Quadratic Optimal Control of Well-Posed Linear Systems

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Abstract

We study the infinite horizon quadratic cost minimization problem for a well-posed linear system in the sense of Salamon and Weiss. The quadratic cost function that we seek to minimize need not be positive, but it is convex and bounded from below. We assume the system to be jointly stabilizable and detectable, and give a feedback solution to the cost minimization problem. Moreover, we connect this solution to the computation of either a (J,S)-inner or an S-normalized coprime factorization of the transfer function, depending on how the problem is formulated. In the case where the system is regular it is possible to show that the feedback operator can be computed from the Riccati operator, and that the Riccati operator is a stabilizing self-adjoint solution of an algebraic Riccati equation. This Riccati equation is nonstandard in the sense that the weighting operator in the quadratic term differs from the expected one, and the computation of the correct weighting operator is a nontrivial task. We apply the general theory to get factorization versions of the bounded and positive real lemmas.

1 Notations

$\mathcal{L}(U;Y)$, $\mathcal{L}(U)$: The set of bounded linear operators from U into Y or from U into itself, respectively.

$A \geq 0$: A is (selfadjoint and) positive definite.

$A >> 0$: $A \geq \epsilon I$ for some $\epsilon > 0$, hence A is invertible.

$\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{R}^-$: $\mathbf{R} = (-\infty,\infty)$, $\mathbf{R}^+ = [0,\infty)$, and $\mathbf{R}^- = (-\infty, 0]$.

$L^2(J;U)$: The set of U-valued L^2-functions on the interval J.

$L^2_\omega(J;U)$: $L^2_\omega(J;U) = \{ u \in L^2_{\mathrm{loc}}(J;U) \mid (t \mapsto \mathrm{e}^{-\omega t}u(t)) \in L^2(J;U) \}$.

$TIC_\omega(U;Y)$, $TI_\omega(U)$: The set of bounded linear time-invariant causal operators from $L^2_\omega(\mathbf{R};U)$ into $L^2_\omega(\mathbf{R};Y)$, or from $L^2_\omega(\mathbf{R};U)$ into itself. $TIC(U;Y) = TIC_0(U;Y)$ and $TIC(U) = TIC_0(U)$.

τ^t: The time shift group $\tau^t u(s) = u(t+s)$ (this is a left-shift when $t > 0$ and a right-shift when $t < 0$).

π_J: $(\pi_J u)(s) = u(s)$ if $s \in J$ and $(\pi_J u)(s) = 0$ if $s \notin J$. Here $J \subset \mathbf{R}$.

π_+, π_-: $\pi_+ = \pi_{\mathbf{R}^+}$ and $\pi_- = \pi_{\mathbf{R}^-}$.

This work is a continuation of [15], and we refer the reader to this paper for additional definitions.

2 Quadratic Cost Minimization

Definition 1 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a well-posed linear system on (U,H,Y), and let $J = J^* \in \mathcal{L}(Y)$. The (nonstandard) quadratic cost minimization problem for Ψ with cost operator J consists of finding, for each $x_0 \in H$, the infimum of the cost*

$$Q(x_0,u) = \langle y, Jy \rangle_{L^2(\mathbf{R}^+;Y)}, \tag{1}$$

over all those $u \in L^2(\mathbf{R}^+;U)$ for which the corresponding observation $y = \mathcal{C}x_0 + \mathcal{D}\pi_+ u$ of Ψ satisfies $y \in L^2(\mathbf{R}^+;Y)$. If there exists an operator $\Pi = \Pi^* \in \mathcal{L}(H)$ such that the optimal cost is given by

$$\inf_{u \in L^2(\mathbf{R}^+,U)} Q(x_0,u) = \langle x_0, \Pi x_0 \rangle_H ,$$

then Π is called the Riccati operator of Ψ with cost operator J.

Definition 2 *Let $J = J^* \in \mathcal{L}(Y)$, and let $\alpha \geq 0$.*

(i) The system $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ on (U, H, Y) is J-coercive if there exist constants $M > 0$ and $\epsilon > 0$ such that the cost Q defined in (1) satisfies

$$
\begin{aligned}
Q(x_0, u) \geq \epsilon \left(\|u\|^2_{L^2(\mathbf{R}^+;U)} + \|y\|^2_{L^2(\mathbf{R}^+;Y)} \right) \\
- M \|x_0\|^2_H
\end{aligned}
\tag{2}
$$

for all those $x_0 \in H$ and $u \in L^2(\mathbf{R}^+; U)$ for which $y = \mathcal{C}x_0 + \mathcal{D}\pi_+ u \in L^2(\mathbf{R}^+; Y)$.

(ii) The operator $\mathcal{D} \in TIC_\alpha(U; Y)$ is J-coercive if there exists a constant $\epsilon > 0$ such that

$$
\begin{aligned}
&\langle \mathcal{D}\pi_+ u, J\mathcal{D}\pi_+ u \rangle_{L^2(\mathbf{R}^+;Y)} \\
&\qquad \geq \epsilon \left(\|u\|^2_{L^2(\mathbf{R}^+;U)} + \|\mathcal{D}\pi_+ u\|^2_{L^2(\mathbf{R}^+;Y)} \right)
\end{aligned}
$$

for all those $u \in L^2(\mathbf{R}^+; U)$ for which $\mathcal{D}\pi_+ u \in L^2(\mathbf{R}^+; Y)$.

Lemma 3 *Let $J = J^* \in \mathcal{L}(Y)$, and let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a jointly stabilizable and detectable [15, Definition 13] well-posed linear system on (U, H, Y). Then Ψ is J-coercive iff its input/output map $\mathcal{D}$ is J-coercive. In this case the quadratic cost minimization problem with cost operator J has a unique minimizing solution $u^{\text{opt}}(x_0) \in L^2(\mathbf{R}^+; U)$ and a bounded Riccati operator $\Pi = \Pi^* \in \mathcal{L}(H)$.*

The idea behind the proof of this lemma is to first stabilize the system by using a preliminary feedback, and to then apply the theory for stable systems presented in [14].

The solution given by Lemma 3 is not yet complete in the sense that it does not give a feedback representation of the solution. Our next task will be to develop such a feedback representation in terms of a right coprime factorization of the input/output map $\mathcal{D}$ with the special property that its numerator is (J, S)-inner. This notion is defined as follows:

Definition 4 *Let $J = J^* \in \mathcal{L}(Y)$, let $S = S^* \in \mathcal{L}(U)$ be invertible, let $\mathcal{D} \in TIC_\alpha(U; Y)$ for some $\alpha \geq 0$, and let $(\mathcal{N}, \mathcal{M})$ be a right coprime factorization of $\mathcal{D}$ in TIC.*

(i) The operator $\mathcal{N} \in TIC(U; Y)$ is (J, S)-inner if $\mathcal{N}^* J \mathcal{N} = S$.

(ii) If $\mathcal{N}$ is (J, S)-inner, then $(\mathcal{N}, \mathcal{M})$ is a (J, S)-inner right coprime factorization of $\mathcal{D}$.

(iii) If $\left[\begin{smallmatrix} \mathcal{N} \\ \mathcal{M} \end{smallmatrix}\right]$ is (I, S)-inner, i.e., if $\mathcal{N}^*\mathcal{N} + \mathcal{M}^*\mathcal{M} = S$, then $(\mathcal{N}, \mathcal{M})$ is an S-normalized right coprime factorization of $\mathcal{D}$.

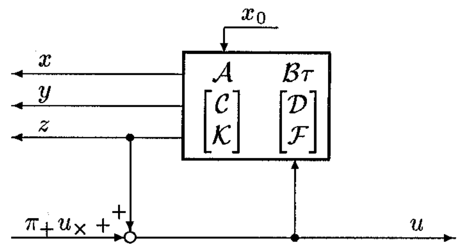

Figure 1: Optimal state feedback connection $\Psi_\times$ in Theorem 5

(iv) In each case S is called the sensitivity operator of $\mathcal{N}$ or of the factorization.

The following is our first main result:

Theorem 5 *Let $J = J^* \in \mathcal{L}(Y)$, let $S \in \mathcal{L}(U)$, $S >> 0$, and let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a J-coercive jointly stabilizable and detectable well-posed linear system on (U, H, Y). Let $x^{\text{opt}}(x_0)$, $y^{\text{opt}}(x_0)$, and $u^{\text{opt}}(x_0)$ be the optimal state, output, and control for the quadratic cost minimization problem for Ψ, and let Π be the corresponding Riccati operator (cf. Lemma 3).*

(i) Let $(\mathcal{N}, \mathcal{M})$ be a (J, S)-inner right coprime factorization of $\mathcal{D}$. Then there is a unique feedback map $\mathcal{K}$ such that $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix} = \begin{bmatrix} \mathcal{K} & (I - \mathcal{M}^{-1}) \end{bmatrix}$ is an stabilizing state feedback pair [15, Definition 8] for Ψ, and

$$
\begin{bmatrix} x^{\text{opt}}(t, x_0) \\ y^{\text{opt}}(x_0) \\ u^{\text{opt}}(x_0) \end{bmatrix} = \begin{bmatrix} \mathcal{A}^t_\times \\ \mathcal{C}_\times \\ \mathcal{K}_\times \end{bmatrix} x_0 = \begin{bmatrix} \mathcal{A}^t + \mathcal{B}\mathcal{M}\tau^t\mathcal{K} \\ \mathcal{C} + \mathcal{N}\mathcal{K} \\ \mathcal{M}\mathcal{K} \end{bmatrix} x_0
$$

is equal to the state and output of the closed loop system $\Psi_\times$ defined by

$$
\Psi_\times = \begin{bmatrix} \mathcal{A}_\times & \mathcal{B}_\times \\ \begin{bmatrix} \mathcal{C}_\times \\ \mathcal{K}_\times \end{bmatrix} & \begin{bmatrix} \mathcal{D}_\times \\ \mathcal{F}_\times \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}\tau\mathcal{M}\mathcal{K} & \mathcal{B}\mathcal{M} \\ \begin{bmatrix} \mathcal{C} + \mathcal{N}\mathcal{K} \\ \mathcal{M}\mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{N} \\ \mathcal{M} - I \end{bmatrix} \end{bmatrix}
$$

with initial value x_0, initial time zero, and zero control $u_\times$ (see Figure 1). The feedback map $\mathcal{K}$ is determined uniquely by the fact that $\mathcal{C}_\times = \mathcal{C} + \mathcal{N}\mathcal{K} \in \mathcal{L}(H; L^2(\mathbf{R}^+; Y))$, $\mathcal{K}_\times = \mathcal{M}\mathcal{K} \in \mathcal{L}(H; L^2(\mathbf{R}^+; U))$, and $\pi_+ \mathcal{N}^* J \mathcal{C}_\times = 0$. Moreover, the Riccati operator of Ψ is given by

$$
\Pi = \mathcal{C}^*_\times J \mathcal{C}_\times = (\mathcal{C} + \mathcal{N}\mathcal{K})^* J (\mathcal{C} + \mathcal{N}\mathcal{K}).
$$

(ii) If $y = \mathcal{C}_\times x_0 + \mathcal{D}_\times \pi_+ u_\times$ is the first output of the optimal closed loop system $\Psi_\times$ in (i) with initial state $x_0 \in H$ and control $u_\times \in L^2(\mathbf{R}^+; U)$ (see Figure 1), then the closed loop cost $Q_\times(x_0, u_\times)$ is

given by

$$\begin{aligned}Q_\times(x_0,u_\times) &= \langle y, Jy\rangle_{L^2(\mathbf{R}^+;Y)} \\ &= \langle x_0, \Pi x_0\rangle_H + \langle u_\times, S u_\times\rangle_{L^2(\mathbf{R}^+;Y)}.\end{aligned}$$

(iii) *If Ψ is exponentially stabilizable and detectable [15, Definition 13], and if $\mathcal{N}$ and $\mathcal{M}$ in (i) are right exponentially coprime [15, Definition 15.1], then the closed loop system $\Psi_\times$ is exponentially stable.*

We also have the following partial converse to Theorem 5:

Theorem 6 *Make the same hypothesis as in Theorem 5. Suppose that the solution to the quadratic cost minimization problem is of state feedback type in the sense that $\left[\begin{smallmatrix} y^{\mathrm{opt}}(x_0) \\ u^{\mathrm{opt}}(x_0)\end{smallmatrix}\right]$ is equal to the output of the closed loop system $\Psi_\times$ with initial value x_0, initial time 0, zero input $u_\times$, and some stabilizing state feedback pair $\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix}$. Define $\mathcal{M} = (I-\mathcal{F})^{-1}$ and $\mathcal{N} = \mathcal{D}\mathcal{M}$. Then there exists a positive invertible operator $S = S^* \in \mathcal{L}(U)$ such that $\mathcal{N}$ is (J,S)-inner, and the claim (ii) in Theorem 5 is true for this closed loop system. If, moreover, $\mathcal{N}$ and $\mathcal{M}$ are right coprime, then $(\mathcal{N},\mathcal{M})$ is a (J,S)-inner right coprime factorization of $\mathcal{D}$. This is, in particular, true whenever Ψ is exponentially stabilizable.*

The minimization problem considered in Theorem 5 leads to an inner coprime factorization. If instead we use the different cost function

$$Q_1(x_0,u) = \|y\|^2_{L^2(\mathbf{R}^+,Y)} + \|u\|^2_{L^2(\mathbf{R}^+;U)}, \qquad (3)$$

then we get a normalized coprime factorization:

Corollary 7 *Let $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be a jointly stabilizable and detectable well-posed linear system on (U,H,Y). Let $x^{\mathrm{opt}}(x_0)$, $y^{\mathrm{opt}}(x_0)$, and $u^{\mathrm{opt}}(x_0)$ be the optimal state, output, and control for the quadratic cost minimization problem described in Definition 1, but with the cost function $Q(x_0,u)$ replaced by the cost function $Q_1(x_0,u)$ in (3). If $S = S^* \in \mathcal{L}(U)$ and $(\mathcal{N},\mathcal{M})$ is an S-normalized right coprime factorization of $\mathcal{D}$ (in the sense of Definition 4), then there is a unique feedback map $\mathcal{K}$ such that $\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix} = \begin{bmatrix}\mathcal{K} & (I-\mathcal{M}^{-1})\end{bmatrix}$ is an admissible stabilizing state feedback pair for Ψ, and*

$$\begin{bmatrix} x^{\mathrm{opt}}(t,x_0) \\ y^{\mathrm{opt}}(x_0) \\ u^{\mathrm{opt}}(x_0)\end{bmatrix} = \begin{bmatrix}\mathcal{A}^t_\times \\ \mathcal{C}_\times \\ \mathcal{K}_\times\end{bmatrix} x_0 = \begin{bmatrix}\mathcal{A}^t + \mathcal{B}\mathcal{M}\tau^t\mathcal{K} \\ \mathcal{C}+\mathcal{N}\mathcal{K} \\ \mathcal{M}\mathcal{K}\end{bmatrix} x_0$$

is equal to the state and output of the closed loop system $\Psi_\times$ defined by

$$\Psi_\times = \begin{bmatrix}\mathcal{A}_\times & \mathcal{B}_\times \\ \begin{bmatrix}\mathcal{C}_\times \\ \mathcal{K}_\times\end{bmatrix} & \begin{bmatrix}\mathcal{D}_\times \\ \mathcal{F}_\times\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\mathcal{A}+\mathcal{B}\tau\mathcal{M}\mathcal{K} & \mathcal{B}\mathcal{M} \\ \begin{bmatrix}\mathcal{C}+\mathcal{N}\mathcal{K} \\ \mathcal{M}\mathcal{K}\end{bmatrix} & \begin{bmatrix}\mathcal{N} \\ \mathcal{M}-I\end{bmatrix}\end{bmatrix}$$

with initial value x_0, initial time zero, and zero input $u_\times$ (see Figure 1). The feedback map $\mathcal{K}$ is determined uniquely by the fact that $\mathcal{C}_\times = \mathcal{C} + \mathcal{N}\mathcal{K} \in \mathcal{L}(H; L^2(\mathbf{R}^+;Y))$, $\mathcal{K}_\times = \mathcal{M}\mathcal{K} \in \mathcal{L}(H; L^2(\mathbf{R}^+;U))$, and

$$\pi_+(\mathcal{N}^*\mathcal{C}_\times + \mathcal{M}^*\mathcal{K}_\times) = 0.$$

Moreover, the Riccati operator of Ψ is given by

$$\begin{aligned}\Pi &= \mathcal{C}^*_\times\mathcal{C}_\times + \mathcal{K}^*_\times\mathcal{K}_\times \\ &= (\mathcal{C}+\mathcal{N}\mathcal{K})^*(\mathcal{C}+\mathcal{N}\mathcal{K}) + (\mathcal{M}\mathcal{K})^*(\mathcal{M}\mathcal{K}).\end{aligned}$$

3 Regular Systems

In order to discuss the algebraic Riccati equation we need a regularity notion introduced by Weiss [18]:

Definition 8

(i) *An operator $\mathcal{D} \in TIC(U;Y)$ is regular if the strong limit*

$$Dv_0 := \lim_{\lambda\to+\infty} \widehat{\mathcal{D}}(\lambda)v_0$$

exists for every $v_0 \in V$; here λ tends to $+\infty$ along the positive real axis and $\widehat{\mathcal{D}}$ is the transfer function (the distribution Laplace transform) of $\mathcal{D}$.

(ii) *The operator $D\colon V \to Y$ defined above is called the feed-through operator of $\mathcal{D}$.*

(iii) *A regular operator $\mathcal{D} \in TIC(V;Y)$ is called strictly proper if its feed-through operator vanishes.*

(iv) *We say that $\mathcal{D}$ is regular together with its adjoint iff, in addition to (i), the strong limit $\lim_{\lambda\to+\infty}\widehat{\mathcal{D}}^*(\lambda)y_0$ exists for every $y_0 \in Y$. (This limit is equal to D^*y_0 whenever it exists.)*

The input/state/output relation of a well-posed linear system can be always written in the form (for smooth inputs u)

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= N(x(t),u(t)), \quad t \geq 0, \\ x(0) &= x_0. \end{aligned} \qquad (4)$$

Here A is the generator of $\mathcal{A}$, $B \in \mathcal{L}(U;\mathrm{range}(B))$ (where $\mathrm{range}(B) \supset H$) is the control operator, and $N \in \mathcal{L}(\mathrm{dom}(N);Y))$ (where $\mathrm{dom}(N) \subset H \times U$) is the combined observation and feed-through operator. In general it impossible to write N in the more familiar form $N = Cx + Du$, due to the structure of $\mathrm{dom}(N)$ (the domain of $x \mapsto N(x,u)$ depends on u, and the domain of $u \mapsto N(x,u)$ depends on x). However, if the system is *regular*, then is possible to split N into $N(x,u) = Cx + Du$, and we get

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{5}$$

where A and B are as above, $C \in \mathcal{L}(\mathrm{dom}(C); Y)$ (where $\mathrm{dom}(A) \subset \mathrm{dom}(C) \subset H$) is (the Weiss extension of) the observation operator, and D is the feed-through operator. We refer to the operators $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$as the *generating operators* of Ψ. If the full system in Figure 1 is regular, then there are, of course, two more generating operators, namely (the Weiss extension of) a feedback operator $K \in \mathcal{L}(\mathrm{dom}(K); Y)$ (where $\mathrm{dom}(A) \subset \mathrm{dom}(K) \subset H$) induced by $\mathcal{K}$ and a feed-through operator F induced by $\mathcal{F}$.

4 The Algebraic Riccati Equation

Theorem 9 *Make the same assumptions and introduce the same notations as in Theorem 5. Extend the system Ψ into*

$$\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{F} \end{bmatrix} \end{bmatrix}$$

by adding the optimal state feedback pair $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix}$, and suppose that this extended system is regular together with its adjoint. Denote the generating operators of Ψ by the same letters as the corresponding operators. Then it is possible to normalize the feed-through operators F of $\mathcal{F}$ to $F = 0$. With this normalization,

(i) the feedback operator K is given by

$$Kx = -S^{-1}\left(B^*\Pi + D^*JC\right)x, \quad x \in \mathrm{dom}(A);$$

(ii) the Riccati operator Π satisfies the algebraic Riccati equation

$$\begin{aligned} &\langle Ax_0, \Pi x_1\rangle_H + \langle x_0, \Pi A x_1\rangle_H + \langle Cx_0, JCx_1\rangle_Y \\ &\quad = \left\langle \left(B^*\Pi + D^*JC\right)x_0, S^{-1}\left(B^*\Pi + D^*JC\right)x_1\right\rangle_U, \\ &\quad x_0, x_1 \in \mathrm{dom}(A). \end{aligned}$$

This theorem differs from the corresponding classical result in the sense that it contains a new parameter, namely the sensitivity operator S. This operator is always invertible, and it can be computed as follows:

Theorem 10 *Make the same assumptions and introduce the same notations as in Theorem 9.*

(i) For all $u_0 \in U$, we have

$$Su_0 = D^*JDu_0 + \lim_{\alpha\to\infty} B^*\Pi(\alpha I - A)^{-1}Bu_0.$$

In particular, $S = D^*JD$ iff the limit above is zero for all $u_0 \in U$.

(ii) If for some $u_0 \in U$ it is true that $Bu_0 \in H$, then

$$Su_0 = D^*JDu_0.$$

(iii) The difference $S - D^*JD$ is positive [negative] definite whenever Π is positive [negative] definite on the reachable subspace.

5 Applications: The Bounded and Positive Real Lemmas

By applying the preceding theory we can derive the first available versions of the strict bounded and positive (real) lemmas for general well-posed linear systems. In these lemmas we need a cost function containing both the output y and the control u. To get such a cost function we adjoin a copy of the control to the output, i.e., we study the augmented system

$$\Psi_{\mathrm{aug}} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ I \end{bmatrix} \end{bmatrix}. \tag{6}$$

To get the positive real lemma we let Ψ be stable, and choose the cost operator J for Ψ_{aug} to be

$$J = \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 I \end{bmatrix},$$

where γ is a real constant. Then the extended system is J-coercive if and only if the input/output map $\mathcal{D}$ satisfies

$$\|\mathcal{D}\|_{TIC(U;Y)} < \gamma. \tag{7}$$

Thus, Theorem 5 applies iff (7) holds. In this case the formulae in Theorem 5 applied to Ψ_{aug} become

$$\begin{aligned} &\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}, \qquad \gamma^2\mathcal{M}^*\mathcal{M} - \mathcal{N}^*\mathcal{N} = S, \\ &\mathcal{K} = S^{-1}\pi_+\mathcal{N}^*\mathcal{C}, \qquad \gamma^2\pi_+\mathcal{M}^*\mathcal{K}_\times = \pi_+\mathcal{N}^*\mathcal{C}_\times, \\ &\begin{bmatrix} \mathcal{C}_\times \\ \mathcal{K}_\times \end{bmatrix} = \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix}\mathcal{K} = \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} S^{-1}\pi_+\mathcal{N}^*\mathcal{C}, \\ &\Pi = \gamma^2\mathcal{K}_\times^*\mathcal{K}_\times - \mathcal{C}_\times^*\mathcal{C}_\times = -\mathcal{C}^*\left(I + \mathcal{N}S^{-1}\pi_+\mathcal{N}^*\right)\mathcal{C}. \end{aligned}$$

The equations in Theorem 9 applied to Ψ_{aug} become (for x_0 and $x_1 \in \mathrm{dom}(A)$)

$$\begin{aligned} &Kx_0 = -S^{-1}\left(B^*\Pi - D^*C\right)x_0, \\ &\langle Ax_0, \Pi x_1\rangle_H + \langle x_0, \Pi A x_1\rangle_H \\ &\qquad = \langle Cx_0, Cx_1\rangle_Y + \langle Kx_0, SKx_1\rangle_U. \end{aligned}$$

Observe that the parameter γ enters these equations only through the sensitivity operator S which is given by the strong limit (for each fixed $u_0 \in U$)

$$Su_0 = \left(\gamma^2 I - D^*D\right)u_0 + \lim_{\alpha\to\infty} B^*\Pi(\alpha I - A)^{-1}Bu_0.$$

We remark that in our setting Π is negative definite; to get the standard setting where Π is positive [2, Theorem 3.7.1] we must replace J by $-J$ and maximize instead of minimize. This will replace S by $-S$ and Π by $-\Pi$.

The *strictly positive (real) lemma* is a statement about a stable system $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ on (U, H, U) (i.e., the output space of this system is equal to its input space). The input/output map $\mathcal{D}$ of Ψ is strictly positive iff

$$\int_{\mathbf{R}^+} \left(\langle (\mathcal{D}\pi_+ u)(s), u(s) \rangle_U \right. \\ \left. + \langle u(s), (\mathcal{D}\pi_+ u)(s) \rangle_U \right) ds \geq \epsilon \|u\|^2_{L^2(\mathbf{R}^+;U)}$$

for all $u \in L^2(\mathbf{R}^+; U)$ and some $\epsilon > 0$. Clearly, $\mathcal{D}$ is strictly positive iff the extended system Ψ_{aug} is J-coercive with respect to the operator

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Thus, Theorem 5 applies with this J to Ψ_{aug} iff $\mathcal{D}$ is strictly positive. The formulae of Theorem 5 become in this case

$$\begin{aligned} &\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}, \qquad \mathcal{M}^*\mathcal{N} + \mathcal{N}^*\mathcal{M} = S, \\ &\mathcal{K} = -S^{-1}\pi_+\mathcal{M}^*\mathcal{C}, \qquad \pi_+ \left(\mathcal{M}^*\mathcal{C}_\times + \mathcal{N}^*\mathcal{K}_\times \right) = 0, \\ &\begin{bmatrix} \mathcal{C}_\times \\ \mathcal{K}_\times \end{bmatrix} = \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} \mathcal{K} = \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} - \begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} S^{-1}\pi_+\mathcal{M}^*\mathcal{C}, \\ &\Pi = \mathcal{K}^*_\times\mathcal{C}_\times + \mathcal{C}^*_\times\mathcal{K}_\times = -\mathcal{K}^*S\mathcal{K} = -\mathcal{C}^*\mathcal{M}S^{-1}\pi_+\mathcal{M}^*\mathcal{C}. \end{aligned}$$

The equations in Theorem 9 applied to Ψ_{aug} become (for x_0 and $x_1 \in \text{dom}(A)$)

$$\begin{aligned} K x_0 &= -S^{-1} \left(B^*\Pi + C \right) x_0, \\ \langle A x_0, \Pi x_1 \rangle_H + \langle x_0, \Pi A x_1 \rangle_H &= \langle K x_0, S K x_1 \rangle_U, \end{aligned}$$

and the sensitivity operator S is given by the strong limit (for each fixed $u_0 \in U$)

$$S u_0 = (D + D^*)\, u_0 + \lim_{\alpha \to \infty} B^*\Pi(\alpha I - A)^{-1} B u_0.$$

Again Π is negative; to get a positive Π we should change the sign of J and maximize instead of minimize [2, Problem 3.25].

6 Proofs and Extensions

We refer the reader to [13, 14, 15, 16] for more details and proofs. In the stable case some of the results presented here were obtained independently by Martin and George Weiss [22]. These results were first presented in [10] and [21].

See [9] and [20] for examples illuminating the correction term to the sensitivity operator S Theorem 10.

A converse of Theorem 9 has been proved in [5].

Extensions to the full information H^∞ problem are given in [16, 17].

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