$$
$$= \mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & A'' & B'' \\ 0 & C'' & D'' \end{bmatrix}\right) = V.$$

Thus $\Sigma^1_{s/s/on}$ is an output nulling representation of Σ.

We next turn to the converse part. Let $\mathcal{Y}$ be an arbitrary complement to $\mathcal{D}(F)$. By part 3) of Proposition 4.2, the operator

$$\begin{bmatrix} G'' & K'' \\ H'' & M'' \end{bmatrix} := \begin{bmatrix} 1_{\mathcal{X}} & B_1''|_{\mathcal{Y}_0} \\ 0 & D_1''|_{\mathcal{Y}_0} \end{bmatrix}\begin{bmatrix} 1_{\mathcal{X}} & B''|_{\mathcal{Y}_0} \\ 0 & D''|_{\mathcal{Y}_0} \end{bmatrix}^{-1}$$

is a bounded linear operator mapping $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{K} \end{smallmatrix}\right]$ one-to-one onto $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{K}_1 \end{smallmatrix}\right]$. It follows from the identity $\left[\begin{smallmatrix} 1_{\mathcal{X}} & B_1''|_{\mathcal{Y}_0} \\ 0 & D_1''|_{\mathcal{Y}_0} \end{smallmatrix}\right] = \left[\begin{smallmatrix} G'' & K'' \\ H'' & M'' \end{smallmatrix}\right]\left[\begin{smallmatrix} 1_{\mathcal{X}} & B''|_{\mathcal{Y}_0} \\ 0 & D''|_{\mathcal{Y}_0} \end{smallmatrix}\right]$ that $G'' = 1_{\mathcal{X}}$ and that $H'' = 0$, and the invertibility of $\left[\begin{smallmatrix} G'' & K'' \\ H'' & M'' \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_{\mathcal{X}} & K'' \\ 0 & M'' \end{smallmatrix}\right]$ implies that M'' is invertible. Thus, (4.10) and (4.11) hold. By (4.8), $\left[\begin{smallmatrix} 1_{\mathcal{X}} & B_1''|_{\mathcal{Y}_0} \\ 0 & D_1''|_{\mathcal{Y}_0} \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} A_1'' \\ C_1'' \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_{\mathcal{X}} & B''|_{\mathcal{Y}_0} \\ 0 & D''|_{\mathcal{Y}_0} \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} A'' \\ C'' \end{smallmatrix}\right]$, hence $\left[\begin{smallmatrix} A_1'' \\ C_1'' \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_{\mathcal{X}} & K'' \\ 0 & M'' \end{smallmatrix}\right]\left[\begin{smallmatrix} A'' \\ C'' \end{smallmatrix}\right]$. Thus equation (4.9) holds.

Finally, we remark that (4.11) determines K'' and M'' uniquely since D'' is surjective. □

Definition 4.4. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal system.

1) By an *unobservable* trajectory of Σ on $[0, n]$ or on $\mathbb{Z}^+$ we mean a trajectory $(x(\cdot), 0)$ (i.e., the signal component of this trajectory is identically zero on $[0, n]$ or on $\mathbb{Z}^+$).
2) The *unobservable subspace* $\mathfrak{U}_n$ of Σ *in time* n is the subspace of the initial states $x(0)$ of all unobservable trajectories $(x(\cdot), 0)$ of Σ on $[0, n]$.
3) The *unobservable subspace* $\mathfrak{U}$ of Σ (in infinite time) is the subspace of the initial states $x(0)$ of all unobservable trajectories $(x(\cdot), 0)$ of Σ on $\mathbb{Z}^+$.
4) The system is (approximately) *observable* if the unobservable subspace is $\{0\}$.

Thus,

$$\mathfrak{U}_{n+1} \subset \mathfrak{U}_n, \quad \mathfrak{U} = \cap_{n \in \mathbb{Z}^+} \mathfrak{U}_n.$$

Observe, in particular, that the subspace $\mathfrak{U}_0$ defined above coincides with the subspace $\mathfrak{U}_0$ defined in (3.7).

The subspaces $\mathfrak{U}_n$ and $\mathfrak{U}$ in Definition 4.4 have the following simple characterizations in terms of an arbitrary output nulling representation of Σ.

Proposition 4.5. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal system and let $\Sigma_{s/s/on} = (\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K})$ be an output nulling representation of this system. Then*

$$\mathfrak{U}_n = \cap_{0 \le k \le n} \mathcal{N}\left(C''(A'')^k\right), \tag{4.12}$$

$$\mathfrak{U} = \cap_{k \in \mathbb{Z}^+} \mathcal{N}\left(C''(A'')^k\right). \tag{4.13}$$

In particular, Σ is observable if and only if

$$\cap_{k \in \mathbb{Z}^+} \mathcal{N}\left(C''(A'')^k\right) = \{0\}. \tag{4.14}$$

Proof. If $x_0 \in \cap_{0 \le k \le n} \mathcal{N}\left(C''(A'')^k\right)$, i.e., if $C''(A'')^k x_0 = 0$ for $0 \le k \le n$, then it follows from (4.3) that $(x(\cdot), w(\cdot))$, where $x(k) = (A'')^k x_0$ and $w(k) = 0$, $0 \le k \le n$, is a trajectory of Σ on the interval $[0, n]$. Thus, $x_0 \in \mathfrak{U}_n$ in this case. Conversely, if $(x(\cdot), w(\cdot))$ is a trajectory of Σ on $[0, n]$ with $x(0) = x_0$ and $w(k) = 0$, $0 \le k \le n$, then by (4.3)

$$x(k+1) = A''x(k)$$
$$0 = C''x(k), \quad 0 \le k \le n,$$

which gives $x_0 \in \mathcal{N}\left(C''(A'')^k\right)$ for all k, $0 \le k \le n$. Thus (4.12) holds. Letting $n \to \infty$ in (4.12) we get (4.13). The final statement follows from (4.13) and the definition of observability. □

5. The input/state/output representation

In this section we shall discuss a third type of representation of a state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ in which trajectories $(x(\cdot), w(\cdot))$ on $\mathbb{Z}^+$ of Σ are described by the usual system of equations (1.2) in the traditional input/state/output theory.

Theorem 5.1. *Let V be a subspace of the product space $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, where $\mathcal{X}$ and $\mathcal{W}$ are Hilbert spaces, and suppose that $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$is the direct sum of two complementary closed subspaces $\mathcal{Y}$ and $\mathcal{U}$. If there exists four operators*

$$A \in \mathcal{B}(\mathcal{X}),\ B \in \mathcal{B}(\mathcal{U}; \mathcal{X}),\ C \in \mathcal{B}(\mathcal{X}, \mathcal{Y}),\ \text{and}\ D \in \mathcal{B}(\mathcal{U}; \mathcal{Y}), \tag{5.1}$$

such that

$$\begin{aligned} V = \mathcal{R}\left(\begin{bmatrix} A & B \\ 1_{\mathcal{X}} & 0 \\ C & D \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) &= \mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & A & 0 & B \\ 0 & C & -1_{\mathcal{Y}} & D \end{bmatrix}\right) \\ &= \left\{ \begin{bmatrix} Ax + Bu \\ x \\ Cx + Du + u \end{bmatrix} \,\middle|\, x \in \mathcal{X},\ u \in \mathcal{U} \right\}, \end{aligned} \tag{5.2}$$

then V has properties (i)–(iv) listed in Definition 2.1, i.e., $(V; \mathcal{X}, \mathcal{W})$ is a state/signal node. Conversely, if V has properties (i)–(iv) listed in Definition 2.1 then V is given by (5.2) for some operators A, B, C, and D satisfying (5.1) for some decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. These operators are uniquely defined by V and by the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$.

Proof. The representation (5.2) has an obvious interpretation as a driving variable representation of V (take $C' = \left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$ and $D' = \left[\begin{smallmatrix} D \\ 1_{\mathcal{U}} \end{smallmatrix}\right]$). Thus, by Lemma 3.1, if V is given by (5.2) for some operators A, B, C, and D satisfying (5.1), then V has properties (i)–(iv).

To prove the converse part we start from an arbitrary driving variable representation of V (e.g., from the one constructed in the proof of the converse part of Lemma 3.1), i.e., we let $\mathcal{L}$ be a Hilbert space, and let A', B', C', and D' satisfy (3.1)–(3.3). Then each $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$ can be written in the form

$$\begin{bmatrix} z \\ x \\ w \end{bmatrix} = \begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix} \begin{bmatrix} x \\ \ell \end{bmatrix},$$

for a unique $\ell \in \mathcal{L}$. Let $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ be an arbitrary decomposition of $\mathcal{W}$ with the property that $P_{\mathcal{U}}^{\mathcal{Y}} D'$ maps $\mathcal{L}$ one-to-one onto $\mathcal{U}$ (for example, we can take $\mathcal{U} = \mathcal{U}_0$, with $\mathcal{U}_0$ defined as in (3.6), and take $\mathcal{Y}$ to be an arbitrary direct complement to $\mathcal{U}_0$). With respect to this decomposition of $\mathcal{W}$ the vector $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right]$ can be written in the form (where we denote $u = P_{\mathcal{U}}^{\mathcal{Y}} w$ and $y = P_{\mathcal{Y}}^{\mathcal{U}} w$)

$$\begin{bmatrix} z \\ x \\ y \\ u \end{bmatrix} = \begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ P_{\mathcal{Y}}^{\mathcal{U}} C' & P_{\mathcal{Y}}^{\mathcal{U}} D' \\ P_{\mathcal{U}}^{\mathcal{Y}} C' & P_{\mathcal{U}}^{\mathcal{Y}} D' \end{bmatrix} \begin{bmatrix} x \\ \ell \end{bmatrix}.$$

Since $P_{\mathcal{U}}^{\mathcal{Y}} D'$ is boundedly invertible, we can solve for ℓ to get the equivalent representation

$$\begin{aligned} \begin{bmatrix} z \\ x \\ y \\ u \end{bmatrix} &= \begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ P_{\mathcal{Y}}^{\mathcal{U}} C' & P_{\mathcal{Y}}^{\mathcal{U}} D' \\ P_{\mathcal{U}}^{\mathcal{Y}} C' & P_{\mathcal{U}}^{\mathcal{Y}} D' \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ P_{\mathcal{U}}^{\mathcal{Y}} C' & P_{\mathcal{U}}^{\mathcal{Y}} D' \end{bmatrix}^{-1} \begin{bmatrix} x \\ u \end{bmatrix} \\ &= \begin{bmatrix} A' - B'(P_{\mathcal{U}}^{\mathcal{Y}} D')^{-1} P_{\mathcal{U}}^{\mathcal{Y}} C' & B'(P_{\mathcal{U}}^{\mathcal{Y}} D')^{-1} \\ 1_{\mathcal{X}} & 0 \\ P_{\mathcal{Y}}^{\mathcal{U}} C' - P_{\mathcal{Y}}^{\mathcal{U}} D'(P_{\mathcal{U}}^{\mathcal{Y}} D')^{-1} P_{\mathcal{U}}^{\mathcal{Y}} C' & P_{\mathcal{Y}}^{\mathcal{U}} D'(P_{\mathcal{U}}^{\mathcal{Y}} D')^{-1} \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \end{aligned}$$

This representation is of the type (5.2) with

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} A' & B' \\ P_{\mathcal{Y}}^{\mathcal{U}} C' & P_{\mathcal{Y}}^{\mathcal{U}} D' \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ P_{\mathcal{U}}^{\mathcal{Y}} C' & P_{\mathcal{U}}^{\mathcal{Y}} D' \end{bmatrix}^{-1} \\ &= \begin{bmatrix} A' - B'(P_{\mathcal{U}}^{\mathcal{Y}} D')^{-1} P_{\mathcal{U}}^{\mathcal{Y}} C' & B'(P_{\mathcal{U}}^{\mathcal{Y}} D')^{-1} \\ P_{\mathcal{Y}}^{\mathcal{U}} C' - P_{\mathcal{Y}}^{\mathcal{U}} D'(P_{\mathcal{U}}^{\mathcal{Y}} D')^{-1} P_{\mathcal{U}}^{\mathcal{Y}} C' & P_{\mathcal{Y}}^{\mathcal{U}} D'(P_{\mathcal{U}}^{\mathcal{Y}} D')^{-1} \end{bmatrix}. \end{aligned} \tag{5.3}$$

The uniqueness of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ follows from the fact that (5.2) is a graph representation of V with respect to the decomposition of $\mathfrak{K}$ into $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ 0 \\ \mathcal{Y} \\ 0 \end{smallmatrix}\right] \dot{+} \left[\begin{smallmatrix} 0 \\ \mathcal{X} \\ 0 \\ \mathcal{U} \end{smallmatrix}\right]$, and the operator appearing in this graph representation is unique. □

We shall call a colligation $\Sigma_{i/s/o} := \big(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\big)$, where $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ and A, B, C, and D satisfy (5.1) and (5.2) an *input/state/output representation of the state/signal node* $\Sigma = (V; \mathcal{X}, \mathcal{W})$. We shall also refer to $\Sigma_{i/s/o}$ as an *input/state/output node*. By the input/state/output *system* $\Sigma_{i/s/o}$ we mean the node $\Sigma_{i/s/o}$ itself together with the set of all trajectories $(x(\cdot), u(\cdot), y(\cdot))$ generated by this node through the equations

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \qquad n_1 \le k \le n_2. \end{aligned} \tag{5.4}$$

The subspace $\mathcal{U}$ considered above is called an *input space*, and the vector $u \in \mathcal{U}$ in (5.2) is called an *input variable*. Analogously, the subspace $\mathcal{Y}$ considered above is called an *output space*, and the vector $y \in \mathcal{Y}$ in (5.2) is called an *output variable*. From each trajectory $(x(\cdot), u(\cdot), y(\cdot))$ of the input/state/output system $\Sigma_{i/s/o}$ we get a trajectory $(x(\cdot), w(\cdot))$ of the state/signal system Σ by taking $w = u + y$, and conversely, from each trajectory $(x(\cdot), w(\cdot))$ of the state/signal system Σ we get a trajectory $(x(\cdot), u(\cdot), y(\cdot))$ of the input/state/output system $\Sigma_{i/s/o}$ by taking $u = P_{\mathcal{U}}^{\mathcal{Y}} w$ and $y = P_{\mathcal{Y}}^{\mathcal{U}} w$.

Remark 5.2. Every input/state/output representation can be interpreted both as a driving variable representation and as an output nulling representation. In both cases we combined u and y into the signal vector $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$. We get a driving variable representation by writing (5.2) in the form

$$\begin{aligned} z &= Ax + Bu, \\ \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} D \\ 1_{\mathcal{U}} \end{bmatrix} u, \end{aligned}$$

with driving variable space $\mathcal{U}$ (the operator $D' = \left[\begin{smallmatrix} D \\ 1_{\mathcal{U}} \end{smallmatrix}\right]$ is injective and has closed range), and we get an output nulling representation by writing it in the form

$$\begin{aligned} z &= Ax + \begin{bmatrix} 0 & B \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \\ 0 &= Cx + \begin{bmatrix} -1_{\mathcal{Y}} & D \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \end{aligned}$$

with error space $\mathcal{Y}$ (the operator $D'' = \begin{bmatrix} -1_{\mathcal{Y}} & D \end{bmatrix}$ is surjective).

Remark 5.3. In the standard input/state/output systems theory one considers trajectories $(x(\cdot), u(\cdot), y(\cdot))$ generated by (5.4), but the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ are not required to be complementary subspaces of a given signal space $\mathcal{W}$. Nevertheless, also in this situation it is possible to introduce the product space $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ with an appropriate inner product, to identify $\mathcal{Y}$ with the subspace

$\left[\begin{smallmatrix}\mathcal{Y}\\0\end{smallmatrix}\right]$ of $\mathcal{W}$, and to identify $\mathcal{U}$ with the subspace $\left[\begin{smallmatrix}0\\\mathcal{U}\end{smallmatrix}\right]$ of $\mathcal{W}$. Then $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$, the triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with V defined by (5.2) is a state/signal node, and the original input/state/output system is an input/state/output representation of this node.

Remark 5.4. Each driving variable representation $\Sigma_{dv/s/s}$ of a state/signal system may be interpreted as an input/state/output system, with the driving variable as input data and the original signal as output data. We can and will therefore apply all notions, notations, and results that we will define or obtain for input/state/output systems to such driving variable representations. In this connection we throughout replace the word "input" by "driving" and the word "output" by "signal". An analogous remark is valid for output nulling representations of state signal systems. When we interpret such representations as input/state/output systems we throughout replace the word "input" by "signal" and the word "output" by "error".

Corollary 5.5. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal system, with an input/state/output representation $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix}A & B\\C & D\end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$.*

1) *The reachable subspaces $\mathfrak{R}_n$ in time n and the reachable subspace $\mathfrak{R}$ are given by*

$$\mathfrak{R}_n = \operatorname{span}\{\mathcal{R}\left(A^k B\right) \mid 0 \le k \le n\}, \quad n \in \mathbb{Z}^+, \tag{5.5}$$

$$\mathfrak{R} = \vee_{k \in \mathbb{Z}^+} \mathcal{R}\left(A^k B\right). \tag{5.6}$$

In particular, Σ is controllable if and only if

$$\mathcal{X} = \vee_{k \in \mathbb{Z}^+} \mathcal{R}\left(A^k B\right). \tag{5.7}$$

2) *The unobservable subspaces $\mathfrak{U}_n$ in time n and the unobservable subspace $\mathfrak{U}$ are given by*

$$\mathfrak{U}_n = \cap_{0 \le k \le n} \mathcal{N}\left(CA^k\right), \tag{5.8}$$

$$\mathfrak{U} = \cap_{k \in \mathbb{Z}^+} \mathcal{N}\left(CA^k\right). \tag{5.9}$$

In particular, Σ is observable if and only if

$$\cap_{k \in \mathbb{Z}^+} \mathcal{N}\left(CA^k\right) = \{0\}. \tag{5.10}$$

Proof. This follows from Propositions 3.5 and 4.5 and Remark 5.2. □

Definition 5.6. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal system. We call the ordered direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ (also denoted by $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{Y}\\\mathcal{U}\end{smallmatrix}\right]$) an *admissible (input/output) decomposition for* Σ if Σ has an input/state/output representation with input space $\mathcal{U}$ and output space $\mathcal{Y}$.

Our following theorem characterizes the set of all admissible input/output decompositions.

Lemma 5.7. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal node, and let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a direct sum decomposition of $\mathcal{W}$. Define $\mathcal{U}_0$ as in* (3.6)*. Then the following statements are equivalent:*

1) $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is an admissible input/output decomposition for* Σ.
2) $P_{\mathcal{U}}^{\mathcal{Y}}|_{\mathcal{U}_0}$ *maps* $\mathcal{U}_0$ *one-to-one onto* $\mathcal{U}$, *i.e.,* $(P_{\mathcal{U}}^{\mathcal{Y}}|_{\mathcal{U}_0})^{-1} \in \mathcal{B}(\mathcal{U};\mathcal{U}_0)$.
3) *The space* $\mathcal{U}_0$ *has the graph representation*

$$\mathcal{U}_0 = \left\{ w = \left[\begin{smallmatrix} D \\ 1_{\mathcal{U}} \end{smallmatrix}\right] u \mid u \in \mathcal{U} \right\}, \tag{5.11}$$

for some $D \in \mathcal{B}(\mathcal{U};\mathcal{Y})$.

If the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is admissible for* Σ, *then the operator* D *in* (5.11) *coincides with the operator* D *in* (5.2).

Proof. Proof of 1) ⇒ 3): If 1) holds, then the representation (5.2) of V gives us a graph space representation of $\mathcal{U}_0$ (with the same operator D as in (5.2)).

Proof of 3) ⇒ 2): If 3) holds, then $P_{\mathcal{U}}^{\mathcal{Y}}$ maps $\mathcal{U}_0$ one-to-one onto $\mathcal{U}$, and $D = P_{\mathcal{Y}}^{\mathcal{U}}(P_{\mathcal{U}}^{\mathcal{Y}}|_{\mathcal{U}_0})^{-1}$.

Proof of 2) ⇒ 1): Let $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ be an arbitrary driving variable representation of Σ. Then $P_{\mathcal{U}}^{\mathcal{Y}}$ maps $\mathcal{U}_0$ one-to-one onto $\mathcal{U}$ and $P_{\mathcal{U}}^{\mathcal{Y}} D'$ maps $\mathcal{L}$ one-to-one onto $\mathcal{U}$. The proof of Theorem 5.1 provides us with an input/state/output representation of Σ with input space $\mathcal{U}$ and output space $\mathcal{Y}$. □

Remark 5.8. According to Lemma 5.7, if $\mathcal{Y}$ is an arbitrary direct complement to the subspace $\mathcal{U}_0$ in (3.6), then $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}_0$ is an admissible decomposition for Σ. For this reason we shall refer to $\mathcal{U}_0$ as the *canonical input space*.

The admissibility of a given decomposition of the signal space of a given state/signal system Σ can also be studied by means of a given driving variable, or output nulling, or input/state/output representation of the given system Σ.

Lemma 5.9. *Let* $\Sigma = (V;\mathcal{X},\mathcal{W})$ *be a state/signal node with the driving variable representation* $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$.

1) $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is an admissible input/output decomposition for* Σ *if and only if*

$$P_{\mathcal{U}}^{\mathcal{Y}} D' \text{ maps } \mathcal{L} \text{ one-to-one onto } \mathcal{U}, \text{ i.e., } (P_{\mathcal{U}}^{\mathcal{Y}} D')^{-1} \in \mathcal{B}(\mathcal{U};\mathcal{L}). \tag{5.12}$$

2) *If the decomposition* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is admissible for* Σ, *then the corresponding operators* A, B, C, *and* D *in* (5.2) *are given by* (5.3).

Proof. In the proof of Theorem 5.1 we constructed an input/state/output representation of Σ under the assumption that (5.12) holds. Thus, (5.12) is sufficient for admissibility. Conversely, suppose that the decomposition is admissible for Σ. Then by Lemma 5.7, $P_{\mathcal{U}}^{\mathcal{Y}}$ maps the canonical input space $\mathcal{U}_0 = \mathcal{R}(D')$ one-to-one onto $\mathcal{U}$, and D' is injective. Thus, (5.12) is also necessary for admissibility. □

Lemma 5.10. *Let* $\Sigma = (V;\mathcal{X},\mathcal{W})$ *be a state/signal node with the output nulling representation* $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}\right)$, *and let* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *be a direct sum decomposition of* $\mathcal{W}$.

1) $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is an admissible input/output decomposition for* Σ *if and only if*

$$D''|_{\mathcal{Y}} \text{ maps } \mathcal{Y} \text{ one-to-one onto } \mathcal{K}, \text{ i.e., } (D''|_{\mathcal{Y}})^{-1} \in \mathcal{B}(\mathcal{K};\mathcal{Y}). \tag{5.13}$$

2) *If the decomposition* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is admissible for* Σ, *then the corresponding operators* A, B, C, *and* D *in* (5.2) *are given by*

$$\begin{aligned}\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{X}} & -B''|_{\mathcal{Y}} \\ 0 & -D''|_{\mathcal{Y}} \end{bmatrix}^{-1} \begin{bmatrix} A'' & B''|_{\mathcal{U}} \\ C'' & D''|_{\mathcal{U}} \end{bmatrix} \\ &= \begin{bmatrix} A'' - B''|_{\mathcal{Y}}(D''|_{\mathcal{Y}})^{-1}C'' & B''|_{\mathcal{U}} - B''|_{\mathcal{Y}}(D''|_{\mathcal{Y}})^{-1}D''|_{\mathcal{U}} \\ -(D''|_{\mathcal{Y}})^{-1}C'' & -(D''|_{\mathcal{Y}})^{-1}D''|_{\mathcal{U}} \end{bmatrix}.\end{aligned} \tag{5.14}$$

Proof. Take an arbitrary $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \mathfrak{K}$. By (4.3), $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$ if and only if

$$\begin{bmatrix} z \\ 0 \end{bmatrix} = \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}.$$

With $u = P_{\mathcal{U}}^{\mathcal{Y}} w$ and $y = P_{\mathcal{Y}}^{\mathcal{U}} w$ this can be written in the equivalent form

$$\begin{bmatrix} z \\ 0 \end{bmatrix} = \begin{bmatrix} A'' & B''|_{\mathcal{Y}} & B''|_{\mathcal{U}} \\ C'' & D''|_{\mathcal{Y}} & D''|_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix}. \tag{5.15}$$

If the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is admissible for Σ, then the condition $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$ determines y uniquely as a continuous function of x and u (by (5.2), $y = Cx + Du$), and therefore the operator $D''|_{\mathcal{Y}}$ in (5.15) must map $\mathcal{Y}$ one-to-one onto $\mathcal{K}$ (recall that the range of D'' is all of $\mathcal{K}$). Thus (5.13) is a necessary condition for admissibility. Conversely, suppose that (5.13) holds. Then (5.15) can be written in the equivalent form

$$\begin{aligned}\begin{bmatrix} z \\ y \end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{X}} & -B''|_{\mathcal{Y}} \\ 0 & -D''|_{\mathcal{Y}} \end{bmatrix}^{-1} \begin{bmatrix} A'' & B''|_{\mathcal{U}} \\ C'' & D''|_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ &= \begin{bmatrix} A'' - B''|_{\mathcal{Y}}(D''|_{\mathcal{Y}})^{-1}C'' & B''|_{\mathcal{U}} - B''|_{\mathcal{Y}}(D''|_{\mathcal{Y}})^{-1}D''|_{\mathcal{U}} \\ -(D''|_{\mathcal{Y}})^{-1}C'' & -(D''|_{\mathcal{Y}})^{-1}D''|_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}.\end{aligned}$$

This is an input/state/output representation with A'', B'', C'', and D'' given by (5.14). Thus, (5.13) is also sufficient for the admissibility of the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. □

Theorem 5.11. *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a state/signal node with the input/state/output representation* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$. *Let* $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ *be a direct sum decomposition of* $\mathcal{W}$, *and define* $\Theta \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{Y}_1 \\ \mathcal{U}_1 \end{smallmatrix}\right])$ *by* (1.6).

1) $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ *is an admissible input/output decomposition for* Σ *if and only if*

$$\begin{aligned}&\Theta_{21}D + \Theta_{22} \text{ maps } \mathcal{U} \text{ one-to-one onto } \mathcal{U}_1, \text{ i.e.,} \\ &(\Theta_{21}D + \Theta_{22})^{-1} \in \mathcal{B}(\mathcal{U}_1; \mathcal{U}).\end{aligned} \tag{5.16}$$

2) *If the decomposition* $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ *is admissible for* Σ, *then the corresponding operators* A_1, B_1, C_1, *and* D_1 *are given by*

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A & B \\ \Theta_{11}C & \Theta_{11}D + \Theta_{12} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21}C & \Theta_{21}D + \Theta_{22} \end{bmatrix}^{-1}, \tag{5.17}$$

or equivalently,

$$\begin{aligned} A_1 &= A - B(\Theta_{21}D + \Theta_{22})^{-1}\Theta_{21}C, \\ B_1 &= B(\Theta_{21}D + \Theta_{22})^{-1}, \\ C_1 &= \Theta_{11}C - (\Theta_{11}D + \Theta_{12})(\Theta_{21}D + \Theta_{22})^{-1}\Theta_{21}C, \\ D_1 &= (\Theta_{11}D + \Theta_{12})(\Theta_{21}D + \Theta_{22})^{-1}. \end{aligned} \tag{5.18}$$

Proof. This follows from Remark 5.2 and Lemma 5.9. □

Theorem 5.12. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal node with the input/state/output representation $\Sigma_{i/s/o} = \big(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\big)$, and let $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ be a direct sum decomposition of $\mathcal{W}$. Define $\widetilde{\Theta} \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{Y}_1 \\ \mathcal{U}_1 \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$ by*

$$\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{Y}}^{\mathcal{U}}|_{\mathcal{Y}_1} & P_{\mathcal{Y}}^{\mathcal{U}}|_{\mathcal{U}_1} \\ P_{\mathcal{U}}^{\mathcal{Y}}|_{\mathcal{Y}_1} & P_{\mathcal{U}}^{\mathcal{Y}}|_{\mathcal{U}_1} \end{bmatrix}. \tag{5.19}$$

1) *$\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ is an admissible input/output decomposition for Σ if and only if*

$$\widetilde{\Theta}_{11} - D\widetilde{\Theta}_{21} \text{ maps } \mathcal{Y}_1 \text{ one-to-one onto } \mathcal{Y}. \tag{5.20}$$

2) *If the decomposition $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ is admissible for Σ, then the corresponding operators A_1, B_1, C_1, and D_1 are given by*

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & -B\widetilde{\Theta}_{21} \\ 0 & \widetilde{\Theta}_{11} - D\widetilde{\Theta}_{21} \end{bmatrix}^{-1} \begin{bmatrix} A & B\widetilde{\Theta}_{22} \\ C & -\widetilde{\Theta}_{12} + D\widetilde{\Theta}_{22} \end{bmatrix}, \tag{5.21}$$

or equivalently,

$$\begin{aligned} A_1 &= A + B\widetilde{\Theta}_{21}(\widetilde{\Theta}_{11} - D\widetilde{\Theta}_{21})^{-1}C, \\ B_1 &= B\widetilde{\Theta}_{22} + B\widetilde{\Theta}_{21}(\widetilde{\Theta}_{11} - D\widetilde{\Theta}_{21})^{-1}(-\widetilde{\Theta}_{12} + D\widetilde{\Theta}_{22}), \\ C_1 &= (\widetilde{\Theta}_{11} - D\widetilde{\Theta}_{21})^{-1}C, \\ D_1 &= (\widetilde{\Theta}_{11} - D\widetilde{\Theta}_{21})^{-1}(-\widetilde{\Theta}_{12} + D\widetilde{\Theta}_{22}). \end{aligned} \tag{5.22}$$

Proof. This follows from Remark 5.2 and Lemma 5.10. □

6. Transfer functions

The (input-output) transfer function of discrete time input/state/output system $\Sigma_{i/s/o} = \big(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\big)$ is defined by the formula

$$\mathfrak{D}(z) = D + zC(1_{\mathcal{X}} - zA)^{-1}B, \quad z \in \Lambda_A, \tag{6.1}$$

where Λ_A is the set of points $z \in \mathbb{C}$ for which $(1_{\mathcal{X}} - zA)$ has a bounded inverse, plus the point at infinity if A is boundedly invertible. The set Λ_A is the maximal domain of analyticity of the function $z\mathfrak{A}(z)$, where $\mathfrak{A}$ is the (Fredholm) resolvent of A, i.e.,

$$\mathfrak{A}(z) = (1_{\mathcal{X}} - zA)^{-1}, \quad z \in \Lambda_A. \tag{6.2}$$

Thus, both $\mathfrak{D}$ and $\mathfrak{A}$ will be defined on the same subset Λ_A of the extended complex plane. The resolvent $\mathfrak{A}$ may have an analytic extension to the point at infinity even if A does not have a bounded inverse, and the transfer function $\mathfrak{D}$ may have an analytic extension to a larger domain, but in this paper we shall not make any use of such extensions. Note that $\mathfrak{D}(z) = D + zC\mathfrak{A}(z)B$, that $\mathfrak{D}(0) = D$ and that $\mathfrak{D}(\infty) = D - CA^{-1}B$ (if A is boundedly invertible).

The function $\mathfrak{D}$ arises in a natural way when one studies the Z-transform of a trajectory $(x(\cdot), u(\cdot), y(\cdot))$ of $\Sigma_{i/s/o}$ on $\mathbb{Z}^+$. Let us denote the formal power series induced by the sequences $\{x(n)\}_{n=0}^{\infty}$, $\{y(n)\}_{n=0}^{\infty}$, and $\{u(n)\}_{n=0}^{\infty}$ by[2]

$$\hat{x}(z) = \sum_{n=0}^{\infty} x(n)z^n, \quad \hat{y}(z) = \sum_{n=0}^{\infty} y(n)z^n, \quad \hat{u}(z) = \sum_{n=0}^{\infty} u(n)z^n.$$

The system of equations (1.2) is then equivalent to the following system of equations for formal power series:

$$\begin{aligned} \hat{x}(z) &= x(0) + zA\hat{x}(z) + zB\hat{u}(z), \\ \hat{y}(z) &= C\hat{x}(z) + D\hat{u}(z). \end{aligned} \tag{6.3}$$

Solving these equations for $\hat{x}$ and $\hat{y}$ in terms of $x(0)$ and $\hat{u}$ we get the more explicit formula

$$\begin{bmatrix} \hat{x}(z) \\ \hat{y}(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(z) \\ \mathfrak{C}(z) \end{bmatrix} x(0) + \begin{bmatrix} \mathfrak{B}(z) \\ \mathfrak{D}(z) \end{bmatrix} \hat{u}(z), \tag{6.4}$$

where the right-hand side should be interpreted as sums and products of (formal) power series of the following type: $x(0)$ is just a constant, $\hat{u}(z)$ is the formal power series induced by the sequence $\{u(n)\}_{n=0}^{\infty}$, and the multipliers $\mathfrak{A}(z)$, $\mathfrak{B}(z)$, $\mathfrak{C}(z)$, and $\mathfrak{D}(z)$, represent the MacLaurin series of the corresponding functions defined by (6.1), (6.2), and by

$$\begin{aligned} \mathfrak{B}(z) &= z(1_{\mathcal{X}} - zA)^{-1}B = z\mathfrak{A}(z)B, \qquad && z \in \Lambda_A, \\ \mathfrak{C}(z) &= C(1_{\mathcal{X}} - zA)^{-1} = C\mathfrak{A}(z), \qquad && z \in \Lambda_A, \end{aligned} \tag{6.5}$$

that is,

$$\begin{aligned} \mathfrak{A}(z) &= \sum_{n=0}^{\infty} A^n z^n, \qquad & \mathfrak{B}(z) &= \sum_{n=0}^{\infty} A^n B z^{n+1}, \\ \mathfrak{C}(z) &= \sum_{n=0}^{\infty} CA^n z^n, \qquad & \mathfrak{D}(z) &= D + \sum_{n=0}^{\infty} CA^n B z^{n+1}. \end{aligned} \tag{6.6}$$

[2]The alternative transform where z is replaced by $1/z$ is also frequently used. The corresponding transfer function is then given by $D + C(z - A)^{-1}B$, defined on the resolvent set of A, including the point at infinity.

The corresponding time-domain formulas are

$$\begin{aligned} x(n) &= A^n x(0) + \sum_{k=0}^{n-1} A^k Bu(n-k-1), \\ y(n) &= CA^n x(0) + Du(n) + \sum_{k=0}^{n-1} CA^k Bu(n-k-1), \qquad n \in \mathbb{Z}^+ \end{aligned} \tag{6.7}$$

(where we interpret an empty sum as zero). From time to time we shall need to refer to the different maps in (6.7), and therefore introduce the following terminology. We define the *state-to-state map* $\check{\mathfrak{A}}\colon \mathcal{X} \to \mathcal{X}^{\mathbb{Z}^+}$, the *input-to-state map* $\check{\mathfrak{B}}\colon \mathcal{U}^{\mathbb{Z}^+} \to \mathcal{X}^{\mathbb{Z}^+}$, the *state-to-output map* $\check{\mathfrak{C}}\colon \mathcal{X} \to \mathcal{Y}^{\mathbb{Z}^+}$, and the *input-to-output map* $\check{\mathfrak{D}}\colon \mathcal{U}^{\mathbb{Z}^+} \to \mathcal{U}^{\mathbb{Z}^+}$ by

$$\begin{aligned} (\check{\mathfrak{A}}x)(n) &= A^n x, && n \in \mathbb{Z}^+, \\ (\check{\mathfrak{B}}u)(n) &= \sum_{k=0}^{n-1} A^k Bu(n-k-1), && n \in \mathbb{Z}^+, \\ (\check{\mathfrak{C}}x)(n) &= CA^n x, && n \in \mathbb{Z}^+, \\ (\check{\mathfrak{D}}u)(n) &= D + \sum_{k=0}^{n-1} CA^k Bu(n-k-1), && n \in \mathbb{Z}^+. \end{aligned} \tag{6.8}$$

It is frequently possible to interpret the above equations as equations between analytic functions defined in a neighborhood of zero rather than formal power series. It suffices to assume that the (formal) power series defining $\hat{u}$ has a strictly positive radius of convergence. This implies that also the series defining $\hat{x}$ and $\hat{y}$ have a positive radius of convergence, that $\hat{u}$, $\hat{z}$, and $\hat{y}$ are analytic functions defined in a neighborhood of zero, and that (6.4) holds with $\mathfrak{A}(z)$, $\mathfrak{B}(z)$, $\mathfrak{C}(z)$, and $\mathfrak{D}(z)$ defined by (6.1), (6.2), and (6.5). In particular, if $x(0) = 0$, then $\hat{y}(z) = \mathfrak{D}(z)\hat{u}(z)$ in a neighborhood of zero, and this explains why the function $\mathfrak{D}$ is called the input-output transfer function. Similar interpretations are valid for the transfer functions $\mathfrak{A}$ (state to state), $\mathfrak{B}$ (input to state), and $\mathfrak{C}$ (state to output).

A more compact way of writing (6.1), (6.2), and (6.5) is

$$\begin{aligned} \begin{bmatrix} z\mathfrak{A}(z) & \mathfrak{B}(z) \\ z\mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} &= \begin{bmatrix} (1/z - A)^{-1} & (1/z - A)^{-1}B \\ C(1/z - A)^{-1} & D + C(1/z - A)^{-1}B \end{bmatrix} \\ &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C & D \end{bmatrix} \begin{bmatrix} 1/z - A & -B \\ 0 & 1_{\mathcal{U}} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1/z - A & 0 \\ -C & 1_{\mathcal{Y}} \end{bmatrix}^{-1} \begin{bmatrix} 1_{\mathcal{X}} & B \\ 0 & D \end{bmatrix}, \quad z \in \Lambda_A, \quad z \neq 0 \end{aligned} \tag{6.9}$$

(the value at infinity is obtained by taking limits as $z \to \infty$, and the corresponding formula for $z = 0$ is trivial).

We shall call

$$\mathfrak{V}(z) := \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix}$$

the *four block input/state/output* transfer function of the system $\Sigma_{i/s/o}$.

A driving-variable/state/signal system $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix} \right]; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$ may be interpreted as an input/state/output system with $\mathcal{L}$ as input space, $\mathcal{X}$ as state space, and $\mathcal{W}$ as output space. The Z-transform $(\hat{x}, \hat{\ell}, \hat{w})$ of a trajectory $(x(\cdot), \ell(\cdot), w(\cdot))$ of this system on $\mathbb{Z}^+$ therefore satisfies

$$\begin{bmatrix} \hat{x}(z) \\ \hat{y}(z) \end{bmatrix} = \mathfrak{V}'(z) \begin{bmatrix} x(0) \\ \hat{u}(z) \end{bmatrix} := \begin{bmatrix} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'(z) & \mathfrak{D}'(z) \end{bmatrix} \begin{bmatrix} x(0) \\ \hat{u}(z) \end{bmatrix}, \tag{6.10}$$

where $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, and $\mathfrak{D}'$ are given by (6.9) with A, B, C, and D replaced by A', B', C', and D'. We shall call $\mathfrak{V}'$ the *four block driving-variable/state/signal* transfer function of the system $\Sigma_{dv/s/s}$. Analogously, the Z-transform $(\hat{x}, \hat{w}, \hat{e})$ of a trajectory $(x(\cdot), w(\cdot), e(\cdot))$ of a signal/state/output nulling system $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix} \right]; \mathcal{X}, \mathcal{W}, \mathcal{K} \right)$ on $\mathbb{Z}^+$ therefore satisfies

$$\begin{bmatrix} \hat{x}(z) \\ \hat{e}(z) \end{bmatrix} = \mathfrak{V}''(z) \begin{bmatrix} x(0) \\ \hat{w}(z) \end{bmatrix} := \begin{bmatrix} \mathfrak{A}''(z) & \mathfrak{B}''(z) \\ \mathfrak{C}''(z) & \mathfrak{D}''(z) \end{bmatrix} \begin{bmatrix} x(0) \\ \hat{w}(z) \end{bmatrix}, \tag{6.11}$$

where $\mathfrak{A}''$, $\mathfrak{B}''$, $\mathfrak{C}''$, and $\mathfrak{D}''$ are given by (6.9) with A, B, C, and D replaced by A'', B'', C'', and D''. We shall call $\mathfrak{V}''$ the *four block signal/state/error* transfer function of the system $\Sigma_{s/s/on}$.

Below we shall study relations between the four block transfer functions $\mathfrak{V}$, $\mathfrak{V}'$, and $\mathfrak{V}''$ that correspond to the three types of representations (input/state/output, driving variable, or output nulling, respectively) of a given state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$.

First we will consider the relationships between the four block driving variable transfer function of two driving-variable representations of a state/signal system.

Theorem 6.1. *Let*

$$\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix} \right]; \mathcal{X}, \mathcal{L}, \mathcal{W} \right) \quad \textit{and} \quad \Sigma^1_{dv/s/s} = \left(\left[\begin{smallmatrix} A'_1 & B'_1 \\ C'_1 & D'_1 \end{smallmatrix} \right]; \mathcal{X}, \mathcal{L}_1, \mathcal{W} \right)$$

be two driving variable representations of the state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$. *Denote the four block transfer functions of* $\Sigma_{dv/s/s}$ *and* $\Sigma^1_{dv/s/s}$ *by* $\left[\begin{smallmatrix} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'(z) & \mathfrak{D}'(z) \end{smallmatrix} \right]$ *and* $\left[\begin{smallmatrix} \mathfrak{A}'_1(z) & \mathfrak{B}'_1(z) \\ \mathfrak{C}'_1(z) & \mathfrak{D}'_1(z) \end{smallmatrix} \right]$, *respectively, and let* $K' \in \mathcal{B}(\mathcal{X}; \mathcal{L})$ *and* $M' \in \mathcal{B}(\mathcal{L}_1; \mathcal{L})$ *be the operators in Theorem* 3.3, *uniquely determined by* (3.12).

1) *The operator* $1_{\mathcal{L}} - K'\mathfrak{B}'(z)$ (*defined on* $\Lambda_{A'}$) *has a bounded inverse if and only if* $z \in \Lambda_{A'} \cap \Lambda_{A'_1}$.
2) *For all* $z \in \Lambda_{A'} \cap \Lambda_{A'_1}$,

$$\begin{bmatrix} \mathfrak{A}'_1(z) & \mathfrak{B}'_1(z) \\ \mathfrak{C}'_1(z) & \mathfrak{D}'_1(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'(z) & \mathfrak{D}'(z) \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ -K'\mathfrak{A}'(z) & 1_{\mathcal{L}} - K'\mathfrak{B}'(z) \end{bmatrix}^{-1} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & M' \end{bmatrix}, \tag{6.12}$$

or equivalently,[3]

$$\begin{aligned}
\mathfrak{A}_1'(z) &= (1_{\mathcal{X}} - \mathfrak{B}'(z)K')^{-1}\mathfrak{A}'(z),\\
\mathfrak{B}_1'(z) &= (1_{\mathcal{X}} - \mathfrak{B}'(z)K')^{-1}\mathfrak{B}'(z)M',\\
\mathfrak{C}_1'(z) &= \mathfrak{C}'(z) + \mathfrak{D}'(z)K'(1_{\mathcal{X}} - \mathfrak{B}'(z)K')^{-1}\mathfrak{A}'(z),\\
\mathfrak{D}_1'(z) &= \mathfrak{D}'(z)(1_{\mathcal{L}} - K'\mathfrak{B}'(z))^{-1}M'.
\end{aligned} \tag{6.13}$$

Proof. The case where $z = 0$ is trivial, so in the sequel we assume that $z \neq 0$.

Assume first that $z \in \Lambda_{A'} \cap \Lambda_{A_1'}$, with $z \neq 0$. Since $z \in \Lambda_{A_1'}$, we get from (6.9),

$$\begin{aligned}
\begin{bmatrix} z\mathfrak{A}_1'(z) & \mathfrak{B}_1'(z) \\ z\mathfrak{C}_1'(z) & \mathfrak{D}_1'(z)\end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C_1' & D_1'\end{bmatrix}\begin{bmatrix} 1/z - A_1' & -B_1' \\ 0 & 1_{\mathcal{L}_1}\end{bmatrix}^{-1}\\
&= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C_1' & D_1'\end{bmatrix}\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ K' & M'\end{bmatrix}^{-1}\left(\begin{bmatrix} 1/z - A_1' & -B_1' \\ 0 & 1_{\mathcal{L}_1}\end{bmatrix}\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ K' & M'\end{bmatrix}^{-1}\right)^{-1}\\
&= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C' & D'\end{bmatrix}\begin{bmatrix} 1/z - A' & -B' \\ -(M')^{-1}K' & (M')^{-1}\end{bmatrix}^{-1}.
\end{aligned}$$

Observe, in particular, that the last block matrix above is boundedly invertible. Since also $z \in \Lambda_{A'}$, we can factor

$$\begin{bmatrix} 1/z - A' & -B' \\ -(M')^{-1}K' & (M')^{-1}\end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ -(M')^{-1}K'z\mathfrak{A}'(z) & (M')^{-1}(1_{\mathcal{L}} - K'\mathfrak{B}'(z))\end{bmatrix} \times \begin{bmatrix} 1/z - A' & -B' \\ 0 & 1_{\mathcal{L}}\end{bmatrix}. \tag{6.14}$$

As we noticed above, the left-hand side in boundedly invertible, and hence also the operator $1_{\mathcal{L}} - K'\mathfrak{B}'(z)$ must be boundedly invertible. Substituting this factorization into the formula above we get

$$\begin{aligned}
\begin{bmatrix} z\mathfrak{A}_1'(z) & \mathfrak{B}_1'(z) \\ z\mathfrak{C}_1'(z) & \mathfrak{D}_1'(z)\end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C' & D'\end{bmatrix}\begin{bmatrix} 1/z - A' & -B' \\ 0 & 1_{\mathcal{L}}\end{bmatrix}^{-1}\\
&\quad\times \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ -(M')^{-1}K'z\mathfrak{A}'(z) & (M')^{-1}(1_{\mathcal{L}} - K'\mathfrak{B}'(z))\end{bmatrix}^{-1}\\
&= \begin{bmatrix} z\mathfrak{A}'(z) & \mathfrak{B}'(z) \\ z\mathfrak{C}'(z) & \mathfrak{D}'(z)\end{bmatrix}\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ -K'z\mathfrak{A}'(z) & 1_{\mathcal{L}} - K'\mathfrak{B}'(z)\end{bmatrix}^{-1}\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & M'\end{bmatrix}.
\end{aligned}$$

Multiplying this identity to the right by $\left[\begin{smallmatrix} 1/z & 0 \\ 0 & 1\end{smallmatrix}\right]$ we get (6.12). We have now proved assertion 2) and one half of assertion 1).

To prove the other half of assertion 1) we assume that $z \in \Lambda_{A'}$, $z \neq 0$, and that $1_{\mathcal{L}} - K'\mathfrak{B}'(z)$ is boundedly invertible. Then the block operator matrix on the left-hand side of (6.14) is also boundedly invertible. As we noticed above,

[3]Note that, by Lemma 10.1, $1_{\mathcal{L}} - K'\mathfrak{B}'(z)$ has a bounded inverse if and only if $1_{\mathcal{X}} - \mathfrak{B}'(z)K'$ has a bounded inverse.

this matrix factors into $\left[\begin{smallmatrix} 1/z-A_1' & -B_1' \\ 0 & 1_{\mathcal{L}_1}\end{smallmatrix}\right]\left[\begin{smallmatrix} 1_{\mathcal{X}} & 0 \\ K' & M'\end{smallmatrix}\right]^{-1}$, and hence $1/z - A_1'$ must be boundedly invertible, i.e., $z \in \Lambda_{A_1'}$.

Theorem 6.2. *Let*

$$\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D''\end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}\right) \quad \text{and} \quad \Sigma^1_{s/s/on} = \left(\left[\begin{smallmatrix} A_1'' & B_1'' \\ C_1'' & D_1''\end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}_1\right)$$

be two output nulling representations of the state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$. Denote the four block transfer functions of $\Sigma_{s/s/on}$ and $\Sigma^1_{s/s/on}$ by $\left[\begin{smallmatrix} \mathfrak{A}''(z) & \mathfrak{B}''(z) \\ \mathfrak{C}''(z) & \mathfrak{D}''(z)\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathfrak{A}_1''(z) & \mathfrak{B}_1''(z) \\ \mathfrak{C}_1''(z) & \mathfrak{D}_1''(z)\end{smallmatrix}\right]$, respectively, and let K'' and M'' be the operators in Theorem 4.3*, uniquely determined by* (4.11)*.*

1) *The operator $1_{\mathcal{K}} - z\mathfrak{C}''(z)K''$ (defined on $\Lambda_{A''}$) has a bounded inverse if and only if $z \in \Lambda_{A''} \cap \Lambda_{A_1''}$.*
2) *For all $z \in \Lambda_{A''} \cap \Lambda_{A_1''}$,*

$$\begin{bmatrix} \mathfrak{A}_1''(z) & \mathfrak{B}_1''(z) \\ \mathfrak{C}_1''(z) & \mathfrak{D}_1''(z)\end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & M''\end{bmatrix}\begin{bmatrix} 1_{\mathcal{X}} & -z\mathfrak{A}''(z)K'' \\ 0 & 1_{\mathcal{K}} - z\mathfrak{C}''(z)K''\end{bmatrix}^{-1}\begin{bmatrix} \mathfrak{A}''(z) & \mathfrak{B}''(z) \\ \mathfrak{C}''(z) & \mathfrak{D}''(z)\end{bmatrix}, \tag{6.15}$$

or equivalently,[4]

$$\begin{aligned} \mathfrak{A}_1''(z) &= \mathfrak{A}''(z)(1_{\mathcal{X}} - zK''\mathfrak{C}''(z))^{-1}\mathfrak{C}''(z), \\ \mathfrak{B}_1''(z) &= \mathfrak{B}''(z) + z\mathfrak{A}''(z)(1_{\mathcal{X}} - zK''\mathfrak{C}''(z))^{-1}K''\mathfrak{D}''(z), \\ \mathfrak{C}_1''(z) &= M''\mathfrak{C}''(z)(1_{\mathcal{X}} - zK''\mathfrak{C}''(z))^{-1}, \\ \mathfrak{D}_1''(z) &= M''(1_{\mathcal{K}} - z\mathfrak{C}''(z)K'')^{-1}\mathfrak{D}''(z). \end{aligned} \tag{6.16}$$

The proof of this theorem is similar to the proof of Theorem 6.1, and we leave it to the reader.

Lemma 6.3. *Let $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ and $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D'\end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ be an input/state/output and a driving variable representation, respectively, of the state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$. Denote the four block transfer functions of $\Sigma_{i/s/o}$ and $\Sigma_{dv/s/s}$ by $\left[\begin{smallmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z)\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'(z) & \mathfrak{D}'(z)\end{smallmatrix}\right]$, respectively.*

1) *The operator $P^{\mathcal{Y}}_{\mathcal{U}}\mathfrak{D}'(z)$ (defined on $\Lambda_{A'}$) has a bounded inverse if and only if $z \in \Lambda_A \cap \Lambda_{A'}$.*
2) *For all $z \in \Lambda_A \cap \Lambda_{A'}$,*

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z)\end{bmatrix} = \begin{bmatrix} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ P^{\mathcal{U}}_{\mathcal{Y}}\mathfrak{C}'(z) & P^{\mathcal{U}}_{\mathcal{Y}}\mathfrak{D}'(z)\end{bmatrix}\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ P^{\mathcal{Y}}_{\mathcal{U}}\mathfrak{C}'(z) & P^{\mathcal{Y}}_{\mathcal{U}}\mathfrak{D}'(z)\end{bmatrix}^{-1} \tag{6.17}$$

[4] Note that, by Lemma 10.1, $1_{\mathcal{K}} - z\mathfrak{C}''(z)K''$ has a bounded inverse if and only if $1_{\mathcal{X}} - zK''\mathfrak{C}''(z)$ has a bounded inverse.

or equivalently,

$$\begin{aligned}\mathfrak{A}(z) &= \mathfrak{A}'(z) - \mathfrak{B}'(z)(P_{\mathcal{U}}^{\mathcal{Y}}\mathfrak{D}'(z))^{-1}P_{\mathcal{U}}^{\mathcal{Y}}\mathfrak{C}'(z)\\ \mathfrak{B}(z) &= \mathfrak{B}'(z)(P_{\mathcal{U}}^{\mathcal{Y}}\mathfrak{D}'(z))^{-1}\\ \mathfrak{C}(z) &= P_{\mathcal{Y}}^{\mathcal{U}}\mathfrak{C}'(z) - P_{\mathcal{Y}}^{\mathcal{U}}\mathfrak{D}'(z)(P_{\mathcal{U}}^{\mathcal{Y}}\mathfrak{D}'(z))^{-1}P_{\mathcal{U}}^{\mathcal{Y}}\mathfrak{C}'(z)\\ \mathfrak{D}(z) &= P_{\mathcal{Y}}^{\mathcal{U}}\mathfrak{D}'(z)(P_{\mathcal{U}}^{\mathcal{Y}}\mathfrak{D}'(z))^{-1}.\end{aligned} \tag{6.18}$$

Proof. We interpret $\Sigma_{i/s/o}$ as a driving variable representation

$$\Sigma^1_{dv/s/s} = \left(\begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix}; \mathcal{X}, \mathcal{L}_1, \mathcal{W}\right)$$

with $\mathcal{L}_1 = \mathcal{U}$ and

$$\left[\begin{array}{c|c} A_1' & B_1' \\ \hline C_1' & D_1' \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ 0 & 1_{\mathcal{U}} \end{array}\right];$$

see Remark 5.2. The corresponding block decomposition of $\Sigma_{dv/s/s}$ is given by

$$\left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right] = \left[\begin{array}{c|c} A' & B' \\ \hline P_{\mathcal{Y}}^{\mathcal{U}}C' & P_{\mathcal{Y}}^{\mathcal{U}}D' \\ P_{\mathcal{U}}^{\mathcal{Y}}C' & P_{\mathcal{U}}^{\mathcal{Y}}D' \end{array}\right].$$

To these two driving variable representations we apply Theorem 6.1. By comparing the two representations to each other we find that the operators $K' \in \mathcal{B}(\mathcal{X};\mathcal{L})$ and $M' \in \mathcal{B}(\mathcal{U};\mathcal{L})$ are given by

$$M' = [P_{\mathcal{U}}^{\mathcal{Y}}D']^{-1}, \qquad K' = -[P_{\mathcal{U}}^{\mathcal{Y}}D']^{-1}P_{\mathcal{U}}^{\mathcal{Y}}C'.$$

The operator $1_{\mathcal{L}} - K'\mathfrak{B}'(z)$ in part 1) Theorem 6.1 is given by

$$\begin{aligned}1_{\mathcal{L}} - K'\mathfrak{B}'(z) &= 1_{\mathcal{L}} + [P_{\mathcal{U}}^{\mathcal{Y}}D']^{-1}P_{\mathcal{U}}^{\mathcal{Y}}C'\mathfrak{B}'(z)\\ &= [P_{\mathcal{U}}^{\mathcal{Y}}D']^{-1}(P_{\mathcal{U}}^{\mathcal{Y}}D' + P_{\mathcal{U}}^{\mathcal{Y}}C'\mathfrak{B}'(z))\\ &= [P_{\mathcal{U}}^{\mathcal{Y}}D']^{-1}P_{\mathcal{U}}^{\mathcal{Y}}\mathfrak{D}(z),\end{aligned}$$

and it is boundedly invertible if and only if $P_{\mathcal{U}}^{\mathcal{Y}}\mathfrak{D}(z)$ is boundedly invertible. Substituting the above values into (6.12) we get (6.17). □

Lemma 6.4. *Let* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *and* $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}\right)$ *be an input/state/output and a output nulling representation, respectively, of the state/signal system* $\Sigma = (V; \mathcal{X}, \mathcal{W})$. *Denote the four block transfer functions of* $\Sigma_{i/s/o}$ *and* $\Sigma_{s/s/on}$ *by* $\left[\begin{smallmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} \mathfrak{A}''(z) & \mathfrak{B}''(z) \\ \mathfrak{C}''(z) & \mathfrak{D}''(z) \end{smallmatrix}\right]$, *respectively.*

1) *The operator* $\mathfrak{D}''(z)|_{\mathcal{Y}}$ *(defined on* $\Lambda_{A''}$*) has a bounded inverse if and only if* $z \in \Lambda_A \cap \Lambda_{A''}$.
2) *For all* $z \in \Lambda_A \cap \Lambda_{A''}$,

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & -\mathfrak{B}''(z)|_{\mathcal{Y}} \\ 0 & -\mathfrak{D}''(z)|_{\mathcal{Y}} \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{A}''(z) & \mathfrak{B}''(z)|_{\mathcal{U}} \\ \mathfrak{C}''(z) & \mathfrak{D}''(z)|_{\mathcal{U}} \end{bmatrix}, \tag{6.19}$$

or equivalently

$$\begin{aligned}\mathfrak{A}(z) &= \mathfrak{A}''(z) - \mathfrak{B}''(z)|_{\mathcal{Y}}(\mathfrak{D}''(z)|_{\mathcal{Y}})^{-1}\mathfrak{C}''(z),\\ \mathfrak{B}(z) &= \mathfrak{B}''(z)|_{\mathcal{U}} - \mathfrak{B}''(z)|_{\mathcal{Y}}(\mathfrak{D}''(z)|_{\mathcal{Y}})^{-1}\mathfrak{D}''(z)|_{\mathcal{U}},\\ \mathfrak{C}(z) &= -(\mathfrak{D}''(z)|_{\mathcal{Y}})^{-1}\mathfrak{C}''(z),\\ \mathfrak{D}(z) &= -(\mathfrak{D}''(z)|_{\mathcal{Y}})^{-1}\mathfrak{D}''(z)|_{\mathcal{U}}.\end{aligned} \tag{6.20}$$

Proof. This lemma is proved in the same way as Lemma 6.3, but this time we interpret $\Sigma_{i/s/o}$ as an output nulling representation of Σ (as in Remark 5.2) and use Theorem 6.2 instead of Theorem 6.1. □

Theorem 6.5. *Let* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *and* $\Sigma^1_{i/s/o} = \left(\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1\end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1\right)$ *be two input/state/output representations of the state/signal system* $\Sigma = (V; \mathcal{X}, \mathcal{W})$. *Denote the four block transfer functions of* $\Sigma_{i/s/o}$ *and* $\Sigma^1_{i/s/o}$ *by* $\left[\begin{smallmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z)\end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} \mathfrak{A}_1(z) & \mathfrak{B}_1(z) \\ \mathfrak{C}_1(z) & \mathfrak{D}_1(z)\end{smallmatrix}\right]$, *respectively. Define* $\Theta \in \mathcal{B}(\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]; \left[\begin{smallmatrix}\mathcal{Y}_1\\ \mathcal{U}_1\end{smallmatrix}\right])$ *and* $\widetilde{\Theta} \in \mathcal{B}(\left[\begin{smallmatrix}\mathcal{Y}_1\\ \mathcal{U}_1\end{smallmatrix}\right]; \left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right])$ *by* (1.6) *and* (5.19), *respectively.*

1) *For each* $z \in \Lambda_A$ *the following conditions are equivalent:*
   (a) $z \in \Lambda_{A_1}$.
   (b) *The operator* $\Theta_{21}\mathfrak{D}(z) + \Theta_{22}$ *has a bounded inverse.*
   (c) *The operator* $\widetilde{\Theta}_{11} - \mathfrak{D}(z)\widetilde{\Theta}_{21}$ *has a bounded inverse.*
2) *For all* $z \in \Lambda_A \cap \Lambda_{A_1}$,

$$\begin{bmatrix}\mathfrak{A}_1(z) & \mathfrak{B}_1(z)\\ \mathfrak{C}_1(z) & \mathfrak{D}_1(z)\end{bmatrix} = \begin{bmatrix}\mathfrak{A}(z) & \mathfrak{B}(z)\\ \Theta_{11}\mathfrak{C}(z) & \Theta_{11}\mathfrak{D}(z)+\Theta_{12}\end{bmatrix}\begin{bmatrix}1_{\mathcal{X}} & 0\\ \Theta_{21}\mathfrak{C}(z) & \Theta_{21}\mathfrak{D}(z)+\Theta_{22}\end{bmatrix}^{-1}, \tag{6.21}$$

or equivalently,

$$\begin{aligned}\mathfrak{A}_1(z) &= \mathfrak{A}(z) - \mathfrak{B}(z)(\Theta_{21}\mathfrak{D}(z)+\Theta_{22})^{-1}\Theta_{21}\mathfrak{C}(z),\\ \mathfrak{B}_1(z) &= \mathfrak{B}(z)(\Theta_{21}\mathfrak{D}(z)+\Theta_{22})^{-1},\\ \mathfrak{C}_1(z) &= \Theta_{11}\mathfrak{C}(z) - (\Theta_{11}\mathfrak{D}(z)+\Theta_{12})(\Theta_{21}\mathfrak{D}(z)+\Theta_{22})^{-1}\Theta_{21}\mathfrak{C}(z),\\ \mathfrak{D}_1(z) &= (\Theta_{11}\mathfrak{D}(z)+\Theta_{12})(\Theta_{21}\mathfrak{D}(z)+\Theta_{22})^{-1}.\end{aligned} \tag{6.22}$$

3) *For all* $z \in \Lambda_A \cap \Lambda_{A_1}$,

$$\begin{bmatrix}\mathfrak{A}_1(z) & \mathfrak{B}_1(z)\\ \mathfrak{C}_1(z) & \mathfrak{D}_1(z)\end{bmatrix} = \begin{bmatrix}1_{\mathcal{X}} & -\mathfrak{B}(z)\widetilde{\Theta}_{21}\\ 0 & \widetilde{\Theta}_{11}-\mathfrak{D}(z)\widetilde{\Theta}_{21}\end{bmatrix}^{-1}\begin{bmatrix}\mathfrak{A}(z) & \mathfrak{B}(z)\widetilde{\Theta}_{22}\\ \mathfrak{C}(z) & -\widetilde{\Theta}_{12}+\mathfrak{D}(z)\widetilde{\Theta}_{22}\end{bmatrix}, \tag{6.23}$$

or equivalently,

$$\begin{aligned}\mathfrak{A}_1(z) &= \mathfrak{A}(z) + \mathfrak{B}(z)\widetilde{\Theta}_{21}(\widetilde{\Theta}_{11}-\mathfrak{D}(z)\widetilde{\Theta}_{21})^{-1}\mathfrak{C}(z),\\ \mathfrak{B}_1(z) &= \mathfrak{B}(z)\widetilde{\Theta}_{22} + \mathfrak{B}(z)\widetilde{\Theta}_{21}(\widetilde{\Theta}_{11}-\mathfrak{D}(z)\widetilde{\Theta}_{21})^{-1}(-\widetilde{\Theta}_{12}+\mathfrak{D}(z)\widetilde{\Theta}_{22}),\\ \mathfrak{C}_1(z) &= (\widetilde{\Theta}_{11}-\mathfrak{D}(z)\widetilde{\Theta}_{21})^{-1}\mathfrak{C}(z),\\ \mathfrak{D}_1(z) &= (\widetilde{\Theta}_{11}-\mathfrak{D}(z)\widetilde{\Theta}_{21})^{-1}(-\widetilde{\Theta}_{12}+\mathfrak{D}(z)\widetilde{\Theta}_{22}).\end{aligned} \tag{6.24}$$

Proof. Assertion 2) follows from Lemma 6.3, assertion 3) from Lemma 6.4, and for assertion 1) we need both of these lemmas. For the proof of 2) we interpret $\Sigma^1_{i/s/o}$ as a driving variable representation, and for the proof of 3) we interpret $\Sigma^1_{i/s/o}$ as an output nulling representation, as explained in Remark 5.2. □

7. Signal behaviors, external equivalence, and similarity

The behavioral approach to systems theory was introduced by Willems, and has been developed extensively by him and others (see, e.g., [PW98] for a recent presentation of behavioral theory). The vast majority of the literature on behaviors deals with finite-dimensional systems, and the existing extensions to the infinite-dimensional case seem to ignore state space representations of the type that we have introduced above. Below we shall consider the problem of realization of a given behavior on a Hilbert space $\mathcal{W}$ by a state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$.

In order to motivate out definition of a signal behavior we first take a closer look at the signal parts of all externally generated trajectories of a state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$. Let $\mathfrak{W}$ be the set of all the signal sequences $w(\cdot)$, defined on $\mathbb{Z}^+$ with values in $\mathcal{W}$, that are the signal components of externally generated trajectories $(x(\cdot), w(\cdot))$ of Σ on $\mathbb{Z}^+$. It is easy to see that this set $\mathfrak{W}$ is a closed right-shift invariant subspace of the Fréchet space $\mathcal{W}^{\mathbb{Z}^+}$ of all $\mathcal{W}$-valued sequences on $\mathbb{Z}^+$.

We now turn the above property into a definition.

Definition 7.1. Let $\mathcal{W}$ be a Hilbert space.[5] By a (causal signal) *behavior* on the signal space $\mathcal{W}$ we mean a closed right-shift invariant subspace of $\mathcal{W}^{\mathbb{Z}^+}$.

This is a special case of a "manifest behavior", as described, e.g., in [PW98, Definition 1.2.9], but our choice of this particular subclass of behaviors is not a standard one. A similar definition was used by Ball and Staffans [BS05] in continuous time (with an extra growth restriction at infinity that was appropriate in their setting).

A behavior that is induced by a state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ as explained above is called *realizable*, and the state/signal system Σ that induces this behavior is called a *realization* of the behavior $\mathfrak{W}$.

Definition 7.2. Two state/signal systems with the same signal space are called *externally equivalent* if they induce the same behavior.

A behavior induced by a state/signal system has both an *image representation* and a *kernel representation* of the following type:

Lemma 7.3. *Let $\mathfrak{W}$ be the behavior induced by a state/signal system $\Sigma = (V; \mathcal{X}; \mathcal{W})$. Then*

[5] We make only indirect use of the fact that $\mathcal{W}$ is a Hilbert space. See the footnote to Definition 2.1.

1) $\mathfrak{W}$ *is the range of the driving-to-signal map* $\check{\mathfrak{D}}'$ *of every driving variable representation of* Σ*, and*
2) $\mathfrak{W}$ *is the kernel of the signal-to-error map* $\check{\mathfrak{D}}''$ *of every output nulling representation of* Σ.

We leave the easy proof to the reader.

After introducing the above notions we face the following tasks:

1) find criteria of realizability of a given behavior on $\mathcal{W}$;
2) find criteria of external equivalence between two state/signal systems with the same signal space.

The solutions of these problems will be given in this section. These solutions involve some additional notation. If $\mathfrak{W}$ is a behavior on $\mathcal{W}$, then the set

$$\mathfrak{W}(0) = \{w(0) \mid w \in \mathfrak{W}\}. \tag{7.1}$$

is a closed subspace of $\mathcal{W}$. We call this subspace the *zero section* of $\mathfrak{W}$. Observe that, if $\mathfrak{W}$ is induced by a state/signal system, then $\mathfrak{W}(0)$ coincides with the canonical input space $\mathcal{U}_0$ in (3.6).

Definition 7.4. Let $\mathfrak{W}$ be a behavior on $\mathcal{W}$. An ordered direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ (also denoted by $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$) is called an *admissible (input/output) decomposition for* $\mathfrak{W}$ if it has the following two properties:

1) For any sequence $u(\cdot) \in \mathcal{U}^{\mathbb{Z}^+}$ there exists at least one sequence $w(\cdot) \in \mathfrak{W}$ such that $u(n) = P_{\mathcal{U}}^{\mathcal{Y}} w(n)$ for all $n \in \mathbb{Z}^+$ (that is, the projection of $\mathfrak{W}$ onto $\mathcal{U}^{\mathbb{Z}^+}$ along $\mathcal{Y}^{\mathbb{Z}^+}$ is surjective).
2) There exists positive constants M and r such that

$$\sum_{n=0}^{T} \|r^n w(n)\|^2 \leq M^2 \sum_{n=0}^{T} \|r^n P_{\mathcal{U}}^{\mathcal{Y}} w(n)\|^2 \tag{7.2}$$

for all $w(\cdot) \in \mathfrak{W}$ and all $T \in \mathbb{Z}^+$.

Theorem 7.5. *Let* $\mathfrak{W}$ *be a behavior on* $\mathcal{W}$.

1) *The following conditions are equivalent:*
   (a) *The behavior* $\mathfrak{W}$ *is realizable by a state/signal system.*
   (b) *There exists at least one admissible input/output decomposition* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *for* $\mathfrak{W}$.
   (c) *For some direct complement* $\mathcal{Y}_0$ *to the zero section* $\mathfrak{W}(0)$ *the decomposition* $\mathcal{W} = \mathcal{Y}_0 \dotplus \mathfrak{W}(0)$ *is admissible for* $\mathfrak{W}$.
   (d) *For every direct complement* $\mathcal{Y}_0$ *to the zero section* $\mathfrak{W}(0)$ *the decomposition* $\mathcal{W} = \mathcal{Y}_0 \dotplus \mathfrak{W}(0)$ *is admissible for* $\mathfrak{W}$.
2) *Assume that* $\mathfrak{W}$ *is realizable by the state/signal system* $\Sigma = (V; \mathcal{X}, \mathcal{W})$*. Then a direct sum decomposition* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is admissible for* $\mathfrak{W}$ *if and only if it is admissible for* Σ.

Proof. We begin by proving one half of assertion 2). Suppose first that the behavior $\mathfrak{W}$ is realized by the state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$. Consider some admissible input/output decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ for the state/signal system Σ. Let $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ be the input/state/output representation of Σ corresponding to this decomposition. Then, for every externally generated trajectory $(x(\cdot), w(\cdot))$ of Σ on $\mathbb{Z}^+$ we have $w(n) = y(n) + u(n)$, where $u(n) = P_{\mathcal{U}}^{\mathcal{Y}} w(n)$ and $y(n) = P_{\mathcal{Y}}^{\mathcal{U}} w(n)$. Clearly, the projection of $\mathcal{W}$ onto $\mathcal{U}^{\mathbb{Z}^+}$ is surjective (this is the first requirement of an admissible input/output decomposition for $\mathfrak{W}$). To prove that also (7.2) holds we choose some $r > 0$ and rewrite (1.2) in the form

$$\begin{aligned} x_r(n+1) &= rAx_r(n) + rBu_r(n), \\ y_r(n) &= Cx_r(n) + Du_r(n), \qquad n \in \mathbb{Z}^+, \\ x(0) &= 0, \end{aligned} \tag{7.3}$$

where $x_r(n) = r^n x(n)$, $u_r(n) = r^n u(n)$, and $y_r(n) = r^n y(n)$. Choose r so small that $\|rA\| < 1$. By (6.7) and by the standard fact that the convolution of an ℓ^1-sequence and an ℓ^2-sequence belongs to ℓ^2,

$$\sum_{n=0}^{T} \|y_r(n)\|^2 \le M_1^2 \sum_{n=0}^{T} \|u_r(n)\|^2,$$

where $M_1 = \|D\| + \|C\|(1 - \|rA\|)^{-1}\|B\|$. Clearly this implies (7.2) with a larger constant M (which depends, among others, on the norms of $P_{\mathcal{Y}}^{\mathcal{U}}$). Thus, the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is admissible for $\mathfrak{W}$, and we have proved one direction of assertion 2). In addition, we have proved the implication (a) $\Rightarrow$ (d), since the decomposition in (d) is admissible for Σ (see Lemma 5.7). Trivially (d) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b). Thus, it remains to prove the other half of assertion 2) and the implication (b) $\Rightarrow$ (a).

Suppose now that $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is an admissible decomposition for the behavior $\mathfrak{W}$. Let r and M be the constants in (7.2). For each $w(\cdot) \in \mathfrak{W}$ we define $w_r(n) = r^n w(n)$, $u_r(n) = r^n P_{\mathcal{U}}^{\mathcal{Y}} w(n)$, and $y_r(n) = r^n P_{\mathcal{Y}}^{\mathcal{U}} w$, $n \in \mathbb{Z}^+$. Then (7.2) implies that the mapping from u_r to y_r is a continuous right-shift invariant mapping from $\ell^2(\mathbb{Z}^+;\mathcal{U})$ to $\ell^2(\mathbb{Z}^+;\mathcal{Y})$. As is well known, this implies that this mapping has a multiplier representation given in terms of Z-transforms by

$$\hat{y}_r(z) = \mathfrak{D}_r(z)\hat{u}_r(z)$$

for some bounded holomorphic $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued function in the unit disk $\mathbb{D}$, satisfying $\sup_{z\in\mathbb{D}} \|\mathfrak{D}_r(z)\| \le M$. This function $\mathfrak{D}_r$ can be realized as the input/output transfer function of an input/state/output system $\Sigma_r = \left(\left[\begin{smallmatrix} A_r & B_r \\ C_r & D_r \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$; see [Aro74, Theorem 3], [Fuh74], or [Hel74, Theorem 3c.1]. We then define

$$\Sigma_{i/s/o} = \left(\begin{bmatrix} r^{-1}A_r & r^{-1}B_r \\ C_r & D_r \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right).$$

This system is an input/state/output representation of a state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, and the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is admissible for this system. The system Σ is a state/signal realization of the given behavior $\mathfrak{W}$. This proves the implication (b) $\Rightarrow$ (a), and completes the proof of assertion 1).

It only remains to prove the second half of the assertion 2), namely that every decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ that is admissible for the behavior $\mathfrak{W}$ is also admissible for its realization Σ. To do this we use the characterization given in Lemma 5.7. Let $u_0 \in \mathcal{U}$, and take some arbitrary $u(\cdot) \in \mathcal{U}^{\mathbb{Z}^+}$ with $u(0) = u_0$. Then there is a corresponding signal $w(\cdot) \in \mathfrak{W}$ such that $P_{\mathcal{U}}^{\mathcal{Y}} w(\cdot) = u(\cdot)$. In particular, $u_0 = P_{\mathcal{U}}^{\mathcal{Y}} w(0)$, where $w(0) \in \mathfrak{W}(0) = \mathcal{U}_0$. Thus $P_{\mathcal{U}}^{\mathcal{Y}}$ maps $\mathcal{U}_0$ onto $\mathcal{U}$. That $P_{\mathcal{U}}^{\mathcal{Y}}|_{\mathcal{U}_0}$ is injective follows from (7.2). By Lemma 5.7, the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is admissible for Σ. □

Corollary 7.6. *Let $\mathfrak{W}$ be a realizable behavior on $\mathcal{W}$, let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a direct sum decomposition of $\mathcal{W}$. Then the following conditions are equivalent.*

1) *$\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is an admissible input/output decomposition for $\mathfrak{W}$.*
2) *$P_{\mathcal{U}}^{\mathcal{Y}}$ maps $\mathfrak{W}(0)$ one-to-one onto $\mathcal{U}$, i.e., $(P_{\mathcal{U}}^{\mathcal{Y}})^{-1} \in \mathcal{B}(\mathcal{U}; \mathfrak{W}(0))$.*
3) *The space $\mathfrak{W}(0)$ has the graph representation*
$$\mathfrak{W}(0) = \left\{ w = \begin{bmatrix} D \\ 1_{\mathcal{U}} \end{bmatrix} u \mid u \in \mathcal{U} \right\}, \tag{7.4}$$
for some $D \in \mathcal{B}(\mathcal{U}; \mathcal{Y})$.

If the decomposition is admissible, then the operator D in (7.4) *is the feedthrough operator of every input/state/output realization of $\mathfrak{W}$ with $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$.*

This follows from Lemma 5.7 and part 2) of Theorem 7.5 (recall that $\mathfrak{W}(0) = \mathcal{U}_0$).

Theorem 7.7. *Let Σ and Σ^1 be two state/signal systems with the common signal space $\mathcal{W}$.*

1) *If Σ and Σ^1 have a common admissible input/output decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ and the corresponding input/output transfer functions coincide in a neighborhood of zero, then the two systems are externally equivalent.*
2) *Conversely, if Σ and Σ^1 are externally equivalent, then any direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is admissible for Σ if and only if it is admissible for Σ^1, and the corresponding input/output transfer functions coincide in the (connected) component of $\Lambda_A \cap \Lambda_{A_1}$ which contains zero. In particular, the feedthrough operators also coincide.*

Proof. Proof of 1): We denote the input/state/output representations of Σ and Σ_1 by $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$, respectively, $\Sigma^1_{i/s/o} = \left(\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$, and the behaviors induced by Σ and Σ_1 by $\mathfrak{W}$, respectively, $\mathfrak{W}_1$. Let $w(\cdot) \in \mathfrak{W}$. Then there exists a sequence $x(\cdot)$ with $x(0) = 0$ such that $(x(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$. Equivalently, $(x(\cdot), u(\cdot), y(\cdot))$, with $u(\cdot) = P_{\mathcal{U}}^{\mathcal{Y}} w(\cdot)$ and $y(\cdot) = P_{\mathcal{Y}}^{\mathcal{U}} w(\cdot)$ is a trajectory of $\Sigma_{i/s/o}$ on $\mathbb{Z}^+$ with $x(0) = 0$. Let $(x_1(\cdot), u(\cdot), y_1(\cdot))$ be the trajectory of $\Sigma^1_{i/s/o}$ on $\mathbb{Z}^+$ which has $x_1(0) = 0$ and the same input sequence u as above. We claim that $y_1(\cdot) = y(\cdot)$. To prove this is suffices to show that the two input-to-output map (the map $\check{\mathfrak{D}}$ in (6.8)) are the same for the two systems $\Sigma_{i/s/o}$ and $\Sigma^1_{i/s/o}$, i.e., that $D = D_1$ and that $CA^kB = C_1A_1^kB$ for all $k \in \mathbb{Z}^+$. However, these are the Taylor coefficients of the corresponding transfer functions $\mathfrak{D}$ and $\mathfrak{D}_1$ at the origin, and since we assume that the two transfer functions coincide in a neighborhood of the origin, these Taylor coefficients are the same, too. Thus, $y(\cdot) = y_1(\cdot)$, as claimed. This means that $(x_1(\cdot), w(\cdot))$ is an externally generated trajectory of Σ_1 on $\mathbb{Z}^+$. The above argument shows that $\mathfrak{W} \subset \mathfrak{W}_1$. By interchanging the roles of the two systems Σ and Σ_1 we conclude by the same argument that $\mathfrak{W}_1 \subset \mathfrak{W}$. Thus, the two systems Σ and Σ_1 are externally equivalent.

Proof of 2). Suppose that Σ and Σ^1 are externally equivalent. Then they induce the same behavior $\mathfrak{W}$. By part 2) of Theorem 7.5, the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is admissible for Σ if and only if it is admissible for $\mathfrak{W}$, and this is true if and only if it is admissible for Σ^1. Assume that the decomposition is admissible (for both systems), and denote the corresponding transfer functions by $\mathfrak{D}$, respectively, $\mathfrak{D}_1$. Let $u(\cdot) \in \mathcal{U}^{\mathbb{Z}^+}$, and suppose that the Z-transform of $u(\cdot)$ has a nonzero radius of convergence. Choose some $w(\cdot) \in \mathfrak{W}$ such that $P_{\mathcal{U}}^{\mathcal{Y}} w(\cdot) = u(\cdot)$. Define $y(\cdot) = P_{\mathcal{Y}}^{\mathcal{U}} w(\cdot)$. Then we have in some (possibly smaller) neighborhood of zero,

$$\hat{y}(z) = \mathfrak{D}(z)\hat{u}(z) = \mathfrak{D}_1\hat{u}(z).$$

This being true for all $u(\cdot) \in \mathcal{U}^{\mathbb{Z}^+}$ whose Z-transform of $u(\cdot)$ has a nonzero radius of convergence, this implies that $\mathfrak{D}(z) = \mathfrak{D}_1(z)$ in some neighborhood of zero. By analytic extension, these two transfer functions must coincide in the connected component of $\Lambda_A \cap \Lambda_{A_1}$ which contains zero. That the feedthrough operators coincide follows from the fact that they are the values of the transfer functions at zero. □

Instead of testing the external equivalence of two state/signal systems by using input/state/output representations of these systems it is also possible to use driving variable or output nulling representations.

Proposition 7.8. *Let Σ and Σ^1 be two state/signal systems with the common signal space $\mathcal{W}$. Let $\Sigma_{i/s/o}$ and $\Sigma^1_{i/s/o}$ be two input/state/output representations of Σ, respectively, Σ_1 corresponding to the same admissible decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$, let $\Sigma_{dv/s/s}$ and $\Sigma^1_{dv/s/s}$ be two driving variable representations of Σ, respectively, Σ_1, and let $\Sigma_{s/s/on}$ and $\Sigma^1_{s/s/on}$ be two output nulling variable representations of Σ, respectively, Σ_1. Then the following conditions are equivalent:*

1) *Σ and Σ^1 are externally equivalent.*
2) *The input-to-output maps $\check{\mathfrak{D}}$ and $\check{\mathfrak{D}}_1$ of $\Sigma_{i/s/o}$, respectively, $\Sigma^1_{i/s/o}$ coincide.*
3) *The driving-to-signal maps $\check{\mathfrak{D}}'$ and $\check{\mathfrak{D}}'_1$ of $\Sigma_{dv/s/s}$, respectively, $\Sigma^1_{dv/s/s}$ have the same ranges.*
4) *The signal-to-error maps $\check{\mathfrak{D}}''$ and $\check{\mathfrak{D}}''_1$ of $\Sigma_{s/s/on}$, respectively, $\Sigma^1_{s/s/on}$ have the same kernels.*

Proof. This follows from Lemma 7.3, Theorem 7.7, and the fact that the input/output transfer function determines the input-to-output map uniquely. □

The rest of this section is devoted to a study of similarity and pseudo-similarity of state/signal systems.

Definition 7.9. Two state/signal systems $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ with the same signal space $\mathcal{W}$ are similar if there exists a boundedly invertible operator $R \in \mathcal{B}(\mathcal{X}; \mathcal{X}_1)$, called the *similarity operator*, such that $(x(\cdot), w(\cdot))$ is a trajectory of Σ if and only if $(x_1(\cdot), w(\cdot)) = (Rx(\cdot), w(\cdot))$ is a trajectory of Σ_1.

From this definition follows that two similar state/signal systems are externally equivalent.

The corresponding similarity notion is well known for input/state/output systems. Two input/state/output systems $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ and $\Sigma^1_{i/s/o} = \left(\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}\right)$ with the same input and output spaces are similar if there exists a boundedly invertible operator $R \in \mathcal{B}(\mathcal{X}; \mathcal{X}_1)$ such that

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} RAR^{-1} & RB \\ CR^{-1} & D \end{bmatrix}.$$

We shall apply the same similarity notion to driving variable and output nulling representations, too, interpreting them as input/state/output systems (as explained in Remark 5.4).

Proposition 7.10. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be two state/signal systems with the same signal space $\mathcal{W}$. Then the following conditions are equivalent.*

1) *Σ and Σ_1 are similar with similarity operator R.*
2) $V_1 = \left[\begin{smallmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{smallmatrix}\right] V$.
3) *Σ and Σ_1 have driving variable representations $\Sigma_{dv/s/s}$ and $\Sigma^1_{dv/s/s}$, respectively, which are similar with similarity operator R.*
4) *To each driving variable representation $\Sigma_{dv/s/s}$ of Σ there is a (unique) driving variable representation $\Sigma^1_{dv/s/s}$ of Σ_1 such that these representations are similar with similarity operator R.*
5) *Σ and Σ_1 have output nulling representations $\Sigma_{s/s/on}$ and $\Sigma^1_{s/s/on}$, respectively, which are similar with similarity operator R.*
6) *To each output nulling representation $\Sigma_{s/s/on}$ of Σ there is a (unique) output nulling representation $\Sigma^1_{s/s/on}$ of Σ_1 such that these representations are similar with similarity operator R.*
7) *There exists some decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of $\mathcal{W}$ which is admissible both for Σ and for Σ_1, and the corresponding input/state/output representations $\Sigma_{i/s/o}$ and $\Sigma^1_{i/s/o}$ are similar with similarity operator R.*
8) *The systems Σ and Σ_1 have the same set of admissible decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of $\mathcal{W}$, and for every such decomposition the corresponding input/state/output representations $\Sigma_{i/s/o}$ and $\Sigma^1_{i/s/o}$ are similar with similarity operator R.*

We leave the easy proof to the reader.

Various partial converses to the statement that two similar systems are externally equivalent is also valid. Some additional conditions are always needed. One such condition is that both the systems are controllable and observable. In this case they need not actually be similar but only pseudo-similar. Two state/signal systems $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ are called *pseudo-similar* if there exists an injective densely defined closed linear operator $R\colon \mathcal{X} \to \mathcal{X}_1$ with dense range such that the following conditions hold:

> If $(x(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$ with $x(0) \in \mathcal{D}(R)$, then $x(n) \in \mathcal{D}(R)$ for all $n \in \mathbb{Z}^+$ and $(Rx(\cdot), w(\cdot))$ is a trajectory of Σ_1 on $\mathbb{Z}^+$, and conversely, if $(x_1(\cdot), w(\cdot))$ is a trajectory of Σ_1 on $\mathbb{Z}^+$ with $x_1(0) \in \mathcal{R}(R)$, then $x_1(n) \in \mathcal{D}(R)$ for all $n \in \mathbb{Z}^+$ and $(R^{-1}x_1(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$.

Proposition 7.11. *Two controllable and observable state/signal systems $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ with the same signal space $\mathcal{W}$ are externally equivalent if and only if they are pseudo-similar.*

Proof. In one direction the assertion is obvious: if Σ and Σ_1 are pseudo-similar, then they induce the same behavior (take $x(0) = 0$ and $x_1(0) = 0$).

Conversely, suppose that Σ and Σ and are controllable and observable state/signal systems which are externally equivalent. Then they have the same set of admissible input/output decompositions of the signal space $\mathcal{W}$. Let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be such a decomposition, and denote the corresponding input/state/output representations of Σ and Σ_1 by $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ and $\Sigma^1_{i/s/o} = \left(\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}\right)$, respectively. Then both $\Sigma_{i/s/o}$ and $\Sigma^1_{i/s/o}$ are controllable and observable, and also externally equivalent. This means that their input/output transfer functions coincide a neighborhood of zero. By [Aro79, Proposition 6], these two systems are pseudo-similar in the following sense: there exists an injective densely defined closed linear operator $R\colon \mathcal{X} \to \mathcal{X}_1$ with dense range such that

$$\begin{aligned} &\mathcal{R}(B) \subset \mathcal{D}(R), \ A\mathcal{D}(R) \subset \mathcal{D}(R), \ A_1\mathcal{R}(R) \subset \mathcal{R}(R), \\ &A_1 R = RA|_{\mathcal{D}(R)}, \ B_1 = RB, \ C_1 R = C|_{\mathcal{D}(R)}, \ D_1 = D. \end{aligned} \tag{7.5}$$

If $(x(\cdot), w(\cdot))$ and $(x_1(\cdot), w(\cdot))$ are externally generated trajectories of Σ and Σ_1, respectively, with $x(0) \in \mathcal{D}(R)$, $x_1(0) \in \mathcal{R}(R)$, and $x_1(0) = Rx(0)$, then for all $n \in \mathbb{Z}^+$,

$$\begin{aligned} x(n) &= A^n x(0) + \sum_{k=0}^{n-1} A^k B u(n-k-1), \\ x_1(n) &= A_1^n R x(0)(0) + \sum_{k=0}^{n-1} A_1^k B_1 u(n-k-1), \end{aligned} \tag{7.6}$$

where $u(n) = P_{\mathcal{U}}^{\mathcal{Y}} w(n)$. This combined with (7.5) gives $x_1(n) = Rx(n)$ for all $n \in \mathbb{Z}^+$. Thus, Σ and Σ_1 are pseudo-similar. □

8. Dilations of state/signal systems

In the classical finite-dimensional input/state/output systems theory a system is called *minimal* if the dimension of its state space is minimal among all systems with the same transfer function. By a classical result due to Kalman, such a finite-dimensional input/state/output system is minimal if and only if it is controllable and observable. We can reformulate this result is the state/signal setting as follows: a state/signal system with a finite-dimensional state space has a state space with minimal dimension among all externally equivalent systems if and only if it is controllable and observable.

In the case where the state space is infinite-dimensional the requirement that its state space should have minimal dimension becomes obscure (all infinite-dimensional separable Hilbert spaces has the same dimension). It is therefore necessary to define minimality in terms of some other property. One natural solution is to study *dilations* and *compressions* of systems. In the finite-dimensional case the minimality of the dimension of the state space is equivalent to the statement that the system cannot be compressed into a "smaller" system, and this characterization has a natural infinite-dimensional analogue. The notions of dilations and compressions of operators and of input/state/output systems have attracted a great deal of attention and it plays an important role in many works, see, e.g., [Aro79], [SF70], and [LP67] for Hilbert space versions, and [BGK79] and [Sta05] for Banach space versions.

Definition 8.1. The state/signal system $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ is a *dilation along* $\mathcal{Z}$ of the state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, or equivalently, the state/signal system Σ is a *compression along* $\mathcal{Z}$ *onto* $\mathcal{X}$ of the state/signal system $\widetilde{\Sigma}$, if the following conditions hold:

1) $\widetilde{\mathcal{X}} = \mathcal{X} \dot{+} \mathcal{Z}$,
2) If $(\tilde{x}(\cdot), w(\cdot))$ is a trajectory of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with $\tilde{x}(0) \in \mathcal{X}$, then $(P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$.
3) There is at least one decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ of $\mathcal{W}$ which is admissible for both $\widetilde{\Sigma}$ and Σ.

Note that, whereas the compressed system is determined uniquely by the dilated system and by the decomposition $\widetilde{\mathcal{X}} = \mathcal{X} \dot{+} \mathcal{Z}$, the converse is clearly not true.

Lemma 8.2. *Let the state/signal system* $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ *be a dilation along* $\mathcal{Z}$ *of* $\Sigma = (V; \mathcal{X}, \mathcal{W})$. *Then the following claims hold.*

1) *To each trajectory* $(x(\cdot), w(\cdot))$ *of* Σ *on* $\mathbb{Z}^+$ *there is a unique trajectory* $(\tilde{x}(\cdot), \tilde{w}(\cdot))$ *of* $\widetilde{\Sigma}$ *on* $\mathbb{Z}^+$ *satisfying* $\tilde{x}(0) = x(0)$ *and* $\tilde{w}(\cdot) = w(\cdot)$. *This trajectory has the additional property that* $x(\cdot) = P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(\cdot)$.
2) $\widetilde{\Sigma}$ *and* Σ *are externally equivalent. In particular, they have the same admissible input/output decompositions of the signal space, and the input/output transfer functions and the input-to-output maps of the corresponding input/state/output representations of* $\widetilde{\Sigma}$ *and* Σ *coincide.*

Proof. Let $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ be a decomposition which is admissible both for $\widetilde{\Sigma}$ and for Σ, and denote the corresponding input/state/output representations of $\widetilde{\Sigma}$ and Σ by $\widetilde{\Sigma}_{i/s/o}$ and $\Sigma_{i/s/o}$, respectively. Let $(x(\cdot), w(\cdot))$ be a trajectory of Σ on $\mathbb{Z}^+$. Define $u(\cdot) = P_{\mathcal{U}}^{\mathcal{Y}} w(\cdot)$ and $y(\cdot) = P_{\mathcal{Y}}^{\mathcal{U}} w(\cdot)$. Then $(x(\cdot), u(\cdot), y(\cdot))$ is a trajectory of $\Sigma_{i/s/o}$, and $\widetilde{\Sigma}_{i/s/o}$ has a unique trajectory $(\tilde{x}(\cdot), u(\cdot), \tilde{y}(\cdot))$ on $\mathbb{Z}^+$ satisfying $\tilde{x}(0) = x(0)$. Define $\tilde{w}(\cdot) = \tilde{y}(\cdot) + u(\cdot)$. Then $(\tilde{x}(\cdot), \tilde{w}(\cdot))$ is a trajectory of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$. According to property 2) in Definition 8.1, $(P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(\cdot), \tilde{w}(\cdot))$ must be a trajectory of Σ, and hence, if we define $\tilde{y}(\cdot) = P_{\mathcal{Y}}^{\mathcal{U}} \tilde{w}(\cdot)$, then $(P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(\cdot), u(\cdot), \tilde{y}(\cdot))$ is a trajectory of $\Sigma_{i/s/o}$. But a trajectory of $\Sigma_{i/s/o}$ is determined uniquely by its initial state and input data, and therefore we must have $x(\cdot) = P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(\cdot)$ and $\tilde{y}(\cdot) = y(\cdot)$. This proves assertion 1). Assertion 2) follows immediately from property 2) in Definition 8.1 together with assertion 1) . □

Observability and controllability are preserved under compressions (but not under dilations).

Lemma 8.3. *Let the state/signal system* $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ *be a dilation along* $\mathcal{Z}$ *of* $\Sigma = (V; \mathcal{X}, \mathcal{W})$. *Let* $\widetilde{\mathfrak{R}}$ *and* $\mathfrak{R}$ *be the reachable subspaces and let* $\widetilde{\mathfrak{U}}$ *and* $\mathfrak{U}$ *be the unobservable subspaces of* $\widetilde{\Sigma}$ *and* Σ, *respectively. Then* $\mathfrak{U} = \widetilde{\mathfrak{U}} \cap \mathcal{X}$ *and* $\mathfrak{R}$ *is the closure of* $P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{\mathfrak{R}}$. *In particular, if* $\widetilde{\Sigma}$ *is controllable or observable, then* Σ *is controllable or observable, respectively.*

We leave the easy proof to the reader.

In order to be able to study the relationship between the two systems $\widetilde{\Sigma}$ and Σ in Definition 8.1 in more detail we need the following two invariance notions.[6]

Definition 8.4. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal system.

1) A closed subspace $\mathcal{Z}$ of $\mathcal{X}$ is *outgoing invariant* for Σ if to each $x_0 \in \mathcal{Z}$ there is a (unique) trajectory $(x(\cdot), 0)$ of Σ on $\mathbb{Z}^+$ with $x(0) = x_0$ satisfying $x(n) \in \mathcal{Z}$ for all $n \in \mathbb{Z}^+$.
2) A closed subspace $\mathcal{Z}$ of $\mathcal{X}$ is *strongly invariant* for Σ if every trajectory $(x(\cdot), w(\cdot))$ of Σ on $\mathbb{Z}^+$ with $x(0) \in \mathcal{Z}$ satisfies $x(n) \in \mathcal{Z}$ for all $n \in \mathbb{Z}^+$.

These invariance properties can also be described in terms of the generating subspace V as follows.

Lemma 8.5. *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a state/signal system, and let* $\mathcal{Z}$ *be a closed subspace of* $\mathcal{X}$.

1) $\mathcal{Z}$ *is outgoing invariant for* Σ *if and only if the following condition holds:*
$$\text{To each } x \in \mathcal{Z} \text{ there is a (unique) } z \in \mathcal{Z} \text{ such that } \begin{bmatrix} z \\ x \\ 0 \end{bmatrix} \in V. \tag{8.1}$$
2) $\mathcal{Z}$ *is strongly invariant for* Σ *if and only if it the following implication is true:*
$$\text{If } \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \text{ and } x \in \mathcal{Z}, \text{ then } z \in \mathcal{Z}. \tag{8.2}$$

[6]The connections between these notions and the unobservable and reachable subspaces are explained in Lemma 8.6 below.

Proof. Proof of 1): The necessity of (8.1) for outgoing invariance is immediate (the solution $(x(\cdot),0)$ mentioned in part 1) of Definition 8.4 satisfies $\begin{bmatrix} x(1) \\ x(0) \\ 0 \end{bmatrix} \in V$.)

Conversely, suppose that (8.1) holds. Let $x_0 \in \mathcal{Z}$. Then (8.1) with x replaced by x_0 gives the existence of $x(1) \in \mathcal{Z}$ such that $\begin{bmatrix} x(1) \\ x_0 \\ 0 \end{bmatrix} \in V$. Applying (8.1) once more with x replaced by $x(1)$ we get the existence of $x(2) \in \mathcal{Z}$ such that $\begin{bmatrix} x(2) \\ x(1) \\ 0 \end{bmatrix} \in V$. Continuing in the same way we get a sequence $x(\cdot)$ such that $x(0) = x_0$ and $(x(\cdot),0)$ is a trajectory of Σ on $\mathbb{Z}^+$. According to Definition 8.4, $\mathcal{Z}$ is outgoing invariant.

Proof of 2): To see that (8.2) is necessary for $\mathcal{Z}$ to be strongly invariant we argue as follows. By part 1) of Proposition 2.2, the condition $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$ implies that there exists a trajectory $(x(\cdot),w(\cdot))$ of Σ on $\mathbb{Z}^+$ with $x(0) = x_0$, $w(0) = w_0$, and $x(1) = z_0$. If, furthermore, $x_0 \in \mathcal{Z}$, then the strong invariance of $\mathcal{Z}$ implies that $x(n) \in \mathcal{Z}$ for all $n \in \mathbb{Z}^+$. In particular, $z_0 = x(1) \in \mathcal{Z}$.

The proof of the converse part is similar to the proof of the converse part of assertion 1), and it is left to the reader. □

The two main examples of outgoing invariant and strongly invariant subspaces are the following:

Lemma 8.6. *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a state/signal system.*

1) *The unobservable subspace is the maximal outgoing invariant subspace for Σ, i.e., it is outgoing invariant, and it contains every other outgoing invariant subspace.*
2) *The reachable subspace is the minimal closed strongly invariant subspace for Σ, i.e., it is strongly invariant, and it is contained in every other closed strongly invariant subspace.*

We leave the easy proof to the reader.

The following theorem is the main result of this section.

Theorem 8.7. *Let $\widetilde{\Sigma} = \big(\widetilde{V};\widetilde{\mathcal{X}},\mathcal{W}\big)$ and $\Sigma = \big(V;\mathcal{X},\mathcal{W}\big)$ be two state/signal systems with $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$ (and with the same signal space). Then $\widetilde{\Sigma}$ is a dilation along $\mathcal{Z}$ of Σ if and only if the following conditions hold:*

1) *V is given by*

$$V = \left\{ \begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}} \tilde{z} \\ x \\ w \end{bmatrix} \;\middle|\; x \in \mathcal{X} \text{ and } \begin{bmatrix} \tilde{z} \\ x \\ w \end{bmatrix} \in \widetilde{V} \right\}. \tag{8.3}$$

2) *$\mathcal{Z}$ has a decomposition $\mathcal{Z} = \mathcal{Z}_o \dotplus \mathcal{Z}_i$ where $\mathcal{Z}_o$ is outgoing invariant for $\widetilde{\Sigma}$ and $\mathcal{Z}_o \dotplus \mathcal{X}$ is strongly invariant for $\widetilde{\Sigma}$.*

One possible choice of the subspaces $\mathcal{Z}_o$ and $\mathcal{Z}_i$ in 2) is to take $\mathcal{Z}_o = \mathcal{Z}_o^{\max}$ and to take $\mathcal{Z}_i$ to be an arbitrary direct complement of $\mathcal{Z}_o^{\max}$ in $\mathcal{Z}$, where

$$\mathcal{Z}_o^{\max} = \left\{ \tilde{x}_0 \in \widetilde{X} \;\middle|\; \begin{array}{l} \text{there exists a trajectory } (\tilde{x}(\cdot), 0) \text{ of } \widetilde{\Sigma} \text{ on } \mathbb{Z}^+ \text{ with} \\ \tilde{x}(0) = \tilde{x}_0 \text{ satisfying } P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(n) = 0 \text{ for all } n \in \mathbb{Z}^+ \end{array} \right\}. \tag{8.4}$$

The subspace $\mathcal{Z}_o^{\max}$ is maximal in the sense that it contains every other space $\mathcal{Z}_o$ that can be used in the decomposition in 2).

We shall call $\mathcal{Z}_o$ an *outgoing subspace* and $\mathcal{Z}_i$ an *incoming subspace* of $\widetilde{\Sigma}$.[7]

Proof. We begin by proving necessity of 1) and 2), assuming that $\widetilde{\Sigma}$ is a dilation of Σ, and begin with condition 1). Let $\left[\begin{smallmatrix} \tilde{z}_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in \widetilde{V}$ with $x_0 \in \mathcal{X}$. By Proposition 2.2, $\widetilde{\Sigma}$ has a trajectory $(\tilde{x}(\cdot), \tilde{w}(\cdot))$ on $\mathbb{Z}^+$ with $\tilde{x}(1) = \tilde{z}_0$, $\tilde{x}(0) = x_0$, and $w(0) = w_0$. By condition 2) in Definition 8.1, $(x(\cdot), \tilde{w}(\cdot))$ with $x(\cdot) = P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(\cdot)$ is a trajectory of Σ. In particular, $\left[\begin{smallmatrix} x(1) \\ x(0) \\ w_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} P_{\mathcal{X}}^{\mathcal{Z}} \tilde{z}_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$. This shows that the right-hand side of (8.3) is contained in V. The opposite inclusion follows from a similar argument which replaces condition 2) in Definition 8.1 by part 1) of Lemma 8.2.

To prove the existence of a decomposition of the type described in part 2) we define $\mathcal{Z}_o = \mathcal{Z}_o^{\max}$ by (8.4). It is easy to see that $\mathcal{Z}_o^{\max}$ is a closed subspace of $\mathcal{X}$, and it is contained in $\mathcal{Z}$ since $P_{\mathcal{X}}^{\mathcal{Z}} \mathcal{Z}_o^{\max} = 0$. Let $\mathcal{Z}_i$ be an arbitrary direct complement of $\mathcal{Z}_o^{\max}$ in $\mathcal{Z}$. We claim that this decomposition of $\mathcal{Z}$ has the two properties mentioned in 2).

It is easy to see from Definition 8.4 that $\mathcal{Z}_o^{\max}$ is outgoing invariant for $\widetilde{\Sigma}$, so it remains to show that $\mathcal{Z}_o^{\max} \dotplus \mathcal{X}$ is strongly invariant for $\widetilde{\Sigma}$. Let $(\tilde{x}(\cdot), w(\cdot))$ be a trajectory of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with $\tilde{x}(0) = z_0 + x_0$, where $z_0 \in \mathcal{Z}_o^{\max}$ and $x_0 \in \mathcal{X}$. Since $\mathcal{Z}_o^{\max}$ is outgoing invariant, there is a trajectory $(\tilde{x}_1(\cdot), 0)$ of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with $\tilde{x}_1(0) = z_0$ satisfying $\tilde{x}_1(n) \in \mathcal{Z}_o^{\max}$ for all $n \in \mathbb{Z}^+$. Define $\tilde{x}_2(\cdot) = \tilde{x}(\cdot) - \tilde{x}_1(\cdot)$. Then $(\tilde{x}_2(\cdot), w(\cdot))$ is a trajectory of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with $\tilde{x}_2(0) = x_0 \in \mathcal{X}$. Define $x(\cdot) = P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}_2(\cdot)$. By Condition 2) in Definition 8.1, $(x(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$. In particular, it is also a trajectory on $[1, \infty)$. By assertion 2) of Lemma 8.2, applied to the time interval $[1, \infty)$, there is a trajectory $(\tilde{x}_3(\cdot), w(\cdot))$ of $\widetilde{\Sigma}$ on $[1, \infty)$ satisfying $\tilde{x}_3(1) = x(1)$ and $P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}_3(n) = x(n)$ for all $n \in [1, \infty)$. Define $\tilde{x}_4(\cdot) = \tilde{x}_2(\cdot) - \tilde{x}_3(\cdot)$. Then $(\tilde{x}_4(\cdot), 0)$ is a trajectory of $\widetilde{\Sigma}$ on $[1, \infty)$, and it satisfies $P_{\mathcal{X}}^{\mathcal{Y}} \tilde{x}_4(n) = P_{\mathcal{X}}^{\mathcal{Y}} \tilde{x}_2(n) - P_{\mathcal{X}}^{\mathcal{Y}} \tilde{x}_3(n) = x(n) - x(n) = 0$ for all $n \in [1, \infty)$. It follows from (8.4) (after we have shifted the trajectory $(\tilde{x}(\cdot), 0)$ one step to the left) that $\tilde{x}_4(0) \in \mathcal{Z}_o^{\max}$. Thus, $\tilde{x}(1) = \tilde{x}_1(1) + \tilde{x}_3(1) + \tilde{x}_4(1)$ where $\tilde{x}_1(1) \in \mathcal{Z}_o^{\max}$, $\tilde{x}_3(1) = x(1) \in \mathcal{X}$, and $\tilde{x}_4(1) \in \mathcal{Z}_o^{\max}$, so $\tilde{x}(1) \in \mathcal{Z}_o^{\max} \dotplus \mathcal{X}$. This proves that the implication (8.2) holds with $\mathcal{Z}$ replaced by $\mathcal{Z}_o^{\max} \dotplus \mathcal{X}$. By Lemma 8.5, $\mathcal{Z}_o^{\max} \dotplus \mathcal{X}$ is strongly invariant.

To prove the maximality of $\mathcal{Z}_o^{\max}$ it suffices to observe that if $\mathcal{Z}_o$ is outgoing invariant, then for each $z_0 \in \mathcal{Z}_o$ there is a trajectory $(\tilde{x}(\cdot), 0)$ of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with

[7] The reason for these names will be explained in Part II of this paper.

$\tilde{x}(0) = z_0$ satisfying $\tilde{x}(n) \in \mathcal{Z}_o \subset \mathcal{Z}$ for all $n \in \mathbb{Z}^+$, and hence $P_{\mathcal{X}}^{\mathcal{Z}}\tilde{x}(n) = 0$ for all $n \in \mathbb{Z}^+$. This implies that $z_0 \in \mathcal{Z}_o^{\max}$.

For the converse proof we assume that 1) and 2) hold. It follows from (8.3) that the two systems $\widetilde{\Sigma}$ and Σ have the same canonical input space $\mathcal{U}_0$, so condition 3) of Definition 8.1 is satisfied.

Our proof of the fact that also condition 2) of Definition 8.1 holds is based on the following implication:

$$\text{If } \begin{bmatrix} \tilde{z} \\ \tilde{x} \\ w \end{bmatrix} \in \widetilde{V} \text{ and } \tilde{x} \in \mathcal{Z}_o \dotplus \mathcal{X}, \text{ then } \begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}}\tilde{z} \\ P_{\mathcal{X}}^{\mathcal{Z}}\tilde{x} \\ w \end{bmatrix} \in V. \tag{8.5}$$

The proof of (8.5) goes as follows. Let $\begin{bmatrix} \tilde{z} \\ \tilde{x} \\ w \end{bmatrix} \in \widetilde{V}$ with $\tilde{x} = z_0 + x_0$, where $z_0 \in \mathcal{Z}_o$ and $x_0 \in \mathcal{X}$. Since $\mathcal{Z}_o$ is outgoing invariant, there is some $z_1 \in \mathcal{Z}_o$ such that $\begin{bmatrix} z_1 \\ z_0 \\ 0 \end{bmatrix} \in \widetilde{V}$ (see Lemma 8.5). Since $\widetilde{V}$ is a subspace also $\begin{bmatrix} \tilde{z}-z_1 \\ x_0 \\ w \end{bmatrix} \in \widetilde{V}$. We can now apply (8.3) to conclude that $\begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}}(\tilde{z}-z_1) \\ x_0 \\ w \end{bmatrix} \in V$. But $P_{\mathcal{X}}^{\mathcal{Z}}(\tilde{z} - z_1) = P_{\mathcal{X}}^{\mathcal{Z}}\tilde{z}$ since $z_1 \in \mathcal{Z}_o \subset \mathcal{Z}$ and $x_0 = P_{\mathcal{X}}^{\mathcal{Z}}\tilde{x}$ since $\tilde{x} - x_0 = z_0 \in \mathcal{Z}_o \subset \mathcal{Z}$. Thus, we conclude that $\begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}}\tilde{z} \\ P_{\mathcal{X}}^{\mathcal{Z}}\tilde{x} \\ w \end{bmatrix} \in V$. This proves (8.5).

Let $(\tilde{x}(\cdot), w(\cdot))$ be a trajectory of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with $\tilde{x}(0) \in \mathcal{X}$. Because of the strong invariance of $\mathcal{Z}_o \dotplus \mathcal{X}$, this implies that $\tilde{x}(n) \in \mathcal{Z}_o \dotplus \mathcal{X}$ for all $n \in \mathbb{Z}^+$. Define $x(\cdot) = P_{\mathcal{X}}^{\mathcal{Z}}\tilde{x}(\cdot)$. Then it follows from (8.5) that $(x(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$. Thus, condition 2) in Definition 8.1 holds, and we conclude that $\widetilde{\Sigma}$ is a dilation of Σ. □

Let us record the following fact which we observed in the preceding proof.

Corollary 8.8. *Let the state/signal system $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ be a dilation along $\mathcal{Z}$ of $\Sigma = (V; \mathcal{X}, \mathcal{W})$, and let $\widetilde{\mathcal{X}} = \mathcal{Z}_o \dotplus \mathcal{X} \dotplus \mathcal{Z}_i$ be the decomposition of $\widetilde{\mathcal{X}}$ given in Theorem 8.7. Denote $\mathcal{Z}_o \dotplus \mathcal{X}$ by $\mathcal{X}_o$. Then V is given by*

$$V = \left\{ \begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}}\tilde{z} \\ P_{\mathcal{X}}^{\mathcal{Z}}\tilde{x} \\ w \end{bmatrix} \,\middle|\, \tilde{x} \in \mathcal{X}_o \text{ and } \begin{bmatrix} \tilde{z} \\ \tilde{x} \\ w \end{bmatrix} \in \widetilde{V} \right\}. \tag{8.6}$$

This follows from (8.3) and (8.5).

Corollary 8.9. *Let $\widetilde{\Sigma} = \big(\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W}\big)$ be a state/signal system. Assume that $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$, and define V by* (8.3). *Then $\Sigma = \big(V; \mathcal{X}, \mathcal{W}\big)$ is a state/signal node. It is a compression along $\mathcal{Z}$ onto $\mathcal{X}$ of $\widetilde{\Sigma}$ if and only if $\mathcal{Z}$ can be decomposed into $\mathcal{Z} = \mathcal{Z}_o \dotplus \mathcal{Z}_i$ in such a way that $\mathcal{Z}_o$ is outgoing invariant for $\widetilde{\Sigma}$ and $\mathcal{Z}_o \dotplus \mathcal{X}$ is strongly invariant for $\widetilde{\Sigma}$.*

Proof. If V is given by (8.3), then V clearly has properties (i) and (iii) in Definition 2.1. That it also has properties (i) and (iv) follows from Lemma 2.4, because if we denote the operator in part 3) of Lemma 2.3 corresponding to $\widetilde{V}$ and V by $\widetilde{F}$ and F, respectively, then $F = P_{\mathcal{X}}^{\mathcal{Z}}\widetilde{F}$ with $\mathcal{D}(F) = \mathcal{D}(\widetilde{F})$. Thus Σ is a state/signal node. The remaining claims follow from Theorem 8.7. □

Remark 8.10. It is possible to reformulate condition 2) in Theorem 8.7 by focusing on the subspace $\mathcal{X}_o := \mathcal{Z}_o \dotplus \mathcal{X}$ instead of focusing on $\mathcal{Z}_o$. We claim that condition 2) in Theorem 8.7 is equivalent to the following condition:

2′) $\widetilde{\mathcal{X}}$ has a decomposition $\widetilde{\mathcal{X}} = \mathcal{X}_o \dotplus \mathcal{Z}_i$, where $\mathcal{Z}_i \subset \mathcal{Z}$, $\mathcal{X} \subset \mathcal{X}_o$, $\mathcal{X}_o$ is strongly invariant for $\widetilde{\Sigma}$, and $\mathcal{X}_o \cap \mathcal{Z}$ is outgoing invariant for $\widetilde{\Sigma}$.

Clearly, 2′) follows from from 2) if we take $\mathcal{X}_o = \mathcal{Z}_o \dotplus \mathcal{X}$. It is almost as easy to derive 2) from 2′), with $\mathcal{Z}_o = \mathcal{X}_o \cap \mathcal{Z}$; the only slightly nontrivial part is to show that $\widetilde{\mathcal{X}} = \mathcal{Z}_o \dotplus \mathcal{X} \dotplus \mathcal{Z}_i$, or equivalently, that $\mathcal{X}_o = (\mathcal{X}_o \cap \mathcal{Z}) \dotplus \mathcal{X}$. However, this follows from the assumptions that $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z} = \mathcal{X}_o \dotplus \mathcal{Z}_i$ where $\mathcal{Z}_i \subset \mathcal{Z}$ and $\mathcal{X} \subset \mathcal{X}_o$, which implies that $P_{\mathcal{Z}}^{\mathcal{X}} - P_{\mathcal{Z}_i}^{\mathcal{X}_o}$ is a projection with kernel $\mathcal{X} \dotplus \mathcal{Z}_i$ and range $\mathcal{X}_o \cap \mathcal{Z}$ (we leave the proof of this to the reader). The same replacement of 2) by 2′) can be carried out in Corollary 8.9, too. The final conclusion of Theorem 8.7 says that if $\mathcal{Z}_o$ is an arbitrary subspace of $\mathcal{Z}$ satisfying the properties listed in 2), then $\mathcal{Z}_o \subset \mathcal{Z}_o^{\max}$. This result implies that the subspace $\mathcal{X}_o^{\max} := \mathcal{Z}_o^{\max} \dotplus \mathcal{X}$ has an analogous maximality property: if $\mathcal{X}_o$ is an arbitrary subspace of $\widetilde{\mathcal{X}}$ satisfying the properties listed in 2′), then $\mathcal{X}_o \subset \mathcal{X}_o^{\max}$. A similar argument shows that all the subspaces $\mathcal{Z}_o$ in 2) and all the subspaces $\mathcal{X}_o$ in 2′) satisfy $\mathcal{Z}_o^{\min} \subset \mathcal{Z}_o$ and $\mathcal{X}_o^{\min} \subset \mathcal{X}_o$, where $\mathcal{Z}_o^{\min}$ and $\mathcal{X}_o^{\min}$ are defined in Theorem 8.11 below.

Theorem 8.11. *Among all the decompositions $\widetilde{\mathcal{X}} = \mathcal{Z}_o \dotplus \mathcal{X} \dotplus \mathcal{Z}_i$ in Theorem* 8.7 *there is one for which the outgoing subspace $\mathcal{Z}_o$ is the smallest possible, i.e., there is an outgoing invariant subspace $\mathcal{Z}_o = \mathcal{Z}_o^{\min}$ which can be used in this decomposition, and which is contained in every outgoing subspace $\mathcal{Z}_o$ for every other choice of decomposition. The subspace $\mathcal{Z}_o^{\min}$ can be constructed as follows: Let $\mathcal{X}_o^{\min}$ be the closure in $\widetilde{\mathcal{X}}$ of all the possible values of the state components $\widetilde{x}(\cdot)$ of all trajectories $(\tilde{x}(\cdot), w(\cdot))$ of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ satisfying $\tilde{x}(0) \in \mathcal{X}$, and define $\mathcal{Z}_o^{min} = \mathcal{X}_o^{\min} \cap \mathcal{Z}$.*

Proof. Define $\mathcal{X}_o^{\min}$ and $\mathcal{Z}_o^{\min}$ as described in Theorem 8.11, and let $\mathcal{Z}_i$ be an arbitrary direct complement to $\mathcal{Z}_o^{\min}$ in $\mathcal{Z}$. Then $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z} = \mathcal{X} \dotplus \mathcal{Z}_o^{\min} \dotplus \mathcal{Z}_i$. We have both $\mathcal{X} \subset \mathcal{X}_o^{\min}$ and $\mathcal{Z}_o^{\min} \subset \mathcal{X}_o^{\min}$, so $\mathcal{Z}_o^{\min} \dotplus \mathcal{X} \subset \mathcal{X}_o^{\min}$. To see that we actually have $\mathcal{Z}_o^{\min} \dotplus \mathcal{X} = \mathcal{X}_o^{\min}$ it suffices to show that $\mathcal{X}_o^{\min} \cap \mathcal{Z}_i = \{0\}$ (since $\mathcal{X}_o^{\min} \subset \widetilde{\mathcal{X}} = (\mathcal{Z}_o^{\min} \dotplus \mathcal{X}) \dotplus \mathcal{Z}_i$). But this is true because

$$\mathcal{X}_o^{\min} \cap \mathcal{Z}_i = (\mathcal{X}_o^{\min} \cap \mathcal{Z}) \cap \mathcal{Z}_i = \mathcal{Z}_o^{\min} \cap \mathcal{Z}_i = \{0\}.$$

Thus $\mathcal{X}_o^{\min} = \mathcal{Z}_o^{\min} \dotplus \mathcal{X}$ and $\widetilde{\mathcal{X}} = \mathcal{X}_o^{\min} \dotplus \mathcal{Z}_i = \mathcal{Z}_o^{\min} \dotplus \mathcal{X} \dotplus \mathcal{Z}_i$.

It is easy to see that $\mathcal{X}_o^{\min}$ is the smallest (closed) strongly invariant subspace of $\widetilde{\mathcal{X}}$ which contains $\mathcal{X}$. In particular, for each decomposition $\widetilde{\mathcal{X}} = \mathcal{X}_o \dotplus \mathcal{Z}_i$ satisfying condition 2′) in Remark 8.10 we must have $\mathcal{X}_o^{\min} \subset \mathcal{X}_o$. As we saw in Remark 8.10, this implies that if $\mathcal{Z}_o$ is an arbitrary subspace which satisfies the conditions listed in 2) of Theorem 8.7, then $\mathcal{Z}_o^{\min} \subset \mathcal{Z}_o$. This proves the claim about the minimality of $\mathcal{Z}_o^{\min}$ (and of $\mathcal{X}_o^{\min}$). It only remains to show that $\mathcal{Z}_o^{\min}$ is outgoing invariant for $\widetilde{\Sigma}$. To do this we argue as follows.

Choose some arbitrary decomposition $\widetilde{\mathcal{X}} = \mathcal{Z}_o \dotplus \mathcal{X} \dotplus \mathcal{Z}_i$ of the type given in Theorem 8.7, and define $\mathcal{X}_o := \mathcal{Z}_o \dotplus \mathcal{X}$. Since $\mathcal{X}_o^{\min}$ is the smallest (closed) strongly invariant subspace of $\widetilde{\mathcal{X}}$ which contains $\mathcal{X}$ we must have $\mathcal{X} \subset \mathcal{X}_o^{\min} \subset \mathcal{X}_o$. It follows from (8.3) and (8.6) that (8.6) also holds if we replace $\mathcal{X}_o$ by $\mathcal{X}_o^{\min}$. Take some arbitrary $z_0 \in \mathcal{Z}_0^{\min} = \mathcal{X}_o^{\min} \cap \mathcal{Z} \subset \mathcal{X}_o \cap \mathcal{Z} = \mathcal{Z}_o$. Then there exists some trajectory $(\tilde{x}(\cdot), w(\cdot))$ of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with $\tilde{x}(0) = z_0$. By (8.6) with $\mathcal{X}_o$ replaced by $\mathcal{X}_o^{\min}$, if we define $x(\cdot) = P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(\cdot)$, then $(x(\cdot), w(\cdot))$ is a trajectory of Σ. Observe that $x(0) = 0$. By part 1) Lemma 8.2, there exists a (unique) trajectory $(\tilde{x}_1(\cdot), w(\cdot))$ of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with $P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}_1(\cdot) = x(\cdot)$ (in particular, $\tilde{x}_1(0) = 0$). Define $\tilde{x}_2(\cdot) = \tilde{x}(\cdot) - \tilde{x}_1(\cdot)$. Then $(\tilde{x}_2(\cdot), 0)$ is a trajectory of $\widetilde{\Sigma}$ with $\tilde{x}_2(0) = z_0$ and $P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}_2(\cdot) = x(\cdot) - x(\cdot) = 0$. Thus $\tilde{x}_2(n) \subset \mathcal{Z}$ for all $n \in \mathbb{Z}^+$. But on the other hand, by the strong invariance of $\mathcal{X}_o^{\min}$, $\tilde{x}_2(n) \subset \mathcal{X}_o^{\min}$ for all $n \in \mathbb{Z}^+$. Thus, $\tilde{x}_2(n) \subset \mathcal{Z} \cap \mathcal{X}_o^{\min} = \mathcal{Z}_o^{\min}$ for all $n \in \mathbb{Z}^+$ and, as we recall, $\tilde{x}_2(0) = z_0$. This proves that $\mathcal{Z}_o^{\min}$ is outgoing invariant. □

It is often useful to split a compression or dilation into the product of two successive dilations or compression.

Lemma 8.12. *Let $\widehat{\Sigma} = (\widehat{V}; \widehat{\mathcal{X}}, \mathcal{W})$ be a compression of $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ along $\widehat{\mathcal{Z}}$ onto $\widehat{\mathcal{X}}$, and let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a compression of $\widehat{\Sigma}$ along $\mathcal{Z}$ onto $\mathcal{X}$. Then $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a compression of $\widetilde{\Sigma}$ along $\widehat{\mathcal{Z}} \dotplus \mathcal{Z}$ onto $\mathcal{X}$, and $P_{\mathcal{X}}^{\widehat{\mathcal{Z}} \dotplus \mathcal{Z}} = P_{\mathcal{X}}^{\mathcal{Z}} P_{\widehat{\mathcal{X}}}^{\widehat{\mathcal{Z}}}$.*

The easy proof is left to the reader.

Two particularly simple types of dilations are those where one of the two subspaces $\mathcal{Z}_o$ and $\mathcal{Z}_i$ in Theorem 8.7 can be taken to be zero.

Definition 8.13. The state/signal system $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ is an *outgoing dilation along $\mathcal{Z}$* of the state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, or equivalently, the state/signal system Σ is an *outgoing compression along $\mathcal{Z}$ onto $\mathcal{X}$* of the state/signal system $\widetilde{\Sigma}$, if the following conditions hold:

1) $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$,
2) If $(\tilde{x}(\cdot), w(\cdot))$ is a trajectory of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$, then $(P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$.
3) There is at least one decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of $\mathcal{W}$ which is admissible for both $\widetilde{\Sigma}$ and Σ.

Clearly, every outgoing dilation is also a dilation.

Lemma 8.14. *Let $\widetilde{\Sigma} = \big(\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W}\big)$ and $\Sigma = \big(V; \mathcal{X}, \mathcal{W}\big)$ be two state/signal systems with $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$ (and with the same signal space). Then the following conditions are equivalent.*

1) *$\widetilde{\Sigma}$ is an outgoing dilation along $\mathcal{Z}$ of Σ,*
2) *V is given by*

$$V = \begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}} & 0 & 0 \\ 0 & P_{\mathcal{X}}^{\mathcal{Z}} & 0 \\ 0 & 0 & 1_{\mathcal{X}} \end{bmatrix} \widetilde{V} = \left\{ \begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}} \tilde{z} \\ P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x} \\ w \end{bmatrix} \;\middle|\; \begin{bmatrix} \tilde{z} \\ \tilde{x} \\ w \end{bmatrix} \in \widetilde{V} \right\}. \tag{8.7}$$

3) (8.3) *holds and $\mathcal{Z}$ is outgoing invariant for $\widetilde{\Sigma}$.*

Proof. The proof of the fact that 1) implies 2) is essentially the same as the proof of the necessity of (8.3) in Theorem 8.7, and the proof of the converse implication is a simplified version of the sufficiency part of the proof of the same theorem. That 1) and 2) together imply 3) is a simplified version of the final paragraph of the proof of Theorem 8.11 (replace $\mathcal{Z}_o^{\min}$ by $\mathcal{Z}$, replace $\mathcal{X}_o^{\min}$ by $\widetilde{\mathcal{X}}$, and use the facts that $\widetilde{\Sigma}$ is a dilation of Σ and that (8.6) now holds with $\mathcal{X}_o$ replaced by $\widetilde{\mathcal{X}}$). Finally, that 3) implies 2) follows from Corollary 8.8. □

Definition 8.15. The state/signal system $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ is an *incoming dilation along* $\mathcal{Z}$ of the state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, or equivalently, the state/signal system Σ is an *incoming compression along* $\mathcal{Z}$ *onto* $\mathcal{X}$ of the state/signal system $\widetilde{\Sigma}$, if the following conditions hold:

1) $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$,
2) If $(\tilde{x}(\cdot), w(\cdot))$ is a trajectory of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with $\tilde{x}(0) \in \mathcal{X}$, then $\tilde{x}(n) \in \mathcal{X}$ for all $n \in \mathbb{Z}^+$ and $(x(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$.
3) There is at least one decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of $\mathcal{W}$ which is admissible for both $\widetilde{\Sigma}$ and Σ.

Lemma 8.16. *Let* $\widetilde{\Sigma} = \big(\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W}\big)$ *and* $\Sigma = \big(V; \mathcal{X}, \mathcal{W}\big)$ *be two state/signal systems with* $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$ (*and with the same signal space*). *Then the following conditions are equivalent.*

1) $\widetilde{\Sigma}$ *is an incoming dilation along* $\mathcal{Z}$ *of* Σ,
2) V *is given by*
$$V = \widetilde{V} \cap \begin{bmatrix} \widetilde{\mathcal{X}} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} = \left\{ \begin{bmatrix} \tilde{z} \\ x \\ w \end{bmatrix} \in \widetilde{V} \;\middle|\; x \in \mathcal{X} \text{ and } \begin{bmatrix} \tilde{z} \\ x \\ w \end{bmatrix} \in \widetilde{V} \right\}. \tag{8.8}$$
3) (8.3) *holds and* $\mathcal{X}$ *is strongly invariant for* $\widetilde{\Sigma}$.

This proof is similar to the proof of Lemma 8.14 and it is left to the reader.

Definition 8.17. A state/signal system is *minimal* if it is not a (nontrivial) dilation of any other state/signal system (along any direction).

Theorem 8.18. *An state/signal system is minimal if and only if it is controllable and observable.*

Proof. Let $\widetilde{\Sigma}$ be state/signal system, and let Σ be a compression of $\widetilde{\Sigma}$. If $\widetilde{\Sigma}$ is observable, then the outgoing subspace $\mathcal{Z}_o$ in the decomposition in Theorem 8.7 is trivial (since it is part of the unobservable subspace), and if $\widetilde{\Sigma}$ is controllable, then the incoming subspace $\mathcal{Z}_i$ in the decomposition in Theorem 8.7 is trivial (since $\mathcal{Z}_o \dotplus \mathcal{X}$ contains the reachable subspace). Thus, if $\widetilde{\Sigma}$ is both controllable and observable, then it does not have any nontrivial dilation.

The converse claim follows from Theorem 8.19 below (which shows that every non-observable or non-controllable system has a nontrivial compression). □

Theorem 8.19. $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a state/signal system. Denote the reachable subspace of* Σ *by* $\mathfrak{R}$ *and the unobservable subspace of* Σ *by* $\mathfrak{U}$.

1) *Let $\mathfrak{O}$ be a direct complement to $\mathfrak{U}$ in $\mathcal{X}$, define $\mathcal{X}_\circ := \overline{P_{\mathfrak{O}}^{\mathfrak{U}}\mathfrak{R}}$, and let $\mathfrak{O}_i$ be a direct complement to $\mathcal{X}_\circ$ in $\mathfrak{O}$. Define $V_\circ$ by*

$$V_\circ := \left\{ \begin{bmatrix} P_{\mathcal{X}_\circ}^{\mathfrak{U}\dotplus\mathfrak{O}_i} z \\ x \\ w \end{bmatrix} \,\middle|\, x \in \mathcal{X}_\circ,\ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}. \tag{8.9}$$

Then $\Sigma_\circ = (V_\circ; \mathcal{X}_\circ, \mathcal{W})$ is a minimal state/signal systems which is a compression of Σ along $\mathfrak{U} \dotplus \mathfrak{O}_i$. Here $\mathfrak{U}$ is outgoing invariant for Σ and $\mathfrak{U} \dotplus \mathcal{X}_\circ$ is strongly invariant for Σ, so that we can take $\mathcal{Z}_o = \mathfrak{U}$ and $\mathcal{Z}_i = \mathfrak{O}_i$ in the decomposition in Theorem 8.7.

2) *Let $\mathfrak{Q}$ be a direct complement to $\mathfrak{R}$ in $\mathcal{X}$, define $\mathfrak{R}_o = \mathfrak{R} \cap \mathfrak{U}$, and let $\mathcal{X}_\bullet$ be a direct complement to $\mathfrak{R}_o$ in $\mathfrak{R}$. Define*

$$V_\bullet := \left\{ \begin{bmatrix} P_{\mathcal{X}_\bullet}^{\mathfrak{R}_o\dotplus\mathfrak{Q}} z \\ x \\ w \end{bmatrix} \,\middle|\, x \in \mathcal{X}_\bullet,\ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}. \tag{8.10}$$

Then $\Sigma_\bullet = (V_\bullet; \mathcal{X}_\bullet, \mathcal{W})$ is a minimal state/signal systems which is a compression of Σ along $\mathfrak{R}_o \dotplus \mathfrak{Q}$ onto $\mathcal{X}_\bullet$. Here $\mathfrak{R}_o$ is outgoing invariant for Σ and $\mathfrak{R}_o \dotplus \mathcal{X}_\bullet$ is strongly invariant for Σ, so that we can take $\mathcal{Z}_o = \mathfrak{R}_o$ and $\mathcal{Z}_i = \mathfrak{Q}$ in the decomposition in Theorem 8.7.

Proof. Proof of 1). We begin by performing an outgoing compression of Σ along $\mathfrak{U}$ onto $\mathfrak{O}$, i.e., we define

$$V_\circ^1 := \left\{ \begin{bmatrix} P_{\mathfrak{O}}^{\mathfrak{U}} z \\ x \\ w \end{bmatrix} \,\middle|\, x \in \mathfrak{O},\ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}.$$

According to Lemma 8.6, $\mathfrak{U}$ is outgoing invariant for Σ, so by Corollary 8.9, $\Sigma_\circ^1 := (V_\circ^1; \mathfrak{O}, \mathcal{W})$ is a compression of Σ along $\mathfrak{U}$. Moreover, it follows from Lemma 8.3 that $\Sigma_\circ^1$ is observable.

We continue by performing an incoming compression of $\Sigma_\circ^1$ along $\mathfrak{O}_i$ onto its reachable subspace, which according to Lemma 8.3 is equal to $\mathcal{X}_\circ$. Thus, we define

$$V_\circ := \left\{ \begin{bmatrix} P_{\mathcal{X}_\circ}^{\mathfrak{O}_i} z \\ x \\ w \end{bmatrix} \,\middle|\, x \in \mathcal{X}_\circ,\ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V_\circ^1 \right\}.$$

The subspace $\mathcal{X}_\circ$ is strongly invariant for $\Sigma_\circ^1$ (see Lemma 8.6), so by Corollary 8.9, $\Sigma_\circ := (V_\circ; \mathcal{X}_\circ, \mathcal{W})$ is a compression of $\Sigma_\circ^1$ along $\mathfrak{O}_i$. By Lemma 8.12, this system is the same one which we defined in Part 1), and by Lemma 8.3, $\Sigma_\circ$ is both controllable and observable.

It remains to show that $\mathfrak{U} \dotplus \mathcal{X}_\circ$ is strongly invariant for Σ. However, this follows from the fact that the maximal outgoing subspace $\mathcal{Z}_o^{\max}$ defined in (8.4) always is contained in the unobservable subspace $\mathfrak{U}$, and in this particular case it coincides with $\mathfrak{U}$. Thus, $\mathfrak{U} \dotplus \mathcal{X}_\circ$ coincides with the space $\mathcal{Z}_o^{\max} \dotplus \mathcal{X}_\circ$, and it must therefore be strongly invariant.

Proof of 2). We begin by performing an incoming compression of Σ along $\mathfrak{Q}$ onto $\mathfrak{R}$, i.e., we define

$$V_\bullet^1 := \left\{ \begin{bmatrix} P_{\mathfrak{R}}^{\mathfrak{Q}} z \\ x \\ w \end{bmatrix} \,\middle|\, x \in \mathfrak{R},\ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}.$$

According to Lemma 8.6, $\mathfrak{R}$ is strongly invariant for Σ, so by Corollary 8.9, $\Sigma^1_\bullet := (V^1_\bullet; \mathfrak{R}, \mathcal{W})$ is a compression of Σ along $\mathfrak{Q}$. Moreover, it follows from Lemma 8.3 that $\Sigma^1_\bullet$ is controllable.

We continue by performing an outgoing compression of $\Sigma^1_\bullet$ along its unobservable subspace, which according to Lemma 8.3 is equal to $\mathfrak{R}_o$. That is, we define

$$V_\bullet := \left\{ \begin{bmatrix} P^{\mathfrak{R}_o}_{\mathcal{X}_\bullet} z \\ x \\ w \end{bmatrix} \,\Big|\, x \in \mathcal{X}_\bullet,\ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V^1_\bullet \right\}.$$

The subspace $\mathfrak{R}_o$ is outgoing invariant for $\Sigma^1_\bullet$ (see Lemma 8.6), so by Corollary 8.9, $\Sigma_\bullet := (V_\bullet; \mathcal{X}_\bullet, \mathcal{W})$ is a compression of $\Sigma^1_\bullet$ along $\mathfrak{R}_o$. By Lemma 8.12, this system is the same one which we defined in Part 1), and by Lemma 8.3, $\Sigma_\bullet$ is both controllable and observable. We already observed above that $\mathfrak{R}_o$ is outgoing invariant and that $\mathfrak{R}_o \dotplus \mathcal{X}_\bullet = \mathfrak{R}$ is strongly invariant for Σ. □

Theorem 8.20. *Every realizable signal behavior has a minimal state/signal realization (i.e., the behavior has a state/signal realization which is minimal).*

This follows from Theorem 8.19 (since a compressed system is externally equivalent to the original system).

Up to now we have not used any specific representation of a state/signal system in our study of dilations and compressions. For completeness we interpret some of our results in terms of driving variable, output nulling, and input/state/output representations. We begin with the following description of the crucial formula (8.3) in Theorem 8.7.

Lemma 8.21. *Let $\widetilde{\Sigma} = \big(\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W}\big)$ and $\Sigma = \big(V; \mathcal{X}, \mathcal{W}\big)$ be two state/signal systems with $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$ (and with the same signal space).*

1) *The following conditions are equivalent:*
   (a) *V is given by* (8.3).
   (b) *If $\left(\begin{bmatrix} \widetilde{A}' & \widetilde{B}' \\ \widetilde{C}' & \widetilde{D}' \end{bmatrix}; \widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}, \mathcal{W}\right)$ is a driving variable representation of $\widetilde{\Sigma}$, then $\left(\begin{bmatrix} P^{\mathcal{Z}}_{\mathcal{X}} \widetilde{A}'|_{\mathcal{X}} & P^{\mathcal{Z}}_{\mathcal{X}} \widetilde{B}' \\ \widetilde{C}'|_{\mathcal{X}} & \widetilde{D}' \end{bmatrix}; \mathcal{X}, \widetilde{\mathcal{L}}, \mathcal{W}\right)$ is a driving variable representation of Σ.*
   (c) *If $\left(\begin{bmatrix} \widetilde{A}'' & \widetilde{B}'' \\ \widetilde{C}'' & \widetilde{D}'' \end{bmatrix}; \widetilde{\mathcal{X}}, \mathcal{W}, \widetilde{\mathcal{K}}\right)$ is an output nulling representation of $\widetilde{\Sigma}$, then $\left(\begin{bmatrix} P^{\mathcal{Z}}_{\mathcal{X}} \widetilde{A}''|_{\mathcal{X}} & P^{\mathcal{Z}}_{\mathcal{X}} \widetilde{B}'' \\ \widetilde{C}''|_{\mathcal{X}} & \widetilde{D}'' \end{bmatrix}; \mathcal{X}, \mathcal{W}, \widetilde{\mathcal{K}}\right)$ is an output nulling representation of Σ.*
   (d) *If $\left(\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}\right)$ is an input/state/output representation of $\widetilde{\Sigma}$ corresponding to some admissible input/output decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$, then $\left(\begin{bmatrix} P^{\mathcal{Z}}_{\mathcal{X}} \widetilde{A}|_{\mathcal{X}} & P^{\mathcal{Z}}_{\mathcal{X}} \widetilde{B} \\ \widetilde{C}|_{\mathcal{X}} & \widetilde{D} \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ is an input/state/output representation of Σ corresponding to the same admissible decomposition of $\mathcal{W}$.*
2) *Assume that the equivalent conditions* (a)–(d) *above hold. Then every driving variable representation of Σ is of the form described in* (b)*, every output nulling representation of Σ is of the form described in* (c)*, and every input/output representation of Σ is of the form described in* (d).

Proof. The equivalence of (a)–(d) follows from (3.3), (4.3), (5.2), and (8.3).

That *every* input/state/output representations of V must be of the type given in (d) follows from the uniqueness of such a representation (see Theorem 5.1). The proof of the claim that all possible output nulling representations of V are of the type (c) is similar to the proof of the claim that all possible driving variable representations of V are of the type (b), so let us only prove the latter claim.

Let $\left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ be an arbitrary driving variable representation of Σ, and let $\left(\left[\begin{smallmatrix} \widetilde{A}' & \widetilde{B}' \\ \widetilde{C}' & \widetilde{D}' \end{smallmatrix}\right]; \widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}, \mathcal{W}\right)$ be the driving variable representation of $\widetilde{\Sigma}$ mentioned in part (b). Then by Theorem 6.1, there exist operators $K' \in \mathcal{B}(\mathcal{X}; \widetilde{\mathcal{L}})$ and $M' \in \mathcal{B}(\mathcal{L}; \widetilde{\mathcal{L}})$, with M' boundedly invertible, such that

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{A}'|_{\mathcal{X}} + P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{B}' K_1' & P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{B}' M' \\ \widetilde{C}'|_{\mathcal{X}} + \widetilde{D}' K_1' & \widetilde{D}' M' \end{bmatrix}.$$

Define $\widetilde{K}' = K' P_{\mathcal{X}}^{\mathcal{Z}}$. Then

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}} (\widetilde{A}' + \widetilde{B}' \widetilde{K}')|_{\mathcal{X}} & P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{B}' M' \\ (\widetilde{C}' + \widetilde{D}' \widetilde{K}')|_{\mathcal{X}} & \widetilde{D}' M' \end{bmatrix}.$$

By Theorem 6.1, $\left(\left[\begin{smallmatrix} \widetilde{A}' + \widetilde{B}'\widetilde{K}' & \widetilde{B}' M' \\ \widetilde{C}' + \widetilde{D}'\widetilde{K}' & \widetilde{D}' M' \end{smallmatrix}\right]; \widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}, \mathcal{W}\right)$ is a driving variable representation of $\widetilde{\Sigma}$, and hence $\left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ is of the type (b). □

Definition 8.1 is very closely related to the following definition of a dilation of a input/state/output system.

Definition 8.22. We say that the input/state/output system

$$\widetilde{\Sigma}_{i/s/o} = \left(\left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{smallmatrix}\right]; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}\right)$$

is a *dilation along* $\mathcal{Z}$ of the input/state/output system

$$\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right),$$

or equivalently, that $\Sigma_{i/s/o}$ is a *compression along* $\mathcal{Z}$ *onto* $\mathcal{X}$ of $\widetilde{\Sigma}_{i/s/o}$, if $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$ and the following condition holds: For each $x_0 \in \mathcal{X}$ and each input sequence $u(\cdot) \in \mathcal{U}^{\mathbb{Z}^+}$ the corresponding trajectories $(\tilde{x}(\cdot), u(\cdot), \tilde{y}(\cdot))$ and $(x(\cdot), u(\cdot), y(\cdot))$ of $\widetilde{\Sigma}_{i/s/o}$, respectively, $\Sigma_{i/s/o}$, with initial state $\tilde{x}(0) = x(0) = x_0$, satisfy $x(\cdot) = P_{\mathcal{X}}^{\mathcal{Z}} \tilde{x}(\cdot)$ and $\tilde{y}(\cdot) = y(\cdot)$.

As usual, we shall call an input/state/output system *minimal* if it is not a (nontrivial) dilation of any other input/state/output system (along any direction).

Lemma 8.23. *Let $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ and $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be two state/signal systems with $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$ (and with the same signal space $\mathcal{W}$).*

1) *Suppose that $\widetilde{\Sigma}$ and Σ have a common admissible input/output decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. Denote the corresponding input/state/output representations by $\widetilde{\Sigma}_{i/s/o}$, respectively, $\Sigma_{i/s/o}$. If $\widetilde{\Sigma}_{i/s/o}$ is a dilation along $\mathcal{Z}$ of $\Sigma_{i/s/o}$, then $\widetilde{\Sigma}$ is a dilation along $\mathcal{Z}$ of Σ.*

2) *Conversely, if* $\widetilde{\Sigma}$ *is a dilation along* $\mathcal{Z}$ *of* Σ*, then the two systems have the same admissible input/output decompositions* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$*, and if we denote the corresponding input/state/output representations by* $\widetilde{\Sigma}_{i/s/o}$ *and* $\Sigma_{i/s/o}$*, respectively, then* $\widetilde{\Sigma}_{i/s/o}$ *is a dilation along* $\mathcal{Z}$ *of* $\Sigma_{i/s/o}$*.*

Proof. Proof of 1). Let $(\widetilde{x}(\cdot), w(\cdot))$ be a trajectory of $\widetilde{\Sigma}$ on $\mathbb{Z}^+$ with $\widetilde{x}(0) \in \mathcal{X}$. Then $(\widetilde{x}(\cdot), u(\cdot), y(\cdot))$ with $u(\cdot) = P^{\mathcal{Y}}_{\mathcal{U}} w(\cdot)$ and $y(\cdot) = P^{\mathcal{U}}_{\mathcal{Y}} w(\cdot)$ is a trajectory of $\widetilde{\Sigma}_{i/s/o}$. By Definition 8.22, $(x(\cdot), u(\cdot), y(\cdot))$ with $x(\cdot) = P^{\mathcal{Z}}_{\mathcal{X}} \widetilde{x}(\cdot)$ is a trajectory of $\Sigma_{i/s/o}$, and hence $(P^{\mathcal{Z}}_{\mathcal{X}} \widetilde{x}(\cdot), w(\cdot))$ is a trajectory of Σ. Thus, $\widetilde{\Sigma}$ is a dilation along $\mathcal{Z}$ of Σ.

Proof of 2). That the two systems have the same admissible input/output decompositions follows from Lemmas 5.7 and 8.2. Let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a decomposition which is admissible both for $\widetilde{\Sigma}$ and for Σ. Let $(\widetilde{x}(\cdot), u(\cdot), y(\cdot))$ be a trajectory of $\widetilde{\Sigma}_{i/s/o}$ on $\mathbb{Z}^+$ with $\widetilde{x}(0) = x_0 \in \mathcal{X}$. Then $(\widetilde{x}(\cdot), w(\cdot))$ with $w(\cdot) = y(\cdot) + u(\cdot)$ is a trajectory of $\widetilde{\Sigma}$, and by Definition 8.1, $(P^{\mathcal{Y}}_{\mathcal{X}} \widetilde{x}(\cdot), w(\cdot))$ is a trajectory of Σ. Hence $(P^{\mathcal{Y}}_{\mathcal{X}} \widetilde{x}(\cdot), u(\cdot), y(\cdot))$ is a trajectory of $\Sigma_{i/s/o}$. More precisely, it is the unique trajectory of Σ with the initial state x_0 and the input data $u(\cdot)$. Thus, $\widetilde{\Sigma}_{i/s/o}$ is a dilation along $\mathcal{Z}$ of $\Sigma_{i/s/o}$. □

Theorem 8.24. *Let* $\widetilde{\Sigma}_{i/s/o} = \left(\left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{smallmatrix}\right]; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}\right)$ *and* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be two input/state/output systems with* $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$ *(and with the same input and output spaces). Then* $\Sigma_{i/s/o}$ *is a compression along* $\mathcal{Z}$ *onto* $\mathcal{X}$ *of* $\widetilde{\Sigma}_{i/s/o}$ *if and only if* $\mathcal{Z}$ *can be decomposed into* $\mathcal{Z} = \mathcal{Z}_o \dotplus \mathcal{Z}_i$ *such that the decomposition of* $\left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{smallmatrix}\right]$ *with respect to the decomposition* $\mathcal{X} = \mathcal{Z}_o \dotplus \mathcal{X} \dotplus \mathcal{Z}_i$ *has the following form (where* $*$ *stands for an irrelevant block)*

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} = \left[\begin{array}{ccc|c} * & * & * & * \\ 0 & A & * & B \\ 0 & 0 & * & 0 \\ \hline 0 & C & * & D \end{array}\right]. \tag{8.11}$$

This is a non-orthogonal version of [Aro79, Proposition 4]. For completeness we include a short proof based on Theorem 8.7.

Proof of Theorem 8.24. Let $\widetilde{\Sigma}$ and Σ be the state/signal systems induced by $\widetilde{\Sigma}_{i/s/o}$ and $\Sigma_{i/s/o}$, respectively.

If $\left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{smallmatrix}\right]$ is of the form (8.11), then it is easy to see that $\mathcal{Z}_o$ is outgoing invariant and $\mathcal{Z}_o \dotplus \mathcal{X}$ is strongly invariant for $\widetilde{\Sigma}$. Moreover, it follows from Lemma 8.21 that (8.3) holds. Thus, by Theorem 8.7, $\widetilde{\Sigma}$ is a dilation along $\mathcal{Z}$ of Σ, and consequently, by Lemma 8.23, $\widetilde{\Sigma}_{i/s/o}$ is a dilation along $\mathcal{Z}$ of $\Sigma_{i/s/o}$.

Conversely, suppose that $\widetilde{\Sigma}_{i/s/o}$ is a dilation along $\mathcal{Z}$ of $\Sigma_{i/s/o}$. Then, by Lemma 8.23, $\widetilde{\Sigma}$ is a dilation along $\mathcal{Z}$ of Σ. Let $\widetilde{\mathcal{X}} = \mathcal{Z}_o \dotplus \mathcal{X} \dotplus \mathcal{Z}_i$ be the decomposition in Theorem 8.7. Then it is easy to see that the fact that $\mathcal{Z}_o$ is outgoing invariant and $\mathcal{Z}_o \dotplus \mathcal{X}$ is strongly invariant imposes the structure (8.11) on $\left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{smallmatrix}\right]$. That the entries in positions $(2,2)$, $(2,4)$, $(4,2)$, and $(4,4)$ are A, B, C, and D follows from (8.3) and Lemma 8.21. □

It is not difficult to see that the decomposition (8.11) of $\left[\begin{smallmatrix}\widetilde{A} & \widetilde{B}\\ \widetilde{C} & \widetilde{D}\end{smallmatrix}\right]$ with respect to the decomposition $\widetilde{\mathcal{X}} = \mathcal{Z}_o \dotplus \mathcal{X} \dotplus \mathcal{Z}_i$ is valid if and only if (we denote $\mathcal{Z}_o \dotplus \mathcal{Z}_i$ by $\mathcal{Z}$)

$$\begin{gathered}\mathcal{R}(\widetilde{B}) \subset \mathcal{Z}_o \dotplus \mathcal{X}, \quad \mathcal{Z}_o \subset \mathcal{N}(\widetilde{C}),\\ \mathcal{R}(\widetilde{A}|_{\mathcal{Z}_o}) \subset \mathcal{Z}_o, \quad \mathcal{R}(\widetilde{A}|_{\mathcal{Z}_o \dotplus \mathcal{X}}) \subset \mathcal{Z}_o \dotplus \mathcal{X},\\ A = P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{A}|_{\mathcal{X}},\ B = P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{B},\ C = \widetilde{C}|_{\mathcal{X}},\ D = \widetilde{D}.\end{gathered} \tag{8.12}$$

Thus, in particular, $\widetilde{A} \in \mathcal{B}(\widetilde{\mathcal{X}})$ is an dilation of $A \in \mathcal{B}(\mathcal{X})$, i.e.,

$$A^n = P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{A}|_{\mathcal{X}}^n, \quad n \in \mathbb{Z}^+. \tag{8.13}$$

Orthogonal dilations (i.e., dilations where $\mathcal{X}$ and $\mathcal{Z}$ are orthogonal) play an essential role in the Nagy–Foiaş theory of harmonic analysis for operators in Hilbert space (see [SF70]) which is intimately connected with the Lax–Phillips scattering theory (see [LP67] and [AA70]).

Theorem 8.25. *An input/state/output system is minimal if and only if it is controllable and observable. Moreover, an input/state/output system Σ which is not minimal can be compressed into a minimal system (i.e., there is a minimal input/state/output system which is an compression of Σ).*

This is a non-orthogonal version of [Aro79, Propositions 3 and 4, p. 151]. It is easy to deduce this theorem from Theorems 8.18 and 8.19 in the same way as we derived Theorem 8.24 from Theorem 8.7. We leave the details to the reader.

Theorem 8.26. *Let Σ be a state/signal system. Then the following conditions are equivalent:*

1) *Σ is minimal.*
2) *Σ is controllable and observable.*
3) *Σ has a minimal input/state/output representation.*
4) *Σ has a controllable driving variable representation and an observable output nulling representation.*
5) *Every input/state/output representation of Σ is minimal.*
6) *Every driving variable representation of Σ is controllable, and every output nulling representation of Σ is observable.*

Proof. This follows from Propositions 3.5 and 4.5, Corollary 5.5, and Theorems 8.18 and 8.25. □

Lemma 8.27. *Let $\widetilde{\Sigma}_{i/s/o} = \left(\left[\begin{smallmatrix}\widetilde{A} & \widetilde{B}\\ \widetilde{C} & \widetilde{D}\end{smallmatrix}\right]; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}\right)$ and $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be two input/state/output systems with $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$. Denote the four block transfer functions of $\widetilde{\Sigma}_{i/s/o}$ and $\Sigma_{i/s/o}$ by $\left[\begin{smallmatrix}\widetilde{\mathfrak{A}}(z) & \widetilde{\mathfrak{B}}(z)\\ \widetilde{\mathfrak{C}}(z) & \widetilde{\mathfrak{D}}(z)\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\mathfrak{A}(z) & \mathfrak{B}(z)\\ \mathfrak{C}(z) & \mathfrak{D}(z)\end{smallmatrix}\right]$, respectively. Then the following conditions are equivalent:*

1) $\widetilde{\Sigma}_{i/s/o}$ *is a dilation along* $\mathcal{Z}$ *of* $\Sigma_{i/s/o}$.
2) *For all* $n \in \mathbb{Z}^+$,

$$\begin{aligned} A^n &= P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{A}^n|_{\mathcal{X}}, & A^n B &= P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{A}^n \widetilde{B}, & & \\ CA^n &= \widetilde{C}\widetilde{A}^n|_{\mathcal{X}}, & CA^n B &= \widetilde{C}\widetilde{A}^n \widetilde{B}, & D &= \widetilde{D}. \end{aligned} \tag{8.14}$$

3) *For all* z *in some neighborhood at zero,*

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{\mathfrak{A}}(z)|_{\mathcal{X}} & P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{\mathfrak{B}}(z) \\ \widetilde{\mathfrak{C}}(z)|_{\mathcal{X}} & \widetilde{\mathfrak{D}}(z) \end{bmatrix}. \tag{8.15}$$

Proof. The equivalence of 1) and 2) follows from (6.7), and the equivalence of 2) and 3) follows from (6.6). □

Theorem 8.28. *Let* $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ *and* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be two state/signal systems with* $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$ *(and with the same signal space* $\mathcal{W}$*).*

1) $\widetilde{\Sigma}$ *is a dilation of* Σ *if and only if there exist driving variable representations* $\widetilde{\Sigma}_{dv/s/s}$ *and* $\Sigma_{dv/s/s}$ *of* $\widetilde{\Sigma}$ *and* Σ*, respectively, with the property that* $\widetilde{\Sigma}_{dv/s/s}$ *is a dilation along* $\mathcal{Z}$ *of* $\Sigma_{dv/s/s}$ *(in the input/state/output sense; in particular they have the same driving variable space).*
2) *If* $\widetilde{\Sigma}$ *is a dilation of* Σ*, then to every driving variable representation* $\Sigma_{dv/s/s}$ *of* Σ *there exists at least one driving variable representation* $\widetilde{\Sigma}_{dv/s/s}$ *of* $\widetilde{\Sigma}$ *such that* $\widetilde{\Sigma}_{dv/s/s}$ *is a dilation along* $\mathcal{Z}$ *of* $\Sigma_{dv/s/s}$ *(in the input/state/output sense).*

Proof. Assertion 1) follows from Remark 5.2 and Lemma 8.23.

To prove assertion 2) we take an arbitrary driving-variable representation $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ of Σ. Let $\widetilde{\Sigma}_{dv/s/s} = \left(\left[\begin{smallmatrix} \widetilde{A}' & \widetilde{B}' \\ \widetilde{C}' & \widetilde{D}' \end{smallmatrix}\right]; \widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}, \mathcal{W}\right)$ be the driving variable representation of $\widetilde{\Sigma}$ mentioned in part 1). Then by Theorem 6.1, there exist operators $K' \in \mathcal{B}(\mathcal{X}; \widetilde{\mathcal{L}})$ and $M' \in \mathcal{B}(\mathcal{L}; \widetilde{\mathcal{L}})$, with M' boundedly invertible, such that

$$\begin{aligned} \begin{bmatrix} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'(z) & \mathfrak{D}'(z) \end{bmatrix} &= \begin{bmatrix} P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{\mathfrak{A}}'(z) & \widetilde{P}_{\mathcal{X}}^{\mathcal{Z}} \mathfrak{B}'(z) \\ \widetilde{\mathfrak{C}}'(z) & \widetilde{\mathfrak{D}}'(z) \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ -K' P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{\mathfrak{A}}'(z) & 1_{\mathcal{L}} - K' P_{\mathcal{X}}^{\mathcal{Z}} \widetilde{\mathfrak{B}}'(z) \end{bmatrix}^{-1} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & M' \end{bmatrix}, \end{aligned}$$

Define $\widetilde{K}' = K' P_{\mathcal{X}}^{\mathcal{Z}}$. Then the right-hand side is the compression along $\mathcal{Z}$ of the function

$$\begin{bmatrix} \widetilde{\mathfrak{A}}'(z) & \widetilde{\mathfrak{B}}'(z) \\ \widetilde{\mathfrak{C}}'(z) & \widetilde{\mathfrak{D}}'(z) \end{bmatrix} \begin{bmatrix} 1_{\widetilde{\mathcal{X}}} & 0 \\ -\widetilde{K}' \widetilde{\mathfrak{A}}'(z) & 1_{\mathcal{L}} - \widetilde{K}' \widetilde{\mathfrak{B}}'(z) \end{bmatrix}^{-1} \begin{bmatrix} 1_{\widetilde{\mathcal{X}}} & 0 \\ 0 & M' \end{bmatrix},$$

which according to Theorem 6.1 is the four-block transfer function of the driving variable representation $\left[\begin{smallmatrix} \widetilde{A}' & \widetilde{B}' \\ \widetilde{C}' & \widetilde{D}' \end{smallmatrix}\right] \left[\begin{smallmatrix} 1_{\widetilde{\mathcal{X}}} & 0 \\ \widetilde{K}' & M' \end{smallmatrix}\right]$ of $\widetilde{\Sigma}$. By Lemma 8.27, $\widetilde{\Sigma}_{dv/s/s}$ is a dilation along $\mathcal{Z}$ of $\Sigma_{dv/s/s}$. □

Theorem 8.29. *Let* $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ *and* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be two state/signal systems with* $\widetilde{\mathcal{X}} = \mathcal{X} \dotplus \mathcal{Z}$ *(and with the same signal space* $\mathcal{W}$*).*

1) $\widetilde{\Sigma}$ *is a dilation of* Σ *if and only if there exist output nulling representations* $\widetilde{\Sigma}_{s/s/on}$ *and* $\Sigma_{s/s/on}$ *of* $\widetilde{\Sigma}$ *and* Σ*, respectively, with the property that* $\widetilde{\Sigma}_{s/s/on}$ *is a dilation along* $\mathcal{Z}$ *of* $\Sigma_{s/s/on}$ *(in the input/state/output sense; in particular they have the same error space).*
2) *If* $\widetilde{\Sigma}$ *is a dilation of* Σ*, then to every output nulling representation* $\Sigma_{s/s/on}$ *of* Σ *there exists at least one output nulling representation* $\widetilde{\Sigma}_{s/s/on}$ *of* $\widetilde{\Sigma}$ *such that* $\widetilde{\Sigma}_{s/s/on}$ *is a dilation along* $\mathcal{Z}$ *of* $\Sigma_{s/s/on}$ *(in the input/state/output sense).*

The proof of this theorem is similar to the proof of Theorem 8.28, and we leave it to the reader.

9. Stability

Below we shall introduce and study different stability notions for state/signal systems. These are related to the stability of different representations of the system. In this connection we interpret each representation as an input/state/output system, and apply the following notion of stability.

Definition 9.1. A input/state/output system is

1) *stable*, if the following implication holds for all its trajectories $(x(\cdot), u(\cdot), y(\cdot))$:
$$u(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{U}) \Rightarrow x(\cdot) \in \ell^\infty(\mathbb{Z}^+; \mathcal{X}) \text{ and } y(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{Y}). \tag{9.1}$$
2) *strongly stable*, if the following implication holds for all its trajectories $(x(\cdot), u(\cdot), y(\cdot))$:
$$u(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{U}) \Rightarrow \lim_{n\to\infty} x(n) = 0 \text{ and } y(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{Y}). \tag{9.2}$$
3) *power stable*, if there exists a constant $r > 1$ such that the following implication holds for all its trajectories $(x(\cdot), u(\cdot), y(\cdot))$:
$$u(\cdot) = 0 \Rightarrow \lim_{n\to\infty} r^n \|x(n)\| = 0. \tag{9.3}$$

It is clear that (9.2) implies (9.1).

Lemma 9.2. *An input/state/output system* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *with the four block transfer function* $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *is stable if and only if the following four conditions hold:*

1) *There is a constant* $C > 0$ *such that* $\|A^n\| \le C$ *for all* $n \in \mathbb{Z}^+$.
2) $\mathfrak{B}(\overline{z})^* x \in H^2(\mathbb{D}; \mathcal{U})$ *for all* $x \in \mathcal{X}$.
3) $\mathfrak{C}(z) x \in H^2(\mathbb{D}; \mathcal{Y})$ *for all* $x \in \mathcal{X}$.
4) $\mathfrak{D} \in H^\infty(\mathbb{D}; \mathcal{U}, \mathcal{Y})$.

This lemma is undoubtedly known, but we have not been able to find an explicit statement in the literature. (A continuous time version of this lemma can easily be derived from [Sta05].) For completeness we therefore include a short proof.

Proof. Clearly, $\Sigma_{i/s/o}$ is stable if and only if the four input-state-output maps listed in (6.8) have the following properties:

1′) $\check{\mathfrak{A}}$ maps $\mathcal{X}$ into $\ell^\infty(\mathbb{Z}^+;\mathcal{X})$;
2′) $\check{\mathfrak{B}}$ maps $\ell^2(\mathbb{Z}^+;\mathcal{U})$ into $\ell^\infty(\mathbb{Z}^+;\mathcal{X})$;
3′) $\check{\mathfrak{C}}$ maps $\mathcal{X}$ into $\ell^2(\mathbb{Z}^+;\mathcal{X})$;
4′) $\check{\mathfrak{D}}$ maps $\ell^2(\mathbb{Z}^+;\mathcal{U})$ into $\ell^2(\mathbb{Z}^+;\mathcal{Y})$.

We claim that each one of these conditions is equivalent to the corresponding condition listed in the statement Lemma 9.2. It is easy to see that all of these operators are always *closed* as operators between the indicated spaces, so by the closed graph theorem, 1′)–4′) are equivalent to the corresponding statements where we require each of these maps to be bounded, i.e.,

1″) $\check{\mathfrak{A}} \in \mathcal{B}(\mathcal{X};\ell^\infty(\mathbb{Z}^+;\mathcal{X}))$;
2″) $\check{\mathfrak{B}} \in \mathcal{B}(\ell^2(\mathbb{Z}^+;\mathcal{U});\ell^\infty(\mathbb{Z}^+;\mathcal{X}))$;
3″) $\check{\mathfrak{C}} \in \mathcal{B}(\mathcal{X};\ell^2(\mathbb{Z}^+;\mathcal{X}))$;
4″) $\check{\mathfrak{D}} \in \mathcal{B}(\ell^2(\mathbb{Z}^+;\mathcal{U});\ell^2(\mathbb{Z}^+;\mathcal{Y}))$.

Obviously, 1) is equivalent to 1″). Condition 1) implies that $\mathfrak{D} \subset \rho(A)$, and hence all the transfer functions listed in 2)–4) are defined and analytic on $\mathbb{D}$. That 3) is equivalent to 3″) follows from the fact that the Z-transform is a bounded linear map from $\ell^2(\mathbb{Z}^+;\mathcal{U})$ onto $H^2(\mathbb{D};\mathcal{Y})$ with a bounded inverse. The equivalence of 4) and 4″) is well known: a causal convolution operator $\check{\mathfrak{D}}$ maps $\ell^2(\mathbb{Z}^+;\mathcal{U})$ into $\ell^2(\mathbb{Z}^+;\mathcal{Y})$ if and only if its symbol $\mathfrak{D}$ belongs to $H^\infty(\mathbb{D};\mathcal{U},\mathcal{Y})$.

The equivalence of 2) and 2″) remains to be established. It is easy to see that 2″) is equivalent to the following condition:

2‴) the sequence $\{\widetilde{\mathfrak{B}}_n\}_{n\in\mathbb{Z}^+}$ of operators defined by $\widetilde{\mathfrak{B}}_n u = \sum_{k=0}^n A^k Bu(-k-1)$ is uniformly bounded in $\mathcal{B}(\ell^2(\mathbb{Z}^-;\mathcal{U});\mathcal{X})$.

Assume that 2‴) holds. Then, for each $u \in \ell^2(\mathbb{Z}^-;\mathcal{U})$, the sequence $\widetilde{\mathfrak{B}}_n u$ is a Cauchy sequence in $\mathcal{X}$ (since the norm in $\ell^2(\mathbb{Z}^-;\mathcal{U})$ of the sequence $\{u(k)\}_{k<m}$ tends to zero as $m \to -\infty$). Denote the limit by $\widetilde{\mathfrak{B}}$. Then $\widetilde{\mathfrak{B}} u = \sum_{k=0}^\infty A^k Bu(-k-1)$ and $\widetilde{\mathfrak{B}} \in \mathcal{B}(\ell^2(\mathbb{Z}^-;\mathcal{U});\mathcal{X})$. By duality, $\widetilde{\mathfrak{B}}^* \in \mathcal{B}(\mathcal{X};\ell^2(\mathbb{Z}^-;\mathcal{U}))$. This is equivalent to the statement that the operator $x \mapsto B^*(A^*)^n x$, $n \in \mathbb{Z}^+$, maps $\mathcal{X}$ into $\ell^2(\mathbb{Z}^+;\mathcal{Y})$, which equivalent to 2) (in the same way as 3) is equivalent to 3′)). Thus, 2‴) $\Rightarrow$ 2). Conversely, if 2) holds, then the operator that we denoted by $\widetilde{\mathfrak{B}}^*$ above is bounded, hence so is $\widetilde{\mathfrak{B}}$, and this implies 2‴). □

Lemma 9.3. *An input/state/output system* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ *is strongly stable if and only if it is stable and* A *is strongly stable, i.e.,* $\lim_{n\to\infty} A^n x = 0$ *for all* $x \in \mathcal{X}$.

Also this lemma must be known, but we have not found an explicit proof in the literature (a proof of the well-posed continuous time version of this lemma is given in [Sta05], and the discrete time proof is the same). For the convenience of the reader we therefore again include a short proof.

Proof. It is easy to see that if $\Sigma_{i/s/o}$ is strongly stable then it is stable, and $\lim_{n\to\infty} A^n x = 0$ for all $x \in \mathcal{X}$. Let us therefore only prove the converse part.

Let $(x(\cdot), u(\cdot), y(\cdot))$ be a trajectory of $\Sigma_{i/s/o}$ on $\mathbb{Z}^+$ with $u \in \ell^2(\mathbb{Z}^+;\mathcal{U})$. Fix $\epsilon > 0$. Choose m large enough so that $\sum_{k=m}^{\infty} \|u(k)\|^2 \le \epsilon^2$. Then we have for all $n \ge m$,

$$x(n) = A^{n-m}x(m) + \sum_{k=0}^{n-m-1} A^{n-k-1}Bu(m+k)$$

Here $A^{n-m}x(m) \to 0$ as $n \to \infty$ (because of the strong stability of A), and the norm of the second term is at most $C\epsilon$, where C is the norm of the mapping $\breve{\mathfrak{B}} \in \mathcal{B}(\ell^2(\mathbb{Z}^+;\mathcal{U}); \ell^\infty(\mathbb{Z}^+;\mathcal{X}))$. Since ϵ was arbitrary, this implies that $x(k) \to 0$ as $k \to \infty$. □

Remark 9.4. As is well known, conditions 2) and 3) in Lemma 9.2 imply that the sums

$$\mathcal{C} := \sum_{n\in\mathbb{Z}^+} A^n BB^*(A^*)^n, \tag{9.4}$$

$$\mathcal{O} := \sum_{n\in\mathbb{Z}^+} (A^*)^n C^*CA^n, \tag{9.5}$$

converge monotonically in the strong sense to nonnegative operators $\mathcal{O} \in \mathcal{B}(\mathcal{X})$ and $\mathcal{C} \in \mathcal{B}(\mathcal{X})$, respectively. These are called the infinite time controllability, respectively, observability Gramians of the system. They are the minimal nonnegative solutions of the Stein equations

$$H - AHA^* = BB^*, \tag{9.6}$$

$$G - A^*GA = C^*C, \tag{9.7}$$

respectively. If A is strongly stable, then the nonnegative solution H of (9.6) is unique (hence $H = \mathcal{C}$), and if A^* is strongly stable (i.e., $(A^*)^n x \to 0$ for all $x \in \mathcal{X}$), then the nonnegative solution G of (9.7) is unique (hence $G = \mathcal{O}$).

Lemma 9.5. *Let* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an input/state/output system. Then the following conditions are equivalent:*

1) $\Sigma_{i/s/o}$ *is power stable;*
2) $\overline{\mathbb{D}} := \{z \in \mathbb{C} \mid |z| \le 1\} \subset \Lambda_A$*;*
3) *There exists constants* $q < 1$ *and* $C > 0$ *such that* $\|A^n\| \le Cq^n$.

Proof. Clearly 2) and 3) are equivalent. It is also clear that 3) implies 1). For the converse implication we observe that condition 1) says that there is some $r > 1$ such that $\lim_{n\to\infty} r^n A^n x = 0$ for all $x \in \mathcal{X}$. By the uniform boundedness principle, $\sup_{n\in\mathbb{Z}^+} r^n \|A^n\| < \infty$. This implies 3) with $\gamma = 1/r$. □

Lemma 9.6. *Every power stable input/state/output system is strongly stable.*

Proof. This follows from Lemmas 9.2, 9.3, and 9.5. □

Thus, power stability implies strong stability, which further implies stability.

We call a driving variable or output nulling representation of a state/signal system stable, or strongly stable, or power stable, if it has this property when it is interpreted as an input/state/output system, as explained in Remark 5.4.

Definition 9.7. A state/signal system is

1) *stabilizable* (or strongly stabilizable, or power stabilizable) if it has a stable (or strongly stable, or power stable, respectively) driving variable representation.
2) *detectable* (or strongly detectable, or power detectable) if it has a stable (or strongly stable, or power stable, respectively) output nulling representation.
3) *LFT-stabilizable*[8] (or strongly LFT-stabilizable, or power LFT-stabilizable), if it has a stable (or strongly stable, or power stable, respectively) input/state/output representation.

Next we shall show that the above notions are closely connected to the corresponding (better known) notions for input/state/output systems.[9]

Definition 9.8. An input/state/output system $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ is

1) *stabilizable* (or strongly stabilizable, or power stabilizable) if there exists an operator $L \in \mathcal{B}(\mathcal{X};\mathcal{U})$, called a *state feedback operator*, such that the new input/state/output system with input $\ell(\cdot)$ and output $w(\cdot) = \left[\begin{smallmatrix} y(\cdot) \\ u(\cdot) \end{smallmatrix}\right]$, described by the system of equations
$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \\ u(n) &= Lx(n) + \ell(n), \qquad z \in \mathbb{Z}^+, \end{aligned} \tag{9.8}$$
is stable (or strongly stable, or power stable, respectively).
2) *detectable* (or strongly detectable, or power detectable) if there exists an operator $H \in \mathcal{B}(\mathcal{Y};\mathcal{X})$, called an *output injection operator*, such that the new input/state/output system with input $w(\cdot) = \left[\begin{smallmatrix} e(\cdot) \\ u(\cdot) \end{smallmatrix}\right]$ and output $y(\cdot)$, described by the system of equations
$$\begin{aligned} x(n+1) &= Ax(n) + Hy(n) + Bu(n), \\ y(n) &= Cx(n) + e(n) + Du(n), \qquad z \in \mathbb{Z}^+, \end{aligned} \tag{9.9}$$
is stable (or strongly stable, or power stable, respectively).
3) *output feedback stabilizable* (or strongly output feedback stabilizable, or power output feedback stabilizable) if there exists an operator $K \in \mathcal{B}(\mathcal{Y};\mathcal{U})$, called a *output feedback operator*, such that $1_{\mathcal{U}} - KD$ has a bounded inverse and the

[8] LFT stands for Linear Fractional Transformation.
[9] A number of slightly different ways of presenting these notions do exist. We have chosen to present a version which makes the connection to the state/signal theory as simple as possible. This is a discrete time analogue of the treatment in [Sta05, Chapter 7].

new input/state/output system with input $\ell(\cdot)$ and output $y(\cdot)$, described by the (implicit) system of equations (where $u(n)$ should be eliminated)

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \\ u(n) &= Ky(n) + \ell(n), \qquad z \in \mathbb{Z}^+, \end{aligned} \tag{9.10}$$

is stable (or strongly stable, or power stable, respectively).

4) *LFT-stabilizable* (or strongly LFT-stabilizable, or power LFT-stabilizable), if there exists Hilbert spaces $\widetilde{\mathcal{Y}}$ and $\widetilde{\mathcal{U}}$ and an operator $\Psi = \left[\begin{smallmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \widetilde{\mathcal{Y}} \\ \widetilde{\mathcal{U}} \end{smallmatrix}\right])$, called an *LFT-feedback operator*, such that both Ψ itself and $\Psi_{21}D+\Psi_{22}$ have bounded inverses, and such that the new input/state/output system with input $u_1(\cdot)$ and output $y_1(\cdot)$, described by the (implicit) system of equations (where $u(n)$ and $y(n)$ should be eliminated)

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \\ y_1(n) &= \Psi_{11}y(n) + \Psi_{12}u(n), \\ u_1(n) &= \Psi_{21}y(n) + \Psi_{22}u(n), \qquad z \in \mathbb{Z}^+, \end{aligned} \tag{9.11}$$

is stable (or strongly stable, or power stable, respectively).

More explicitly, the resulting input/state/output systems have the following structure. If we denote the system in part 1) by $\Sigma^L = \left(\left[\begin{smallmatrix} A^L & B^L \\ C^L & D^L \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]\right)$, then

$$\begin{bmatrix} A^L & B^L \\ C^L & D^L \end{bmatrix} = \left[\begin{array}{c|c} A+BL & B \\ \hline C+DL & D \\ L & 1_{\mathcal{U}} \end{array}\right]. \tag{9.12}$$

If we denote the system in part 2) by $\Sigma^H = \left(\left[\begin{smallmatrix} A^H & B^H \\ C^H & D^H \end{smallmatrix}\right]; \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right], \mathcal{Y}\right)$, then

$$\begin{bmatrix} A^H & B^H \\ C^H & D^H \end{bmatrix} = \left[\begin{array}{c|cc} A+HC & H & B+HD \\ \hline C & 1_{\mathcal{Y}} & D \end{array}\right]. \tag{9.13}$$

If we denote the system in part 3) by $\Sigma^K = \left(\left[\begin{smallmatrix} A^K & B^K \\ C^K & D^K \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$, then

$$\begin{aligned} \begin{bmatrix} A^K & B^K \\ C^K & D^K \end{bmatrix} &= \begin{bmatrix} A+BK\left(1_{\mathcal{Y}}-DK\right)^{-1}C & B\left(1_{\mathcal{U}}-KD\right)^{-1} \\ \left(1_{\mathcal{Y}}-DK\right)^{-1}C & D\left(1_{\mathcal{U}}-KD\right)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1_{\mathcal{K}} & 0 \\ -KC & 1_{\mathcal{U}}-KD \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1_{\mathcal{Y}} & -BK \\ 0 & 1_{\mathcal{Y}}-DK \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \end{aligned} \tag{9.14}$$

If we denote the system in part 4) by $\Sigma^\Psi = \left(\left[\begin{smallmatrix} A^\Psi & B^\Psi \\ C^\Psi & D^\Psi \end{smallmatrix}\right]; \mathcal{X}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}}\right)$, then (see also Lemma 10.1)

$$\begin{bmatrix} A^\Psi & B^\Psi \\ C^\Psi & D^\Psi \end{bmatrix} = \begin{bmatrix} A & B \\ \Psi_{11}C & \Psi_{11}D + \Psi_{12} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Psi_{21}C & \Psi_{21}D + \Psi_{22} \end{bmatrix}^{-1}, \tag{9.15}$$

or equivalently,

$$\begin{aligned} A^\Psi &= A - B(\Psi_{21}D + \Psi_{22})^{-1}\Psi_{21}C, \\ B^\Psi &= B(\Psi_{21}D + \Psi_{22})^{-1}, \\ C^\Psi &= \Psi_{11}C - (\Psi_{11}D + \Psi_{12})(\Psi_{21}D + \Psi_{22})^{-1}\Psi_{21}C, \\ D^\Psi &= (\Psi_{11}D + \Psi_{12})(\Psi_{21}D + \Psi_{22})^{-1}. \end{aligned} \tag{9.16}$$

When we apply Definition 9.8 to various systems it is often more convenient to use the following equivalent characterization:

Lemma 9.9. *Let $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an input/state/output system.*

1) *The system $\Sigma^L = \left(\left[\begin{smallmatrix} A^L & B^L \\ C^L & D^L \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]\right)$ whose coefficient matrix is given by* (9.12) *is stable* (*or strongly stable, or power stable*) *if and only if the system* $\left(\left[\begin{array}{c|c} A^L & B \\ \hline C & 0 \\ L & 0 \end{array}\right]; \mathcal{X}, \mathcal{U}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]\right)$ *has the same property.*
2) *The system $\Sigma^H = \left(\left[\begin{smallmatrix} A^H & B^H \\ C^H & D^H \end{smallmatrix}\right]; \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right], \mathcal{Y}\right)$ whose coefficient matrix is given by* (9.13) *is stable* (*or strongly stable, or power stable*) *if and only if the system* $\left(\left[\begin{array}{c|cc} A^H & H & B \\ \hline C & 0 & 0 \end{array}\right]; \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right], \mathcal{Y}\right)$ *has the same property.*
3) *The system $\Sigma^K = \left(\left[\begin{smallmatrix} A^K & B^K \\ C^K & D^K \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ whose coefficient matrix is given by* (9.14) *is stable* (*or strongly stable, or power stable*) *if and only if the system* $\left(\left[\begin{smallmatrix} A^K & B \\ C & 0 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *has the same property.*
4) *The system $\Sigma^\Psi = \left(\left[\begin{smallmatrix} A^\Psi & B^\Psi \\ C^\Psi & D^\Psi \end{smallmatrix}\right]; \mathcal{X}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}}\right)$ whose coefficient matrix is given by* (9.15) *is stable* (*or strongly stable, or power stable*) *if and only if the system* $\left(\left[\begin{smallmatrix} A^\Psi & B \\ C & 0 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *has the same property.*

Proof. Proof of 1): The latter system differs from Σ^L only in the sense that we have subtracted a multiple of the first input from the second input and modified the feedthrough term, and this does not affect stability.

Proof of 2): The latter system differs from Σ^H only in the sense that we have subtracted a multiple of the second output from the first output and modified the feedthrough term, and this does not affect stability.

Proof of 3): The latter system differs from Σ^K only in the sense that we have multiplied both the input and the output by bounded invertible operators and modified the feedthrough term, and this does not affect stability.

Proof of 4): The latter system differs from Σ^Ψ only in the sense that we have multiplied both the input and the output by bounded invertible operators and modified the feedthrough term, and this does not affect stability. Indeed, the operator multiplying C to the left is invertible, because of the invertibility of Ψ and the following Schur factorization:

$$\begin{bmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{bmatrix}\begin{bmatrix}1_{\mathcal{Y}} & D\\ 0 & 1_{\mathcal{U}}\end{bmatrix}\begin{bmatrix}1_{\mathcal{Y}} & 0\\ -(\Psi_{21}D+\Psi_{22})^{-1}\Psi_{21} & 1_{\mathcal{U}}\end{bmatrix}$$
$$=\begin{bmatrix}\Psi_{11}-(\Psi_{11}D+\Psi_{12})(\Psi_{21}D+\Psi_{22})^{-1}\Psi_{21} & \Psi_{11}D+\Psi_{12}\\ 0 & \Psi_{21}D+\Psi_{22}\end{bmatrix}. \qquad \square$$

Lemma 9.10. *Let $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]; \mathcal{X},\mathcal{U},\mathcal{Y}\right)$ be an input/state/output system.*

1) *If $\Sigma_{i/s/o}$ is output feedback stabilizable (or strongly output feedback stabilizable, or power output feedback stabilizable), then it is also LFT-stabilizable (or strongly LFT-stabilizable, or power LFT-stabilizable, respectively).*
2) *If $\Sigma_{i/s/o}$ is LFT-stabilizable (or strongly LFT-stabilizable) with an LFT-feedback operator $\Psi = \left[\begin{smallmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{smallmatrix}\right]$ where Ψ_{22} has a bounded inverse, then it is also output feedback stabilizable (or strongly output feedback stabilizable, respectively).*
3) *If $\Sigma_{i/s/o}$ is LFT-stabilizable (or strongly LFT-stabilizable) and $D = 0$, then it is also output feedback stabilizable (or strongly output feedback stabilizable, respectively).*
4) *If $\Sigma_{i/s/o}$ is power LFT-stabilizable then it is also power output feedback stabilizable.*
5) *If $\Sigma_{i/s/o}$ is LFT-stabilizable, then it is both stabilizable and detectable.*

Proof. Proof of 1) : Take $\widetilde{\mathcal{Y}} = \mathcal{Y}$, $\widetilde{\mathcal{U}} = \mathcal{U}$, and $\Psi = \left[\begin{smallmatrix}1_{\mathcal{Y}} & 0\\ -K & 1_{\mathcal{U}}\end{smallmatrix}\right]$.

Proof of 2): Use parts 3)–4) of Lemma 9.9, and take $K = -\Psi_{22}^{-1}\Psi_{21}$.

Proof of 3): This follows from 2), since the assumption that $D = 0$ implies that $\Psi_{22} = \Psi_{21}D + \Psi_{22}$ has a bounded inverse.

Proof of 4): The claim 2) remains valid also in the power stabilizable case, with the same proof. However, in the power stabilizable case the spectral radius of the operator A^{Ψ} lies strictly inside the unit disk. This implies that the set of all LFT-feedbacks Ψ which power stabilize $\Sigma_{i/s/o}$ is open. Therefore, if it is nonempty, it must contain some element Ψ for which Ψ_{22} is invertible. Thus, by the power stable version of part 2), $\Sigma_{i/s/o}$ is power output feedback stabilizable.

Proof of 5): Use parts 1), 2) and 4) of Lemma 9.9, and take $L = -(\Psi_{21}D + \Psi_{22})^{-1}\Psi_{21}C$ and $H = -B(\Psi_{21}D+\Psi_{22})^{-1}\Psi_{21}$. Also note that L is a left multiple of C and that H is a right multiple of B, which implies that in this case Σ^L is stable if and only if $\left(\left[\begin{array}{c|c}A^L & B\\ \hline C & 0\end{array}\right]; \mathcal{X},\mathcal{U},\mathcal{Y}\right)$ is stable, and Σ^H is stable if and only if $\left(\left[\begin{array}{c|c}A^H & B\\ \hline C & 0\end{array}\right]; \mathcal{X},\mathcal{U},\mathcal{Y}\right)$ is stable. $\square$

Theorem 9.11. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal node.*

1) *The following conditions are equivalent.*
   (a) *Σ is stabilizable (or strongly stabilizable, or power stabilizable);*
   (b) *Σ has a stabilizable (or strongly stabilizable, or power stabilizable) input/state/output representation;*

(c) *every input/state/output representation of* Σ *is stabilizable* (*or strongly stabilizable, or power stabilizable*).

2) *The following conditions are equivalent.*
   (a) Σ *is detectable* (*or strongly detectable, or power detectable*);
   (b) Σ *has a detectable* (*or strongly detectable, or power detectable*) *input/state/output representation;*
   (c) *every input/state/output representation of* Σ *is detectable* (*or strongly detectable, or power detectable*).

3) *The following conditions are equivalent.*
   (a) Σ *is LFT-stabilizable* (*or strongly LFT-stabilizable, or power LFT-stabilizable*);
   (b) Σ *has a LFT-stabilizable* (*or strongly LFT-stabilizable, or power LFT-stabilizable*) *input/state/output representation;*
   (c) *every input/state/output representation of* Σ *is LFT-stabilizable* (*or strongly LFT-stabilizable, or power LFT-stabilizable*).

Proof. The proofs of the strongly stable and power stable versions of this theorem are identical to the proofs of the basic version, so below we shall only prove the basic "stable" version.

Proof of 1): We prove this by showing that (b) $\Rightarrow$ (a) $\Rightarrow$ (c) (the implication (c) $\Rightarrow$ (b) is trivial).

Let $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be a stabilizable input/state/output representation of Σ, and let $L \in \mathcal{B}(\mathcal{Y}; \mathcal{U})$ be a stabilizing state feedback operator. Then the system Σ^L whose coefficient matrix is given by (9.12) is stable. This system has an obvious interpretation as a driving variable representation of Σ (with driving variable space $\mathcal{U}$). Thus, according to Definition 9.7, Σ is stabilizable.

Conversely, suppose that Σ is stabilizable (in the sense of Definition 9.7). Let $\Sigma_{dv/s/s} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ be a stable driving variable representation of Σ, and let $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an arbitrary input/state/output representation of Σ. We can alternatively interpret this representation, too, as a driving variable representation as explained in Remark 5.2. Split C' and D' into $C' = \left[\begin{smallmatrix} C_1' \\ C_2' \end{smallmatrix}\right]$ and $D' = \left[\begin{smallmatrix} D_1' \\ D_2' \end{smallmatrix}\right]$ in accordance with the splitting $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. Then, by Theorem 3.3, there exist operators $L \in \mathcal{B}(\mathcal{X}; \mathcal{U})$ and $M' \in \mathcal{B}(\mathcal{L}; \mathcal{U})$, with M' boundedly invertible, such that

$$\begin{bmatrix} A' & B' \\ C_1' & D_1' \\ C_2' & D_2' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & M' \end{bmatrix} = \begin{bmatrix} A + BL & BM' \\ C + DL & DM' \\ L & M' \end{bmatrix}. \tag{9.17}$$

This coefficient matrix is identical to the one in (9.12) apart from the fact that the input variable has been multiplied by the invertible operator M'. This means that L is a stabilizing state feedback operator for $\Sigma_{i/s/o}$.

The proof of Part 2) is similar to the proof of Part 1), and it is left to the reader (this time we interpret the input/state/output representation as an output nulling representation as explained in Remark 5.2).

Proof of 3): The implication (a) $\Rightarrow$ (c) follows from Theorem 5.11 (take Ψ to be the operator Θ defined in (1.6)), and the implication (c) $\Rightarrow$ (b) is trivial. Thus, it remains to prove the implication (b) $\Rightarrow$ (a).

Let $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an input/state/output representation of Σ with a LFT-stabilizing feedback operator $\Psi \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \widetilde{\mathcal{Y}} \\ \widetilde{\mathcal{U}} \end{smallmatrix}\right])$, and let $\Sigma^{\Psi} = \left(\left[\begin{smallmatrix} A^{\Psi} & B^{\Psi} \\ C^{\Psi} & D^{\Psi} \end{smallmatrix}\right]; \mathcal{X}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}}\right)$ be the stable input/state/output system whose coefficient matrix is given by (9.15). We claim that there exists an admissible input/output decomposition $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ of $\mathcal{W}$ such that the corresponding input/state/output representation is stable. The proof of this claim is by direct construction.

We begin by interpreting Ψ as an operator $\Psi = \left[\begin{smallmatrix} \Psi_1 \\ \Psi_2 \end{smallmatrix}\right] \in \mathcal{B}(\mathcal{W}; \left[\begin{smallmatrix} \widetilde{\mathcal{Y}} \\ \widetilde{\mathcal{U}} \end{smallmatrix}\right])$, where $\Psi_1 = \Psi_{11} P_{\mathcal{Y}}^{\mathcal{U}} + \Psi_{12} P_{\mathcal{U}}^{\mathcal{Y}}$ and $\Psi_2 = \Psi_{21} P_{\mathcal{Y}}^{\mathcal{U}} + \Psi_{22} P_{\mathcal{U}}^{\mathcal{Y}}$. The bounded inverse of this operator belongs to $\mathcal{B}(\left[\begin{smallmatrix} \widetilde{\mathcal{Y}} \\ \widetilde{\mathcal{U}} \end{smallmatrix}\right]; \mathcal{W})$, and it can be decomposed into $\widetilde{\Psi} := \Psi^{-1} = \begin{bmatrix} \widetilde{\Psi}_1 & \widetilde{\Psi}_2 \end{bmatrix}$. Define

$$\mathcal{Y}_1 = \mathcal{N}\left(\left[\Psi_2\right]\right), \quad \mathcal{U}_1 = \mathcal{N}\left(\left[\Psi_1\right]\right).$$

Define $P \in \mathcal{B}(\mathcal{W})$ and $Q \in \mathcal{B}(\mathcal{W})$ by

$$P := \widetilde{\Psi} \begin{bmatrix} \Psi_1 \\ 0 \end{bmatrix}, \quad Q := \widetilde{\Psi} \begin{bmatrix} 0 \\ \Psi_2 \end{bmatrix}.$$

Clearly $P + Q = 1_{\mathcal{W}}$. For all $w \in \mathcal{Y}_1$ we have $Qw = 0$, hence $Pw = w$, and for all $w \in \mathcal{U}_1$ we have $Pw = 0$, hence $Qw = w$. This implies that P and Q are complementary projections in $\mathcal{W}$, with $\mathcal{R}(P) = \mathcal{N}(Q) = \mathcal{Y}_1$ and $\mathcal{N}(P) = \mathcal{R}(Q) = \mathcal{U}_1$, i.e., $P = P_{\mathcal{Y}_1}^{\mathcal{U}_1}$ and $Q = P_{\mathcal{U}_1}^{\mathcal{Y}_1}$. In particular, this implies that $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$. Furthermore, Ψ_1 maps $\mathcal{Y}_1$ one-to-one onto $\widetilde{\mathcal{Y}}$ with the bounded inverse $\widetilde{\Psi}_1$, and Ψ_1 maps $\mathcal{U}_1$ one-to-one onto $\widetilde{\mathcal{U}}$ with the bounded inverse $\widetilde{\Psi}_2$.

Let $\Phi := \begin{bmatrix} P_{\mathcal{Y}_1}^{\mathcal{U}_1}|_{\mathcal{Y}} & P_{\mathcal{Y}_1}^{\mathcal{U}_1}|_{\mathcal{U}} \\ P_{\mathcal{U}_1}^{\mathcal{Y}_1}|_{\mathcal{Y}} & P_{\mathcal{U}_1}^{\mathcal{Y}_1}|_{\mathcal{U}} \end{bmatrix}$. This is the same operator that we find in (1.6), corresponding to the two decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$, and it is explicitly given by

$$\Phi = \begin{bmatrix} \widetilde{\Psi}_1 \Psi_{11} & \widetilde{\Psi}_1 \Psi_{12} \\ \widetilde{\Psi}_2 \Psi_{21} & \widetilde{\Psi}_2 \Psi_{22} \end{bmatrix}.$$

In particular, $\Phi_{12} D + \Phi_{22} = \widetilde{\Psi}_2(\Psi_{12} D + \Psi_{22})$ is invertible, and by Theorem 5.11, the decomposition $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ is admissible. Let us denote the corresponding input/state/output system by $\Sigma_{i/s/o}^1 = \left(\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1\right)$. This system is obtained from Σ^{Ψ} by multiplying the input by $\widetilde{\Psi}_2^{-1}$ and the output by $\widetilde{\Psi}_1$. Thus, $\Sigma_{i/s/o}^1$ is stable. $\square$

10. Appendix

Lemma 10.1. *Let* $A \in \mathcal{B}(\mathcal{X};\mathcal{Z})$ *and* $B \in \mathcal{B}(\mathcal{Z};\mathcal{X})$.

1) $1_{\mathcal{X}} - BA$ *has a bounded inverse if and only if* $1_{\mathcal{Z}} - AB$ *has a bounded inverse.*
2) *If* $1_{\mathcal{X}} - BA$ *has a bounded inverse, then*

$$\begin{aligned} (1_{\mathcal{Z}} - AB)^{-1} &= 1_{\mathcal{X}} + A(1_{\mathcal{X}} - BA)^{-1}B, \\ B(1_{\mathcal{Z}} - AB)^{-1} &= (1_{\mathcal{X}} - BA)^{-1}B. \end{aligned} \tag{10.1}$$

For a proof see, e.g., [Sta05, Appendix A4].

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References

[AA70] Vadim M. Adamyan and Damir Z. Arov, *On unitary couplings of semiunitary operators*, Eleven Papers in Analysis (Providence, R.I.), American Mathematical Society Translations, vol. 95, American Mathematical Society, 1970, pp. 75–129.

[Aro74] Damir Z. Arov, *Scattering theory with dissipation of energy*, Dokl. Akad. Nauk SSSR. **216** (1974), 713–716, Translated in Soviet Math. Dokl. 15 (1974), 848–854.

[Aro79] ———, *Passive linear stationary dynamic systems*, Sibir. Mat. Zh. **20** (1979), 211–228, Translated in Sib. Math. J. 20 (1979), 149-162.

[Bel68] Vitold Belevitch, *Classical network theory*, Holden-Day, San Francisco, Calif.-Cambridge-Amsterdam, 1968.

[BGK79] H. Bart, Israel Gohberg, and Marinus A. Kaashoek, *Minimal factorization of matrix and operator functions*, Operator Theory: Advances and Applications, vol. 1, Birkhäuser-Verlag, Basel Boston Berlin, 1979.

[BS05] Joseph A. Ball and Olof J. Staffans, *Conservative state-space realizations of dissipative system behaviors*, To appear in Integral Equations Operator Theory (2005), 63 pp.

[Fuh74] Paul A. Fuhrmann, *On realization of linear systems and applications to some questions of stability*, Math. Systems Theory **8** (1974), 132–140.

[Hel74] J. William Helton, *Discrete time systems, operator models, and scattering theory*, J. Funct. Anal. **16** (1974), 15–38.

[LP67] P.D. Lax and R.S. Phillips, *Scattering theory*, Academic Press, New York, 1967.

[PW98] Jan Willem Polderman and Jan C. Willems, *Introduction to mathematical systems theory: A behavioral approach*, Springer-Verlag, New York, 1998.

[SF70] Béla Sz.-Nagy and Ciprian Foiaş, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam London, 1970.

[Sta05] Olof J. Staffans, *Well-posed linear systems*, Cambridge University Press, Cambridge and New York, 2005.

[WT02] Jan C. Willems and Harry L. Trentelman, *Synthesis of dissipative systems using quadratic differential forms: Part I*, IEEE Trans. Autom. Control **47** (2002), 53–69.

Damir Z. Arov
Department of Mathematics
South-Ukrainian Pedagogical University
65020 Odessa, Ukraine

Olof J. Staffans
Åbo Akademi University
Department of Mathematics
FIN-20500 Åbo, Finland
e-mail: `http://www.abo.fi/~staffans/`