Coprime Factorizations and Well-Posed Linear Systems

Olof J. Staffans
Åbo Akademi University
Department of Mathematics
FIN-20500 Åbo, Finland
Olof.Staffans@abo.fi
http://www.abo.fi/~staffans/

Abstract

We study the basic notions related to the stabilization of an infinite-dimensional well-posed linear system in the sense of Salamon and Weiss. We first introduce an appropriate stabilizability and detectability notion, and show that if a system is jointly stabilizable and detectable then its transfer function has a doubly coprime factorization in H^∞. The converse is also true: every function with a doubly coprime factorization in H^∞ is the transfer function of a jointly stabilizable and detectable well-posed linear system. We show further that a stabilizable and detectable system is stable if and only if its input/output map is stable. Finally, we construct a dynamic, possibly non-well-posed, stabilizing compensator. The notion of stability that we use is the natural one for the quadratic cost minimization problem, and it does not imply exponential stability.

Keywords: Stabilizability, detectability, input/output stability, dynamic stabilization.

1 Notations

$\mathcal{L}(U;Y)$, $\mathcal{L}(U)$: The set of bounded linear operators from U into Y or from U into itself, respectively.

$\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{R}^-$: $\mathbf{R} = (-\infty,\infty)$, $\mathbf{R}^+ = [0,\infty)$, and $\mathbf{R}^- = (-\infty, 0]$.

$L^2(J;U)$: The set of U-valued L^2-functions on the interval J.

$L^2_\omega(J;U)$: $L^2_\omega(J;U) = \{ u \in L^2_{\mathrm{loc}}(J;U) \mid (t \mapsto \mathrm{e}^{-\omega t}u(t)) \in L^2(J;U) \}$.

$TIC_\omega(U;Y)$, $TI_\omega(U)$: The set of bounded, linear, time-invariant, and causal operators from $L^2_\omega(\mathbf{R};U)$ into $L^2_\omega(\mathbf{R};Y)$, or from $L^2_\omega(\mathbf{R};U)$ into itself. $TIC(U;Y) = TIC_0(U;Y)$ and $TIC(U) = TIC_0(U)$.

τ^t: The time shift group $\tau^t u(s) = u(t+s)$ (this is a left-shift when $t>0$ and a right-shift when $t<0$).

π_J: $(\pi_J u)(s) = u(s)$ if $s \in J$ and $(\pi_J u)(s) = 0$ if $s \notin J$. Here J is a subset of $\mathbf{R}$.

π_+, π_-: $\pi_+ = \pi_{\mathbf{R}^+}$ and $\pi_- = \pi_{\mathbf{R}^-}$.

2 Well-Posed Linear Systems

We begin with a short presentation of the Salamon-Weiss class of well-posed linear systems. This theory has been developed in [1, 5, 10, 11, 12, 16, 17, 18, 19] (and many other papers), and we refer the reader to these sources for additional reading.

Definition 1 *Let U, H, and Y be Hilbert spaces, and let $\omega \in \mathbf{R}$. A (causal) ω-bounded well-posed linear system on (U,H,Y) is a quadruple $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are bounded linear operators of the following type:*

(i) $\mathcal{A}^t\colon H \to H$ is a strongly continuous semigroup of bounded linear operators on H satisfying $\sup_{t\in\mathbf{R}^+} \|\mathrm{e}^{-\omega t}\mathcal{A}^t\| < \infty$;

(ii) $\mathcal{B}\colon L^2_\omega(\mathbf{R};U) \to H$ satisfies $\mathcal{A}^t\mathcal{B}u = \mathcal{B}\tau^t\pi_- u$ for all $u \in L^2_\omega(\mathbf{R};U)$ and $t \in \mathbf{R}^+$;

(iii) $\mathcal{C}\colon H \to L^2_\omega(\mathbf{R};Y)$ satisfies $\mathcal{C}\mathcal{A}^t x = \pi_+\tau^t\mathcal{C}x$ for all $x \in H$ and $t \in \mathbf{R}^+$;

(iv) $\mathcal{D}\colon L^2_\omega(\mathbf{R};U) \to L^2_\omega(\mathbf{R};Y)$ satisfies $\tau^t\mathcal{D}u = \mathcal{D}\tau^t u$, $\pi_-\mathcal{D}\pi_+ u = 0$, and $\pi_+\mathcal{D}\pi_- u = \mathcal{C}\mathcal{B}u$ for all $u \in L^2_\omega(\mathbf{R};U)$ and $t \in \mathbf{R}$.

The system Ψ is stable if it is ω-bounded with $\omega = 0$, and it is strongly stable if, in addition, $\mathcal{A}^t x \to 0$ as $t \to \infty$ for all $x \in H$. It is exponentially stable iff it is ω-bounded for some $\omega < 0$.

The different components of Ψ are named as follows: U is the input space, H the state space, Y the output space, $\mathcal{A}$ the semigroup, $\mathcal{B}$ the controllability map, $\mathcal{C}$ the observability map, and $\mathcal{D}$ the input/output map of Ψ.

It is not difficult to show that an ω-bounded system is α-bounded for all $\alpha > \omega$.

Definition 2 *We call Ψ a well-posed linear system on (U,H,Y) iff it is an ω-bounded well-posed linear system* *on (U, H, Y) for some $\omega \in \mathbf{R}$. The infimum of all the numbers ω for which Ψ is ω-bounded is the exponential growth rate of Ψ. Thus, Ψ is exponentially stable iff its exponential growth rate is negative.*

As Salamon [12] and Weiss [16, 17, 18] have shown, the growth rate of a system Ψ is equal to the growth rate of its semigroup:

Lemma 3 *The exponential growth rate of a well-posed linear system Ψ is equal to the exponential growth rate $\omega = \lim_{t\to\infty} t^{-1}\log(\|\mathcal{A}^t\|)$ of its semigroup. In particular, Ψ is exponentially stable iff its semigroup is exponentially stable.*

The axioms listed in Definition 2 describe standard properties of the corresponding maps induced by "classical" system of the type

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \ge T, \\ x(T) &= x_T. \end{aligned} \tag{1}$$

Here A is the generator of a semigroup $\mathcal{A}$ on a Hilbert space H, $B \in \mathcal{L}(U;H)$, $C \in \mathcal{L}(H;Y)$, and $D \in \mathcal{L}(U;Y)$, where the input space U and the output space Y are Hilbert spaces. Moreover, T is a given initial time and x_T a given initial value. We call u the control, x the state, y the output (or observation), A the generator, B the control operators, C the observation operator, and D the feed-through operator of this classical system. The state x is required to be a strong solution of (1), i.e., the state x and output y are given by

$$x(t) = \mathcal{A}^t x_T + \int_T^t \mathcal{A}^{t-s}Bu(s)\,ds, \quad t \ge T$$

$$y(t) = C\mathcal{A}^t x_T + \int_T^t C\mathcal{A}^{t-s}Bu(s)\,ds + Du(t), \quad t \ge T.$$

In this case we define $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ by

$$\begin{aligned} \mathcal{B}u &= \int_{-\infty}^0 \mathcal{A}^{-s}Bu(s)\,ds, \\ \mathcal{C}x &= \left(t \mapsto C\mathcal{A}^t x, \quad t \in \mathbf{R}^+\right), \\ \mathcal{D}u &= \left(t \mapsto \int_{-\infty}^t C\mathcal{A}^{t-s}Bu(s)\,ds + Du(t), \quad t \in \mathbf{R}\right). \end{aligned}$$

Thus, $\mathcal{B}$ is the mapping from the control $u \in L^2_\omega(\mathbf{R}^-;U)$ to the final state $x(0) \in H$ (take $T = -\infty$, $x_T = 0$, and $t = 0$), $\mathcal{C}$ is the mapping from the initial state $x_0 \in H$ to the output $y \in L^2_\omega(\mathbf{R}^+;Y)$ (take $T = 0$ and $u = 0$), and $\mathcal{D}$ is the mapping from the control $u \in L^2_\omega(\mathbf{R};U)$ to the output $y \in L^2_\omega(\mathbf{R};Y)$ (take $T = -\infty$ and $x_T = 0$).

As a matter of fact, every well-posed linear system can be written in a form similar to (1), namely

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= N(x(t), u(t)), \quad t \ge T, \\ x(T) &= x_T. \end{aligned} \tag{2}$$

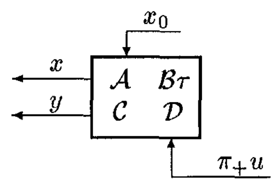

Figure 1: Input/state/output diagram of Ψ

where A, B, and N are unbounded. This equation cannot always be written in the form (1) due to the nature of the domain of N. The system is *regular* if it can be written in the form (1) with unbounded A, B, and C (but bounded D).

The definitions of the *controlled state* and *output* of a well-posed linear system are natural extensions of the state and output of (1):

Definition 4 *Let $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ be a well-posed linear system on (U, H, Y), and let $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$. The controlled state $x(t)$ at time $t \in \mathbf{R}^+$ and the output y of Ψ with initial time zero, initial value x_0, and control u, are given by*

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{A}^t & \mathcal{B}\tau^t \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x_0 \\ \pi_+ u \end{bmatrix} = \begin{bmatrix} \mathcal{A}^t x_0 + \mathcal{B}\tau^t \pi_+ u \\ \mathcal{C}x_0 + \mathcal{D}\pi_+ u \end{bmatrix}.$$

We use diagrams of the type drawn in Figure 1 to represent the relation between the state x, the output y, the initial value x_0, and the control u of Ψ in the initial value setting with initial time zero. In our diagrams we throughout use the following conventions:

(i) Initial states and controls enter at the top or bottom, and they are acted on by all the operators located in the column to which they are attached. In particular, note that x_0 is attached to the first column and u to the second.

(ii) Final states and outputs leave to the left or right, and they are the sums of all the elements in the row to which they are attached. In particular, note that x is attached to the top row, and y to the bottom row.

3 Feedback, Stabilizability and Detectability

The most basic feedback connection is the notion of a (static) output feedback, drawn in Figure 2. Here L is a bounded linear operator from the output space into the input space. This feedback configuration with initial time zero, initial value x_0, and control v, gives us the following formulas for the effective input u, the state $x(t)$ at time $t \ge 0$, the output y, and the feedback

0-7803-4394-8/98 $10.00 (c) 1998 IEEE

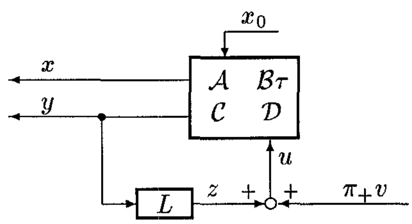

Figure 2: Static output feedback

signal z:

$$\begin{aligned} u &= z + \pi_+ v, \\ x(t) &= \mathcal{A}^t x_0 + \mathcal{B}\tau^t u, \\ y &= \mathcal{C}x_0 + \mathcal{D}u, \\ z &= Ly, \end{aligned}$$

which formally can be solved as

$$\begin{aligned} u &= (I - L\mathcal{D})^{-1}(L\mathcal{C}x_0 + \pi_+ v), \\ x(t) &= \left(\mathcal{A}^t + \mathcal{B}\tau^t L (I - \mathcal{D}L)^{-1}\mathcal{C}\right)x_0 \\ &\quad + \mathcal{B}(I - L\mathcal{D})^{-1}\tau^t \pi_+ v, \qquad (3) \\ y &= (I - \mathcal{D}L)^{-1}(\mathcal{C}x_0 + \mathcal{D}\pi_+ v), \\ z &= (I - L\mathcal{D})^{-1} L (\mathcal{C}x_0 + \mathcal{D}\pi_+ v). \end{aligned}$$

We say that the feedback operator L is admissible whenever these equations are valid:

Definition 5 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a well-posed linear system on (U, H, Y). Then $L \in \mathcal{L}(Y;U)$ is called an admissible output feedback operator for Ψ iff the operator $I - L\mathcal{D}$ has an inverse in $TIC_\alpha(U)$ for some $\alpha \in \mathbf{R}$, or equivalently, iff the operator $I - \mathcal{D}L$ has an inverse in $TIC_\alpha(Y)$ for some $\alpha \in \mathbf{R}$.*

As Weiss [19, Section 6] proved, x and y in (3) can be interpreted as the state and output of another well-posed linear system:

Proposition 6 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a well-posed linear system on (U, H, Y), and let $L \in \mathcal{L}(Y;U)$ be an admissible output feedback operator for Ψ. Then the system*

$$\begin{aligned} \Psi_L &= \begin{bmatrix} \mathcal{A}_L & \mathcal{B}_L\tau \\ \mathcal{C}_L & \mathcal{D}_L \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A} + \mathcal{B}\tau L (I - \mathcal{D}L)^{-1}\mathcal{C} & \mathcal{B}(I - L\mathcal{D})^{-1}\tau \\ (I - \mathcal{D}L)^{-1}\mathcal{C} & \mathcal{D}(I - L\mathcal{D})^{-1} \end{bmatrix} \end{aligned}$$

is another well-posed linear system on (U, H, Y). We call this system the closed loop system with feedback operator L. In the initial value setting with initial time zero, initial value x_0, and control v, the controlled state $x(t)$ at time t and the output y of Ψ_L are given by (3).

We remark that if we in the classical system (1) replace u by $u = Ly + v$, then we get a new well-defined system of the same type iff $I - DL$ is invertible, or equivalently, iff $I - LD$ is invertible. In the new system the operators $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ have been replaced by

$$\begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix} = \begin{bmatrix} A + BL(I - DL)^{-1}C & B(I - LD)^{-1} \\ (I - DL)^{-1}C & D(I - LD)^{-1} \end{bmatrix}$$

Observe the striking similarity between this formula and the one given in Proposition 6.

Definition 7 *The operator $L \in \mathcal{L}(Y;U)$ is a (strongly) [exponentially] stabilizing output feedback operator for Ψ iff L is an admissible output feedback operator for Ψ and the resulting closed loop system Ψ_L is (strongly) [exponentially] stable.*

The notion of a *state feedback* can formally be reduced to the notion of an output feedback. Intuitively, a state feedback means that an additional output is created, and this output is then fed back into the input, as shown in Figure 3. In this figure the original system is represented by $\left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$. We find two additional components, namely a new observability map $\mathcal{K}$ (from the initial state to the new output) and a new input/output map $\mathcal{F}$ (from the original input to the new output). The pair $\left[\mathcal{K} \quad \mathcal{F}\right]$ is *admissible* if the resulting system is well-posed, i.e., if $\left[0 \quad I\right]$ is an admissible output feedback operator for the extended system:

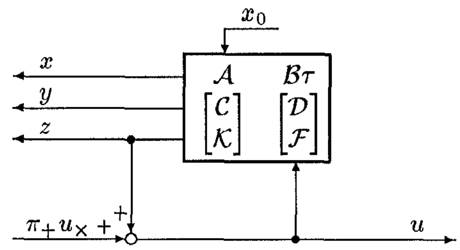

Figure 3: State feedback connection

Definition 8 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a well-posed linear system on (U, H, Y). The pair $\left[\mathcal{K} \quad \mathcal{F}\right]$ is an admissible state feedback pair for Ψ iff the extended system*

$$\Psi_{\mathrm{SF}} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{F} \end{bmatrix} \end{bmatrix}$$

is a well-posed linear system on $(U, H, Y \times U)$ and $\left[0 \quad I\right]$ is an admissible output feedback operator for Ψ_{SF}, i.e., $I - \mathcal{F}$ has an inverse in $TIC_\omega(U)$ for some $\omega \in \mathbf{R}$. It is (strongly) [exponentially] stabilizing if the resulting closed loop system is (strongly) [exponentially] stable.

The notion of an *output injection* is analogous. In this case a new input is created, into which we feed the original output y plus a new perturbation $w^\times$, as shown in Figure 4. The original system is still represented

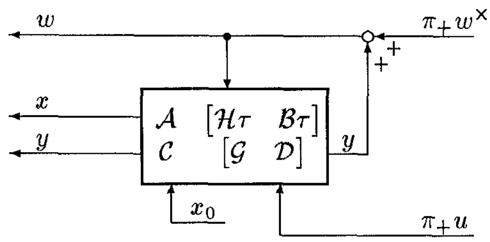

Figure 4: Output injection connection

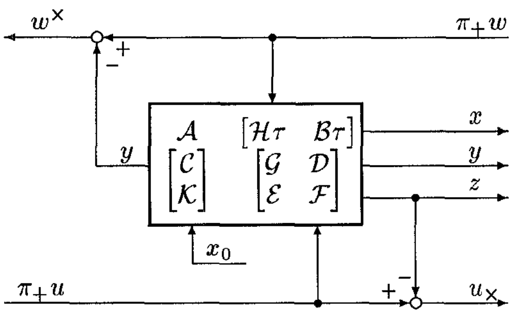

Figure 5: The extended system

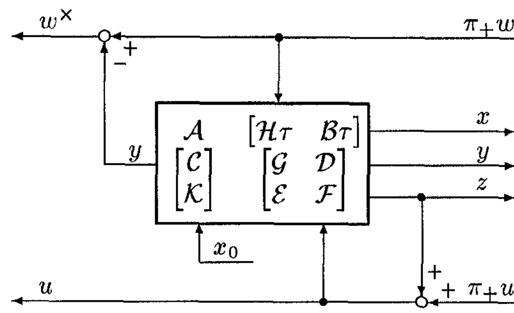

Figure 6: Right coprime factor $\Psi_\times$

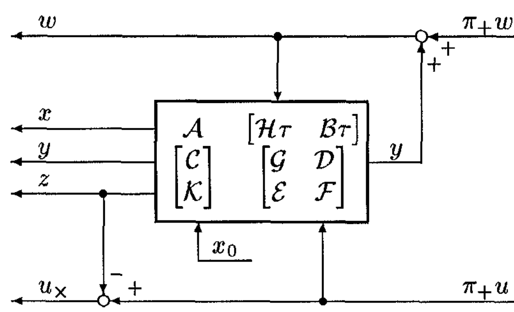

Figure 7: Left coprime factor $\Psi^\times$

by $[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}]$. In this figure we find a new controllability map $\mathcal{H}$ (from the new input to the state) and a new input/output map $\mathcal{G}$ (from the new input to the original output). The pair $[\begin{smallmatrix}\mathcal{H}\\ \mathcal{G}\end{smallmatrix}]$ is *admissible* if the resulting system is well-posed:

Definition 9 *Let $\Psi = [\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}]$ be a well-posed linear system on (U, H, Y). The pair $[\begin{smallmatrix}\mathcal{H}\\ \mathcal{G}\end{smallmatrix}]$ is an admissible output injection pair for Ψ iff the extended system*

$$\Psi_{\mathrm{OI}} = \begin{bmatrix}\mathcal{A} & [\mathcal{H} \;\; \mathcal{B}]\\ \mathcal{C} & [\mathcal{G} \;\; \mathcal{D}]\end{bmatrix}$$

is a well-posed linear system on $(Y \times U, H, Y)$, and $[\begin{smallmatrix}I\\ 0\end{smallmatrix}]$ is an admissible output feedback operator for Ψ_{OI}, i.e., $I - \mathcal{G}$ has an inverse in $TIC_\omega(Y)$ for some $\omega \in \mathbf{R}$. It is (strongly) [exponentially] stabilizing if the resulting closed loop system is (strongly) [exponentially] stable.

In the sequel we shall need to study a case where we at the same time want to add both a state feedback pair $[\mathcal{K} \;\; \mathcal{F}]$ and an output injection pair $[\begin{smallmatrix}\mathcal{H}\\ \mathcal{G}\end{smallmatrix}]$ to a given system $[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}]$. If we try to write a figure similar to Figures 3 and 4, we immediately observe that we need one more input/output map $\mathcal{E}$ (from the output injection input to the state feedback output); see Figure 5. This operator need not always exist, and this forces us to introduce still another definition:

Definition 10 *Let $\Psi = [\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}]$ be a well-posed linear system on (U, H, Y). The pairs $[\mathcal{K} \;\; \mathcal{F}]$ and $[\begin{smallmatrix}\mathcal{H}\\ \mathcal{G}\end{smallmatrix}]$ are called jointly admissible state feedback and output injection pairs for Ψ iff $[\mathcal{K} \;\; \mathcal{F}]$ is an admissible state feedback pair for Ψ, $[\begin{smallmatrix}\mathcal{H}\\ \mathcal{G}\end{smallmatrix}]$ is an admissible output injection pair for Ψ, and in addition, there exists a operator $\mathcal{E}$, called the interaction operator, such that and the combined extended system*

$$\Psi_{\mathrm{ext}} = \begin{bmatrix}\mathcal{A} & \begin{bmatrix}\mathcal{H} & \mathcal{B}\end{bmatrix}\\ \begin{bmatrix}\mathcal{C}\\ \mathcal{K}\end{bmatrix} & \begin{bmatrix}\mathcal{G} & \mathcal{D}\\ \mathcal{E} & \mathcal{F}\end{bmatrix}\end{bmatrix}$$

is a well-posed linear system on $(Y \times U, H, Y \times U)$.

Lemma 11 *Let $\Psi = [\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}]$ be a well-posed linear system on (U, H, Y). Then the following conditions are equivalent:*

(i) The pairs $[\mathcal{K} \;\; \mathcal{F}]$ and $[\begin{smallmatrix}\mathcal{H}\\ \mathcal{G}\end{smallmatrix}]$ are jointly admissible state feedback and output injection pairs with interaction operator $\mathcal{E}$;

(ii) The system Ψ_{ext} in Definition 10 is a well-posed linear system on $(Y \times U, H, Y \times U)$, and both $[\begin{smallmatrix}0 & 0\\ 0 & I\end{smallmatrix}]$ and $[\begin{smallmatrix}I & 0\\ 0 & 0\end{smallmatrix}]$ are admissible output feedback operators for Ψ_{ext}.

(iii) The system Ψ_{ext} in Definition 10 is a well-posed linear system on $(Y \times U, H, Y \times U)$, and $I - \mathcal{F}$ and $I - \mathcal{G}$ have inverses in $TIC_\omega(U)$ respectively $TIC_\omega(Y)$ for some $\omega \in \mathbf{R}$.

So far we have only looked at the *joint admissibility* of state feedback and output injection pairs. If the resulting closed loop systems drawn in Figures 6 and 7 are stable, then we call these pairs *jointly stabilizing*:

Definition 12 *The pairs $[\mathcal{K} \;\; \mathcal{F}]$ and $[\begin{smallmatrix}\mathcal{H}\\ \mathcal{G}\end{smallmatrix}]$ are called jointly (strongly) [exponentially] stabilizing state feedback and output injection pairs for Ψ if they are jointly admissible state feedback and output injection pairs* *with some interaction operator $\mathcal{E}$, and both the closed loop systems $\Psi_\times$ and $\Psi^\times$ corresponding to the feedback operators $\left[\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}I&0\\0&0\end{smallmatrix}\right]$, respectively, are (strongly) [exponentially] stable (see Figures 6 and 7, respectively).*

Definition 13 *Let Ψ be a well-posed linear system.*

(i) Ψ is (strongly) [exponentially] stabilizable iff there exists a (strongly) [exponentially] stabilizing state feedback pair for Ψ.

(ii) Ψ is (strongly) [exponentially] detectable iff there exists a (strongly) [exponentially] stabilizing output injection pair for Ψ.

(iii) Ψ is jointly (strongly) [exponentially] stabilizable and detectable iff there exist some jointly (strongly) [exponentially] stabilizing state feedback and output injection pairs for Ψ.

We do not know if it is possible for a system to be both stabilizable and detectable without being jointly stabilizable and detectable.

There is a simple connection between stability, detectability, and input/output-stability:

Theorem 14 *A (strongly) [exponentially] stabilizable and detectable well-posed linear system is (strongly) [exponentially] stable iff it is input/output stable, i.e., its input/output map belongs to $TIC(U;Y)$.*

4 Coprime Factorizations

Definition 15 *Let U, Y, and Z be Hilbert spaces, and let $\omega \in \mathbf{R}$.*

(i) The operators $\mathcal{N} \in TIC_\omega(U;Y)$ and $\mathcal{M} \in TIC_\omega(U;Z)$ are right ω-coprime iff there exist operators $\widetilde{\mathcal{Y}} \in TIC_\omega(Y;U)$ and $\widetilde{\mathcal{X}} \in TIC_\omega(Z;U)$ that together with $\mathcal{N}$ and $\mathcal{M}$ satisfy the Bezout identity

$$\widetilde{\mathcal{Y}}\mathcal{N} + \widetilde{\mathcal{X}}\mathcal{M} = I$$

in $TIC_\omega(U)$. In the case where $\omega = 0$ we call $\mathcal{N}$ and $\mathcal{M}$ right coprime, and in the case where $\omega < 0$ we call $\mathcal{N}$ and $\mathcal{M}$ exponentially right coprime.

(ii) The operators $\widetilde{\mathcal{N}} \in TIC_\omega(U;Y)$ and $\widetilde{\mathcal{M}} \in TIC_\omega(Z;Y)$ are left ω-coprime iff there exist operators $\mathcal{Y} \in TIC_\omega(Y;U)$ and $\mathcal{X} \in TIC_\omega(Y;Z)$ that together with $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}}$ satisfy the Bezout identity

$$\widetilde{\mathcal{N}}\mathcal{Y} + \widetilde{\mathcal{M}}\mathcal{X} = I$$

in $TIC_\omega(Y)$. In the case where $\omega = 0$ we call $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}}$ left coprime, and in the case where $\omega < 0$ we call $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}}$ exponentially left coprime.

Thus, $\mathcal{N}$ and $\mathcal{M}$ are right coprime iff $\left[\begin{smallmatrix}\mathcal{N}\\ \mathcal{M}\end{smallmatrix}\right]$ has a left inverse in $TIC(Y \times Z;U)$, and $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}}$ are left coprime iff $[\widetilde{\mathcal{N}} \quad \widetilde{\mathcal{M}}]$ has a right inverse in $TIC(Y;U \times Z)$.

Definition 16 *Let U and Y be Hilbert spaces, and let $\mathcal{D} \in TIC_\alpha(U;Y)$ for some $\alpha \in \mathbf{R}$.*

(i) The pair $(\mathcal{N},\mathcal{M})$ is a right [exponentially] coprime factorization of $\mathcal{D}$ if $\mathcal{N} \in TIC(U;Y)$ and $\mathcal{M} \in TIC(U)$ are right [exponentially] coprime, $\mathcal{M}$ has an inverse in $TIC_\alpha(U)$, and $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$.

(ii) The pair $(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}})$ is a left [exponentially] coprime factorization of $\mathcal{D}$ if $\widetilde{\mathcal{M}} \in TIC(Y)$ and $\widetilde{\mathcal{N}} \in TIC(U;Y)$ are left [exponentially] coprime, $\widetilde{\mathcal{M}}$ has an inverse in $TIC_\alpha(Y)$, and $\mathcal{D} = \widetilde{\mathcal{M}}^{-1}\widetilde{\mathcal{N}}$.

(iii) A doubly [exponentially] coprime factorization of $\mathcal{D}$ consists of eight operators in TIC (of the appropriate dimensions) satisfying

$$\begin{bmatrix}\widetilde{\mathcal{M}} & \widetilde{\mathcal{N}}\\ -\widetilde{\mathcal{Y}} & \widetilde{\mathcal{X}}\end{bmatrix}\begin{bmatrix}\mathcal{X} & -\mathcal{N}\\ \mathcal{Y} & \mathcal{M}\end{bmatrix} = \begin{bmatrix}\mathcal{X} & -\mathcal{N}\\ \mathcal{Y} & \mathcal{M}\end{bmatrix}\begin{bmatrix}\widetilde{\mathcal{M}} & \widetilde{\mathcal{N}}\\ -\widetilde{\mathcal{Y}} & \widetilde{\mathcal{X}}\end{bmatrix} = I$$

in $TIC(U \times Y;U \times Y)$, and, in addition, we require that $(\mathcal{N},\mathcal{M})$ is a right and $(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}})$ a left [exponentially] coprime factorization of $\mathcal{D}$.

As the following theorem shows, if a well-posed linear system is jointly stabilizable and detectable, then its input/output map has a doubly coprime factorization. A converse to this statement is true as well.

Theorem 17 *(i) Let $\Psi = \left[\begin{smallmatrix}\mathcal{A}&\mathcal{B}\\ \mathcal{C}&\mathcal{D}\end{smallmatrix}\right]$ be a jointly [exponentially] stabilizable and detectable well-posed linear system. Then, with the notations of Definitions 12 and 16,*

$$\begin{bmatrix}\widetilde{\mathcal{M}} & \widetilde{\mathcal{N}}\\ -\widetilde{\mathcal{Y}} & \widetilde{\mathcal{X}}\end{bmatrix}\begin{bmatrix}\mathcal{X} & -\mathcal{N}\\ \mathcal{Y} & \mathcal{M}\end{bmatrix} = \begin{bmatrix}I+\mathcal{G}^\times & \mathcal{D}^\times\\ -\mathcal{E}^\times & I-\mathcal{F}^\times\end{bmatrix}\begin{bmatrix}I-\mathcal{G}_\times & -\mathcal{D}_\times\\ \mathcal{E}_\times & I+\mathcal{F}_\times\end{bmatrix}$$

is a doubly [exponentially] coprime factorization of $\mathcal{D}$. (Here the left factor is the input/output map of $\Psi^\times$ in Figure 7, and the right factor is the input/output map of $\Psi^\times$ in Figure 6.)

(ii) Conversely, every $\mathcal{D}$ that belongs to $TIC_\alpha(U;Y)$ for some $\alpha \in \mathbf{R}$ and has a doubly [exponentially] coprime factorization can be realized as the input/output map of a jointly strongly [exponentially] stabilizable and detectable well-posed linear system $\Psi = \left[\begin{smallmatrix}\mathcal{A}&\mathcal{B}\\ \mathcal{C}&\mathcal{D}\end{smallmatrix}\right]$.

5 Dynamic Stabilization

Theorem 18 *Let $\Psi = \left[\begin{smallmatrix}\mathcal{A}&\mathcal{B}\\ \mathcal{C}&\mathcal{D}\end{smallmatrix}\right]$ be a jointly [exponentially] stabilizable and detectable well-posed linear system, and let Ψ_{ext} denote the system in Definition 10. Then the system $\Psi_\times^\times$ drawn in Figure 8 defines a possibly non-well-posed dynamic compensator which stabilizes Ψ_{ext} in the sense that if we connect the outputs $w^\times$ and u in Figures 8 and 10 to the inputs with the same labels, then the resulting system is (well-posed and) [exponentially] stable.*

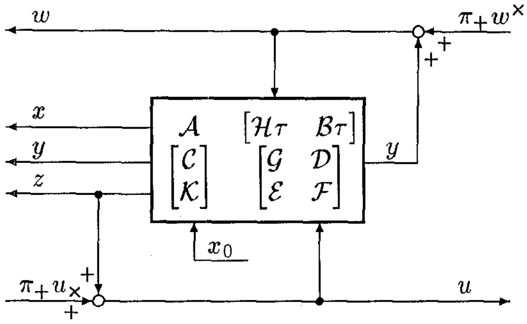

Figure 8: Possibly Non-Well-Posed Stabilizing Dynamic Compensator $\Psi_{\times}^{\times}$

A closely related result has been discovered independently by Curtain, Weiss, and Weiss [4].

Remark 19 *By using Theorem 18 one can easily develop a Youla parameterization of the set of all stabilizing compensators for* Ψ_{ext}. *To get the Youla parameterization we simply connect the Youla parameter Q from* $\tilde{w}$ *to* $\tilde{u}_{\times}$ *in Figure 8 and connect this system to* Ψ_{ext} *as described in Theorem 18. The resulting input/output map from* w *to* $u_{\times}$ *will be equal to Q.*

6 Proofs

We refer the reader to [13] for more details and for proofs of most of the results presented in Sections 3–5. Related results have been obtained independently by Ruth Curtain and George and Martin Weiss in [2, 3, 4]. The exponentially stable version Theorem 14 is due to Weiss and Rebarber [20].

The results presented here were originally developed to support those presented in [14, 15].

References

[1] Ruth F. Curtain and George Weiss. Well posedness of triples of operators (in the sense of linear systems theory). In *Control and Optimization of Distributed Parameter Systems*, pages 401–416, Basel, 1989. Birkhäuser-Verlag.

[2] Ruth F. Curtain and George Weiss. Dynamic stabilization of regular linear systems. *IEEE Transactions on Automatic Control*, 42:1–18, 1996.

[3] Ruth F. Curtain, George Weiss, and Martin Weiss. Coprime factorization for regular linear systems. *Automatica*, 32:1519–1531, 1996.

[4] Ruth F. Curtain, George Weiss, and Martin Weiss. Stabilization of irrational transfer functions by controllers with internal loop. Preprint, 1997.

[5] F. Flandoli, Irena Lasiecka, and Roberto Triggiani. Algebraic Riccati equations with non-smoothing observation arising in hyperbolic and Euler-Bernoulli boundary control problems. *Annali di Matematica Pura ed Applicata*, 153:307–382, 1988.

[6] Birgit Jacob and Hans Zwart. Equivalent conditions for stabilizability of infinte-dimensional systems with admissible control operators. Preprint, 1996.

[7] Hartmut Logemann. Stabilization and regulation of infinite-dimensional systems using coprime factorizations. In Ruth F. Curtain, editor, *Analysis and Optimization of Systems: State and Frequency Domain Approaches for Infinite-Dimensional Systems*, volume 185 of *Lecture Notes in Control and Information Sciences*, pages 102–139, Berlin and New York, 1993. Springer-Verlag.

[8] K. A. Morris. State feedback and estimation of well-posed systems. *Mathematics of Control, Signals, and Systems*, 7:351–388, 1994.

[9] Richard Rebarber. Conditions for the equivalence of internal and external stability for distributed parameter systems. *IEEE Transactions on Automatic Control*, 38:994–998, 1993.

[10] Dietmar Salamon. *Control and Observation of Neutral Systems.* Pitman Publishing Ltd, London, 1984.

[11] Dietmar Salamon. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. *Transactions of the American Mathematical Society*, 300:383–431, 1987.

[12] Dietmar Salamon. Realization theory in Hilbert space. *Mathematical Systems Theory*, 21:147–164, 1989.

[13] Olof J. Staffans. Coprime factorizations and well-posed linear systems. *SIAM Journal on Control and Optimization*, 36:1268–1292, 1998.

[14] Olof J. Staffans. Quadratic optimal control of well-posed linear systems. To appear in SIAM Journal on Control and Optimization, 1998.

[15] Olof J. Staffans. Quadratic optimal control of well-posed linear systems. In *Proceedings of the 37th IEEE Conference on Decision and Control*, Tampa, Florida, December 1998.

[16] George Weiss. Admissibility of unbounded control operators. *SIAM Journal on Control and Optimization*, 27:527–545, 1989.

[17] George Weiss. Admissible observation operators for linear semigroups. *Israel Journal of Mathematics*, 65:17–43, 1989.

[18] George Weiss. Transfer functions of regular linear systems. Part I: Characterizations of regularity. *Transactions of American Mathematical Society*, 342:827–854, 1994.

[19] George Weiss. Regular linear systems with feedback. *Mathematics of Control, Signals, and Systems*, 7:23–57, 1994.

[20] George Weiss and Richard Rebarber. Dynamic stabilization of well-posed linear systems. In *Proceedings of MMAR98*, Miedzyzdroje, Poland, August 1998.