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STATE/SIGNAL LINEAR TIME-INVARIANT SYSTEMS THEORY, PART III: TRANSMISSION AND IMPEDANCE REPRESENTATIONS OF DISCRETE TIME SYSTEMS

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Dedicated to the memory of Tiberiu Constantinescu

ABSTRACT. In this paper we continue the development of the passive linear discrete time invariant s/s (state/signal) systems theory. A pasiv s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has a Hilbert state space $\mathcal{X}$ and a Kreĭn signal space $\mathcal{W}$, and the trajectories $(x(\cdot), w(\cdot))$ of this system are determined by the fact that at each time instant $t \geqslant 0$ the triple $(x(t+1), x(t), w(t))$ belongs to the maximal non-negative subspace V of the Kreĭn space $-X \,[\dot{+}]\, X \,[+]\, \mathcal{W}$. By decomposing the signal space $\mathcal{W}$ into the direct sum $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ we can write each trajectory $(x(\cdot), w(\cdot))$ in the form $(x(\cdot), u(\cdot), y(\cdot))$ where $u(t) \in \mathcal{U}$ and $y(t) \in \mathcal{Y}$ for all t. This decomposition is admissible if all the triples $(x(\cdot), u(\cdot), y(\cdot))$ obtained in this way can be interpreted as trajectories of a standard linear i/s/o (input/state/output) system with bounded coefficients. If $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal in the Kreĭn space $\mathcal{W}$, then the corresponding i/s/o representation of s/s system Σ is called a transmission representation, and its transfer function is called a transmission matrix. If instead $\mathcal{U}$ and $\mathcal{Y}$ are Lagrangian subspaces of $\mathcal{W}$, then the corresponding i/s/o representation is called an impedance representation, and its transfer function is called an impedance matrix. Here we study the properties of these representations and their transfer functions.

KEYWORDS: *Passive, conservative, scattering, impedance, transmission, Krein space.*

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1. INTRODUCTION

This article is a continuation of the articles [8] and [9], which we in the sequel refer to as 'Part I' and 'Part II', respectively. In Part I we developed a linear discrete time-invariant s/s (state/signal) systems theory in a general setting. This theory differs from the standard i/s/o (input/state/output) systems theory in the sense that we do not distinguish between input and output signals, but only between an 'internal' state $x \in \mathcal{X}$ and an 'external' signal $w \in \mathcal{W}$, where the state space $\mathcal{X}$ and signal space $\mathcal{W}$ are vector spaces. In Part I both of these were assumed to be Hilbert spaces, but no use was made of the specific inner product (in particular, we made no use of orthogonality), so that all results in Part I remain valid if we replace the inner product by another equivalent inner product. This makes it possible to apply the results from Part I also in the case where $\mathcal{X}$ and $\mathcal{W}$ are Kreĭn spaces. Here, as in Part II, we still take the state space $\mathcal{X}$ to be a Hilbert space, but the signal space $\mathcal{W}$ will be a Kreĭn space. When we cite a particular result from one of Parts I–II we shall simply add a roman number "I" or "II" to the corresponding number appearing there. Thus, for example, Theorem II.5.6 stands for Theorem 5.6 in Part II.

A *trajectory* $(x(\cdot), w(\cdot))$ of a linear time-invariant s/s system Σ in discrete time consists of a state sequence $x(n) \in \mathcal{X}$ and a signal sequence $w(n) \in \mathcal{W}$, $n \in \mathbb{Z}^+ := (0, 1, 2, \ldots)$, that satisfy the system of equations

$$\begin{aligned} x(n+1) &= F \begin{bmatrix} x(n) \\ w(n) \end{bmatrix}, \quad n \in \mathbb{Z}^+, \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where F is a bounded linear operator with closed domain $\mathcal{D}(F)$ in the product space $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ and range $\mathcal{R}(F) \subset \mathcal{X}$. The domain of F has the property that for every $x \in \mathcal{X}$ there is at least one $w \in \mathcal{W}$ such that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F)$. This property guarantees that for every $x_0 \in \mathcal{X}$ there exists at least one trajectory $(x(\cdot), w(\cdot))$ of the system with initial state $x(0) = x_0$. We remark that x_0 and the sequence $w(\cdot)$ together determine the trajectory $(x(\cdot), w(\cdot))$ uniquely.

A s/s system Σ with a Hilbert state space $\mathcal{X}$ and a Kreĭn space $\mathcal{W}$ is *forward passive* if all trajectories $(x(\cdot), w(\cdot))$ of Σ satisfy the 'energy' inequality

$$\|x(n+1)\|^2_{\mathcal{X}} \leqslant \|x(n)\|^2_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}} \quad n \in \mathbb{Z}^+, \tag{1.2}$$

where $\|\cdot\|_{\mathcal{X}}$ is the norm in the Hilbert space $\mathcal{X}$ and $[\cdot,\cdot]_{\mathcal{W}}$ is the Kreĭn inner product in $\mathcal{W}$. It is *forward conservative* if this inequality holds in the form of an equality, i.e.,

$$\|x(n+1)\|^2_{\mathcal{X}} = \|x(n)\|^2_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+. \tag{1.3}$$

The definitions of backward passive (or backward conservative) and passive (or conservative) s/s systems are based on the notion of the adjoint s/s system Σ_* of a given s/s system Σ introduced and studied in Part II, Section 4. The adjoint system Σ_* has the same state space $\mathcal{X}$ as the original system Σ, but the signal space of Σ_* is $\mathcal{W}_* = -\mathcal{W}$ instead of $\mathcal{W}$. Algebraically, the space $\mathcal{W}_*$ is the same as $\mathcal{W}$, but it has a different inner product. If we denote the identity operator from $\mathcal{W}_*$ to $\mathcal{W}$ by $\mathcal{I}$, then the inner product in $\mathcal{W}_*$ is given by

$$[w_*, w'_*]_{\mathcal{W}_*} := -[\mathcal{I}w_*, \mathcal{I}w'_*]_{\mathcal{W}}, \quad w_*,\ w'_* \in \mathcal{W}_*. \tag{1.4}$$

Instead of identifying $\mathcal{W}$ with itself (as is usually done) we identify the dual of $\mathcal{W}$ with $\mathcal{W}_*$: every bounded linear functional on $\mathcal{W}$ is of the form (for some $w_* \in \mathcal{W}_*$)

$$\langle w, w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} := [w, \mathcal{I}w_*]_{\mathcal{W}} = [\mathcal{I}^* w, w_*]_{\mathcal{W}_*}, \quad w \in \mathcal{W},\ w_* \in \mathcal{W}_*. \tag{1.5}$$

This identity and (1.4) imply that $\mathcal{I}^* = -\mathcal{I}^{-1}$, i.e., $\mathcal{I}$ is an anti-unitary operator. The adjoint system Σ_* is defined in such a way that a sequence $(x_*(\cdot), w_*(\cdot))$, where each $x_*(n) \in \mathcal{X}$ and $w_*(n) \in \mathcal{W}_*$, is a trajectory of Σ_* if and only if

$$(1.6)\qquad \begin{aligned} &-(x(n+1), x_*(0))_{\mathcal{X}} + (x(0), x_*(n+1))_{\mathcal{X}} \\ &\qquad + \sum_{k=0}^{n} \langle w(k), w_*(n-k)\rangle_{\langle \mathcal{W}, \mathcal{W}_*\rangle} = 0, \quad n \in \mathbb{Z}^+, \end{aligned}$$

for every trajectory $(x(\cdot), w(\cdot))$ of Σ.

The forward passivity or conservativity of a s/s system Σ does not imply that Σ_* is forward passive or conservative. We call a system Σ *backward passive* (or *backward conservative*) if Σ_* is forward passive (or forward conservative, respectively). Finally, a s/s system is *passive* (or *conservative*) if it is both forward and backward passive (or forward and backward conservative, respectively).

In the outline of the s/s passive systems theory given above we did not mention 'inputs' and 'outputs'. However, there is a close connection between the s/s passive systems theory and the i/s/o passive systems theory that enable us to transform i/s/o results into a s/s setting, and conversely.

We recall that the trajectory $(x(\cdot), u(\cdot), y(\cdot))$ of an i/s/o system $\Sigma_{i/s/o}$ consists of a state sequence $x(n) \in \mathcal{X}$, an input sequence $u(n) \in \mathcal{U}$, and an output sequence $y(n) \in \mathcal{Y}$, $n \in \mathbb{Z}^+$, where $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, that satisfy the system of equations

$$(1.7)\qquad \begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \quad n \in \mathbb{Z}^+, \\ x(0) &= x_0. \end{aligned}$$

Here $x_0 \in \mathcal{X}$ and the input data $u(\cdot) \in \mathcal{U}^{\mathbb{Z}^+}$ may be taken arbitrarily, after which $x(\cdot) \in \mathcal{X}^{\mathbb{Z}^+}$ and $y(\cdot) \in \mathcal{Y}^{\mathbb{Z}^+}$ are defined by (1.7). We denote this system by $\Sigma_{i/s/o} = (\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. The four block operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a bounded linear operator $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$. We can turn this i/s/o system into a s/s system by considering $w(\cdot) := \left[\begin{smallmatrix} u(\cdot) \\ y(\cdot) \end{smallmatrix}\right]$ to be the 'external' signal in the signal space $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$, which is the product of the topological vector spaces $\mathcal{Y}$ and $\mathcal{U}$. We can then rewrite (1.7) in the form (1.1) by taking

$$\begin{aligned} \mathcal{D}(F) &= \left\{ w = \begin{bmatrix} y \\ u \end{bmatrix} \;\middle|\; u \in \mathcal{U},\ y = Cx + Du \text{ for some } x \in \mathcal{X} \right\}, \\ F\begin{bmatrix} x \\ w \end{bmatrix} &= Ax + Bu, \quad w = \begin{bmatrix} y \\ u \end{bmatrix} \in \mathcal{D}(F). \end{aligned}$$

In this way we obtain a s/s system Σ which has the same state space $\mathcal{X}$ as $\Sigma_{i/s/o}$, and with the signal space $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$. By identifying $\left[\begin{smallmatrix} \mathcal{Y} \\ 0 \end{smallmatrix}\right]$ with $\mathcal{Y}$ and $\left[\begin{smallmatrix} 0 \\ \mathcal{U} \end{smallmatrix}\right]$ with $\mathcal{U}$ we can write $\mathcal{W}$ as the direct sum $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. Thus, to any i/s/o system $\Sigma_{i/s/o}$ with Hilbert state space $\mathcal{X}$, input space $\mathcal{U}$, and output space $\mathcal{Y}$, there is a unique s/s system Σ with the same state space $\mathcal{X}$ and with signal space $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U} = \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ such that $(x(\cdot), u(\cdot), y(\cdot))$ is a trajectory of $\Sigma_{i/s/o}$ if and only if $(x(\cdot), w(\cdot))$ is a trajectory of Σ, where $w(\cdot) = \left[\begin{smallmatrix} y(\cdot) \\ u(\cdot) \end{smallmatrix}\right]$.

As shown in Part I, the above construction can be reversed in the sense that to each s/s system Σ there corresponds not only one, but usually infinitely many i/s/o systems $\Sigma_{i/s/o}$. This reverse construction begins with a splitting of the signal space $\mathcal{W}$ into a direct sum $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. We call this decomposition *admissible* if there is some i/s/o system $\Sigma_{i/s/o}$ with input space $\mathcal{U}$ and output space $\mathcal{Y}$ such that Σ is obtained from $\Sigma_{i/s/o}$ as described above. Clearly, not all decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ are admissible, but there is always at least one admissible decomposition. If in this decomposition both $\mathcal{Y}$ and $\mathcal{U}$ are nontrivial, then there exist infinitely many admissible decompositions, and the i/s/o system corresponding to this decomposition is uniquely determined by Σ and by the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. (Both of these claims follow from Lemma 5.7 in Part I, which gives a complete description of the set of all admissible decompositions.) We call each such i/s/o system an *i/s/o representation* of Σ.

In the discussion above both $\mathcal{Y}$ and $\mathcal{U}$ were Hilbert spaces. It would therefore be possible to interpret $\mathcal{W}$ to be the orthogonal direct sum $\mathcal{Y} \oplus \mathcal{U}$ of $\mathcal{Y}$ and $\mathcal{U}$, but this we shall *not* do. Instead we throughout use a different inner product in $\mathcal{W}$, defined by a formula of the type

$$\left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]_{\mathcal{W}} := \left(\begin{bmatrix} y \\ u \end{bmatrix}, \mathcal{J}\begin{bmatrix} y' \\ u' \end{bmatrix}\right)_{\mathcal{Y}\oplus\mathcal{U}}, \quad \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix} \in \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \tag{1.8}$$

where $\mathcal{J}$ is a self-adjoint linear bounded operator on $\mathcal{Y} \oplus \mathcal{U}$ with a bounded inverse. With this inner product $\mathcal{W}$ becomes a Kreĭn space which has the same strong and weak topologies as $\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ and $\mathcal{Y} \oplus \mathcal{U}$. In the reverse construction we first fix some inner products in $\mathcal{Y}$ and $\mathcal{U}$ which are compatible with the topology of $\mathcal{Y}$ and $\mathcal{U}$ as subspaces of $\mathcal{W}$, after which the Kreĭn space inner product in the signal space $\mathcal{W}$ defines a unique self-adjoint linear bounded operator $\mathcal{J}$ on $\mathcal{Y} \oplus \mathcal{U}$ with a bounded inverse such that (1.8) holds.

In Part I we studied the relationship between the coefficients A, B, C, and D in (1.7) induced by different i/s/o representations of a given s/s system Σ, and also between their *transfer functions*

$$\mathfrak{D}(z) = D + zC(1_{\mathcal{X}} - zA)^{-1}B, \quad z \in \Lambda_A, \tag{1.9}$$

where Λ_A be the set of points $z \in \mathbb{C}$ for which $(1_{\mathcal{X}} - zA)$ has a bounded inverse, plus the point at infinity if A is boundedly invertible. In Part II the s/s system Σ was assumed to be passive or conservative, and we studied primarily i/s/o representations of scattering type, which we simply refer to as *scattering representations*. These representations are induced by *fundamental* decompositions of $\mathcal{W}$, i.e., orthogonal decompositions of the type $\mathcal{W} = -\mathcal{Y} \,[\dotplus]\, \mathcal{U}$, where $-\mathcal{Y}$ is an anti-Hilbert space and $\mathcal{U}$ is a Hilbert space. We recall that if $\mathcal{W}$ is neither a Hilbert space nor an anti-Hilbert space, then there exist infinitely many such decompositions. If we let $\mathcal{Y}$ and $\mathcal{U}$ inherit the inner products induced on them by $\mathcal{W}$, then with respect to these inner products the operator $\mathcal{J}$ is (1.8) will be of the form $\mathcal{J} = \left[\begin{smallmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$, i.e.,

$$\left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]_{\mathcal{W}} = -(y, y')_{\mathcal{Y}} + (u, u')_{\mathcal{U}}, \quad \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix} \in \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}. \tag{1.10}$$

In terms of this representation the forward passivity condition (1.2) becomes

$$\|x(n+1)\|^2_{\mathcal{X}} + \|y(n)\|^2_{\mathcal{Y}} \leqslant \|x(n)\|^2_{\mathcal{X}} + \|u(n)\|^2_{\mathcal{U}}, \quad n \in \mathbb{Z}^+, \tag{1.11}$$

and the forward conservativity (1.3) condition becomes

$$\|x(n+1)\|^2_{\mathcal{X}} + \|y(n)\|^2_{\mathcal{Y}} = \|x(n)\|^2_{\mathcal{X}} + \|u(n)\|^2_{\mathcal{U}}, \quad n \in \mathbb{Z}^+. \tag{1.12}$$

If $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ is the corresponding i/s/o representation, then the conditions (1.11) and (1.12) mean that the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is contractive or isometric, respectively, from the Hilbert space $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] = \mathcal{X} \oplus \mathcal{U}$ to the Hilbert space $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] = \mathcal{X} \oplus \mathcal{Y}$. An i/s/o system satisfying (1.11) is called a scattering passive i/s/o system, and this class of systems has been studied, e.g., in [3, 4, 5, 6], [2], and [19].

The adjoint of a contractive operator between two Hilbert spaces is contractive. Therefore, if $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ is an i/s/o system with a contractive operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, then also $\left[\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right]$ is contractive, meaning that the trajectories $(x_*(\cdot), u_*(\cdot), y_*(\cdot))$ of the adjoint i/s/o system $\Sigma^*_{i/s/o} = \left(\left[\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right]; \mathcal{X}, \mathcal{Y}, \mathcal{U}\right)$ satisfy the corresponding inequality

$$\|x_*(n+1)\|^2_{\mathcal{X}} + \|y_*(n)\|^2_{\mathcal{Y}_*} \leqslant \|x_*(n)\|^2_{\mathcal{X}} + \|u_*(n)\|^2_{\mathcal{U}_*}, \quad n \in \mathbb{Z}^+, \tag{1.13}$$

where $\mathcal{U}_* = \mathcal{Y}$ and $\mathcal{Y}_* = \mathcal{U}$. By Proposition 4.11 in Part II, $\Sigma^*_{i/s/o}$ is an i/s/o representation of the adjoint s/s system Σ_*. Thus, if a s/s system Σ is forward passive and its signal space $\mathcal{W}$ has an admissible fundamental decomposition $\mathcal{W} = -\mathcal{W}_- [\dot{+}] \mathcal{W}_+$, then the corresponding i/s/o representation of Σ is scattering passive and Σ is a passive s/s system. Moreover, it was proved in Part II that for a passive s/s system Σ every fundamental decomposition is admissible. There we also describe the connection between different passive scattering representations $\Sigma_{i/s/o}$ and $\Sigma^1_{i/s/o}$ of a passive s/s system Σ and their transfer functions $\mathfrak{D}$ and $\mathfrak{D}_1$, which in this case are called *scattering matrices*. The latter connection is described by the formula

$$\mathfrak{D}_1(z) = [\Theta_{11}\mathfrak{D}(z) + \Theta_{12}][\Theta_{21}\mathfrak{D}(z) + \Theta_{22}(z)]^{-1}, \quad z \in \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\},$$

where $\Theta = \left[\begin{smallmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{smallmatrix}\right]$ is the four block decomposition of the identity operator in $\mathcal{W}$ with respect to the two given fundamental decompositions $\mathcal{W} = -\mathcal{W}_- [\dot{+}] \mathcal{W}_+$ and $\mathcal{W} = -\mathcal{W}^1_- [\dot{+}] \mathcal{W}^1_+$ of $\mathcal{W}$.

In Part II we also looked to some extent at the more general case where the signal space $\mathcal{W}$ of a passive s/s system Σ has an admissible *orthogonal decomposition* $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$, i.e., $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal in the Kreĭn space $\mathcal{W}$, and this is a direct sum. In this case the inner products in $-\mathcal{Y}$ and $\mathcal{U}$ inherited from $\mathcal{W}$ make both $\mathcal{Y}$ and $\mathcal{U}$ Kreĭn spaces themselves. The corresponding i/s/o representation $\Sigma_{i/s/o}$ is called an *transmission representation* of Σ, and its transfer function is called the *transmission matrix*. At the same time $\mathcal{Y}$ and $\mathcal{U}$ may be considered as Hilbert spaces with inner products $(\cdot,\cdot)_{\mathcal{Y}}$ and $(\cdot,\cdot)_{\mathcal{Y}}$, respectively, defined by

$$[y,y']_{\mathcal{Y}} = (y,\mathcal{J}_{\mathcal{Y}}y')_{\mathcal{Y}}, \quad y,\ y' \in \mathcal{Y}, \tag{1.14}$$

$$[u,u']_{\mathcal{U}} = (u,\mathcal{J}_{\mathcal{U}}u')_{\mathcal{U}}, \quad u,\ u' \in \mathcal{U}, \tag{1.15}$$

where $\mathcal{J}_{\mathcal{Y}}$ is a signature operator (i.e., it is self-adjoint and unitary) in $\mathcal{Y}$ with respect to both $[\cdot,\cdot]_{\mathcal{Y}}$ and $(\cdot,\cdot)_{\mathcal{Y}}$, and $\mathcal{J}_{\mathcal{U}}$ is a signature operator in $\mathcal{U}$ with respect to both $[\cdot,\cdot]_{\mathcal{U}}$ and $(\cdot,\cdot)_{\mathcal{U}}$. Then the operator $\mathcal{J}$ in (1.8) is given by $\mathcal{J} = \begin{bmatrix} -\mathcal{J}_{\mathcal{Y}} & 0 \\ 0 & \mathcal{J}_{\mathcal{U}} \end{bmatrix}$, and it is a signature operator in $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U} = \begin{bmatrix} -\mathcal{Y} \\ \mathcal{U} \end{bmatrix}$ with respect to both $[\cdot,\cdot]_{\mathcal{W}}$ and $(\cdot,\cdot)_{\mathcal{W}}$. In terms of this representation the forward passivity condition (1.2) becomes

$$\|x(n+1)\|^2_{\mathcal{X}} - \|x(n)\|^2_{\mathcal{X}} \leqslant -[y(n),y(n)]_{\mathcal{Y}} + [u(n),u(n)]_{\mathcal{U}}, \quad n \in \mathbb{Z}^+, \tag{1.16}$$

and the forward conservativity (1.3) condition becomes

$$\|x(n+1)\|^2_{\mathcal{X}} - \|x(n)\|^2_{\mathcal{X}} = -[y(n),y(n)]_{\mathcal{Y}} + [u(n),u(n)]_{\mathcal{U}}, \quad n \in \mathbb{Z}^+. \tag{1.17}$$

These conditions are equivalent to the requirements that the operator $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a contraction or an isometry, respectively, from the Kreĭn space $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} = \mathcal{X}\,[\dot{+}]\,\mathcal{U}$ to the Kreĭn space $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} = \mathcal{X}\,[\dot{+}]\,\mathcal{Y}$. Another way to express this is to say that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a $(\mathcal{J}_1;\mathcal{J}_2)$-contraction of $(\mathcal{J}_1;\mathcal{J}_2)$-isometry, respectively, from the Hilbert space $\mathcal{X} \oplus \mathcal{U}$ to the Hilbert space $\mathcal{X} \oplus \mathcal{Y}$, where $\mathcal{J}_1 = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & J_{\mathcal{U}} \end{bmatrix}$ and $\mathcal{J}_2 = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 0 & J_{\mathcal{Y}} \end{bmatrix}$.

There are two essential differences between transmission and scattering representations of a s/s system Σ. The first difference is connected to the fact that the adjoint of a contractive operator between two Hilbert spaces is contractive. We used this fact above to conclude that forward passivity of a scattering representation implies backward passivity, hence passivity of the s/s system Σ. The same statement is not true for transmission representations since the contractivity of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ from the Kreĭn space $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ to the Kreĭn space $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ does not imply that $\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$ is contractive from $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ to $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$. Thus, the existence of a forward passive transmission representation of a s/s system Σ does not yet imply that Σ is passive. The second difference is that for a passive s/s system not every transmission decomposition (i.e., orthogonal decomposition) of the signal space $\mathcal{W}$ need be admissible.

This article is devoted to representations of passive s/s systems. As we mentioned above, Part I contains results about general admissible i/s/o representations (in addition to driving variable and output nulling representations), and Part II contains a fairly detailed discussion of two kinds os i/s/o scattering representations, plus some results on transmission representations. Here we continue our study of transmission representations and their connection to scattering representations, and develop a connection between the transmission and scattering matrices of a passive s/s/ system. We then move on to study i/s/o representations that correspond to certain non-orthogonal admissible decompositions of the signal space. We take $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$, where $\mathcal{F}$ and $\mathcal{E}$ are Lagrangian subspaces of $\mathcal{W}$, i.e., they coincide with their own orthogonal complements in the Kreĭn space $\mathcal{W}$. We call such a decomposition a *Lagrangian decomposition* of $\mathcal{W}$ provided the inner products in $\mathcal{F}$ and $\mathcal{E}$ has been chosen in an appropriate way. If this decomposition is admissible, then we call the corresponding i/s/o system an *impedance representation* of Σ, and we refer to its transfer function as the *impedance matrix* of this representation. We show in Lemma 2.3 below that the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is Lagrangian if and only if (1.8) holds for some off-diagonal signature operator $\mathcal{J} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$, where Ψ is a unitary operator from $\mathcal{U}$ to $\mathcal{Y}$. (In particular, the dimensions of $\mathcal{U}$ and $\mathcal{Y}$ must be the same.) For such $\mathcal{J}$,

$$(1.18) \qquad \left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]_{\mathcal{W}} = (y, \Psi u')_{\mathcal{Y}} + (u, \Psi^* y')_{\mathcal{U}}, \quad \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix} \in \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix},$$

and consequently,

$$(1.19) \qquad \left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y \\ u \end{bmatrix}\right]_{\mathcal{W}} = 2\Re(y, \Psi u)_{\mathcal{Y}} = 2\Re(u, \Psi^* y)_{\mathcal{U}}, \quad \begin{bmatrix} y \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}.$$

Since the operator Ψ plays a very significant role in the theory we include it in the notation and write $\mathcal{W} = \mathcal{Y} \overset{\Psi}{\dotplus} \mathcal{U}$ instead of $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. In terms of this representation the forward passivity condition (1.2) becomes

$$(1.20) \qquad \|x(n+1)\|^2_{\mathcal{X}} - \|x(n)\|^2_{\mathcal{X}} \leqslant 2\Re(y(n), \Psi u(n))_{\mathcal{Y}}, \quad n \in \mathbb{Z}^+,$$

where $u(n) \in \mathcal{U}$ and $y(n) \in \mathcal{Y}$ are taken from the decomposition $w(n) = y(n) + u(n)$. Likewise, the forward conservativity (1.3) condition becomes

$$(1.21) \qquad \|x(n+1)\|^2_{\mathcal{X}} - \|x(n)\|^2_{\mathcal{X}} = 2\Re(y(n), \Psi u(n))_{\mathcal{Y}}, \quad n \in \mathbb{Z}^+,$$

Recall that if the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is admissible, then $(x(\cdot), u(\cdot), y(\cdot))$ in (1.20) and (1.21) are trajectories of the corresponding i/s/o system $\Sigma_{i/s/o}$.

As we shall show below, the impedance case is similar to the scattering case in the sense that if a s/s system Σ is forward passive and has an admissible impedance representation, then it is passive. On the other hand, it is also similar to the transmission case in the sense that even if Σ is passive it need not be true that every Lagrangian decomposition of the signal space is admissible. In Example 5.13 we show that there even exist passive s/s system for which *no Lagrangian decomposition is admissible*. Necessary and sufficient conditions for the admissibility of a Lagrangian decomposition of the signal space of a passive s/s system Σ are given in Theorem 5.8. There we also describe the connections between impedance and scattering representations of a passive s/s system and between their input-state/state-output transfer functions.

NOTATIONS. The following standard notations are used below. $\mathbb{C}$ is the complex plane, $\mathbb{D}$ is the open unit disk in $\mathbb{C}$, and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$.

The space of bounded linear operators from one Kreĭn space $\mathcal{X}$ to another Kreĭn space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X};\mathcal{Y})$, and we abbreviate $\mathcal{B}(\mathcal{X};\mathcal{X})$ to $\mathcal{B}(\mathcal{X})$. The domain, range, and kernel of a linear operator A is denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$, respectively. The restriction of A to some subspace $\mathcal{Z} \subset \mathcal{D}(A)$ is denoted by $A|_{\mathcal{Z}}$. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$. For each $A \in \mathcal{B}(\mathcal{X})$ we let Λ_A be the set of points $z \in \mathbb{C}$ for which $(1_{\mathcal{X}} - zA)$ has a bounded inverse, plus the point at infinity if A is boundedly invertible. We denote the projection onto a closed subspace $\mathcal{Y}$ of a space $\mathcal{X}$ along some complementary subspace $\mathcal{U}$ by $P_{\mathcal{Y}}^{\mathcal{U}}$, and by $P_{\mathcal{Y}}$ if $\mathcal{Y}$ is orthogonal to $\mathcal{U}$ with respect to a Hilbert or Kreĭn space inner product in $\mathcal{X}$.

We denote the ordered product of the two locally convex topological vector spaces $\mathcal{Y}$ and $\mathcal{U}$ by $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. In particular, although $\mathcal{Y}$ and $\mathcal{U}$ may be Hilbert or Kreĭn spaces (in which case the product topology on $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ is induced by an inner product), we shall not require that $\left[\begin{smallmatrix}\mathcal{Y}\\ 0\end{smallmatrix}\right] \perp \left[\begin{smallmatrix}0\\ \mathcal{U}\end{smallmatrix}\right]$ in $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. We identify a vector $\left[\begin{smallmatrix}y\\ 0\end{smallmatrix}\right] \in \left[\begin{smallmatrix}\mathcal{Y}\\ 0\end{smallmatrix}\right]$ with $y \in \mathcal{Y}$ and a vector $\left[\begin{smallmatrix}0\\ u\end{smallmatrix}\right] \in \left[\begin{smallmatrix}0\\ \mathcal{U}\end{smallmatrix}\right]$ with $u \in \mathcal{U}$, and then we sometimes write $\mathcal{Y} \dotplus \mathcal{U}$ instead of $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$, interpreting $\mathcal{Y} \dotplus \mathcal{U}$ as an *ordered direct sum*.

We denote the inner product in the Hilbert space $\mathcal{X}$ by $(\cdot,\cdot)_{\mathcal{X}}$, the inner product in the Kreĭn space $\mathcal{W}$ by $[\cdot,\cdot]_{\mathcal{W}}$. The set of all vectors that are orthogonal to a set $\mathcal{G}$ is denoted by $G^{[\perp]}$ in the case of a Kreĭn space and by $\mathcal{G}^{\perp}$ in the case of a Hilbert space. The orthogonal sum of two Hilbert spaces $\mathcal{Y}$ and $\mathcal{U}$ is denoted by $\mathcal{Y} \oplus \mathcal{U}$, and the orthogonal sum of two Kreĭn spaces $\mathcal{Y}$ and $\mathcal{U}$ is denoted by $\mathcal{Y}\,[\dotplus]\,\mathcal{U}$.

2. DIRECT SUM DECOMPOSITIONS OF A KREĬN SPACE

In this section we shall discuss some questions related to direct sum decompositions of a Kreĭn space and the validity of (1.8). In this discussion we use various results on the geometry of Kreĭn spaces presented in Part II, Section 2. For the convenience of the reader we repeat the most central part of that presentation below.

By a Kreĭn space we mean a linear space $\mathcal{W}$ endowed with an indefinite inner product $[\cdot,\cdot]_{\mathcal{W}}$ containing two subspaces $-\mathcal{W}_-$ and $\mathcal{W}_+$ of $\mathcal{W}$ such that the restriction of $[\cdot,\cdot]_{\mathcal{W}}$ to $\mathcal{W}_+ \times \mathcal{W}_+$ makes $\mathcal{W}_+$ a Hilbert space while the restriction of $-[\cdot,\cdot]_{\mathcal{W}}$ to $\mathcal{W}_- \times \mathcal{W}_-$ makes $\mathcal{W}_-$ a Hilbert space, and $\mathcal{W} = -\mathcal{W}_-\,[\dotplus]\,\mathcal{W}_+$ is a $[\cdot,\cdot]_{\mathcal{W}}$-orthogonal direct sum decomposition of $\mathcal{W}$. In this case the decomposition $\mathcal{W} = -\mathcal{W}_-\,[\dotplus]\,\mathcal{W}_+$ is said to form a *fundamental decomposition* for the Kreĭn space $\mathcal{W}$. A choice of fundamental decomposition $\mathcal{W} = -\mathcal{W}_-\,[\dotplus]\,\mathcal{W}_+$ determines a

Hilbert space norm on $\mathcal{W}$ by

$$\|w_- + w_+\|^2_{\mathcal{W}_-\oplus\mathcal{W}_+} = -[w_-, w_-]_{\mathcal{W}} + [w_+, w_+]_{\mathcal{W}}, \quad w_- \in \mathcal{W}_-,\ w_+ \in \mathcal{W}_+. \tag{2.1}$$

While the norm $\|\cdot\|_{\mathcal{W}_-\oplus\mathcal{W}_+}$ itself depends on the choice of fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ for $\mathcal{W}$, all these norms are equivalent and the resulting strong and weak topologies are each independent of the choice of the fundamental decomposition. Any norm on $\mathcal{W}$ arising in this way from some choice of fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ for $\mathcal{W}$ we shall call an *admissible norm* on $\mathcal{W}$, and we shall refer to the corresponding positive inner product on $\mathcal{W}_- \oplus \mathcal{W}_+$ as an *admissible Hilbert space inner product* on $\mathcal{W}$.

For each Kreĭn space $\mathcal{W}$ we define its *anti-space* $-\mathcal{W}$ to be algebraically and topologically the same space as $\mathcal{W}$ but with a change of sign in the inner product: $[\cdot,\cdot]_{-\mathcal{W}} = -[\cdot,\cdot]_{\mathcal{W}}$. If $\mathcal{W}$ is a Hilbert space, then we call $-\mathcal{W}$ an *anti-Hilbert space*. Observe that a Kreĭn space and its anti-space have the same admissible norms and admissible Hilbert space inner products.

A subspace $\mathcal{G}$ of a Kreĭn space is said to be *nonnegative*, *neutral* or *nonpositive* if $[g,g]_{\mathcal{W}} \geqslant 0$ for all $g \in \mathcal{G}$, $[g,g]_{\mathcal{W}} = 0$ for all $g \in \mathcal{G}$, or $[g,g]_{\mathcal{W}} \leqslant 0$ for all $g \in \mathcal{G}$, respectively. Subspaces of these types are called *semi-definite*. In each semi-definite subspace $\mathcal{G}$ the Cauchy inequality $|[g,g']_{\mathcal{W}}|^2 \leqslant [g,g][g',g']_{\mathcal{W}}$ holds for all $g, g' \in \mathcal{G}$. In particular, in each neutral subspace we have $[g,g']_{\mathcal{W}} = 0$ for all $g, g' \in \mathcal{G}$. A subspace is *maximal nonnegative* (respectively, *maximal nonpositive*) if it is nonnegative (nonpositive) and if it is not properly contained in any other nonnegative (nonpositive) subspace. If $[g,g]_{\mathcal{W}} > 0$ for all $g \in \mathcal{G}$ with $g \neq 0$ we say that $\mathcal{G}$ is *positive*; similarly, $\mathcal{G}$ is *negative* if $[g,g]_{\mathcal{W}} < 0$ for all $g \in \mathcal{G}$ with $g \neq 0$.

A bounded linear operator A on a Kreĭn space $\mathcal{W}$ is called *nonnegative* (and we write $A \geqslant 0$) or *nonpositive* ($A \leqslant 0$) if $[w, Aw]_{\mathcal{W}} \geqslant 0$ or $[w, Aw]_{\mathcal{W}} \leqslant 0$, respectively, for all $w \in \mathcal{W}$. It is *positive* ($A > 0$) or *negative* ($A < 0$) if $[w, Aw]_{\mathcal{W}} > 0$ or $[w, Aw]_{\mathcal{W}} < 0$, respectively, for all nonzero $w \in \mathcal{W}$, and A is *uniformly positive* if $[w, Aw] \geqslant \varepsilon\|w\|^2_{\mathcal{W}}$ for some $\varepsilon > 0$ and some admissible norm $\|\cdot\|_{\mathcal{W}}$. By $A \leqslant B$, where both A and B are bounded linear operators, we mean that $A - B \leqslant 0$, etc.

The orthogonal complement $\mathcal{G}^{[\perp]}$ of an arbitrary subset $\mathcal{G} \subset \mathcal{W}$ with respect to the Kreĭn space inner product $[\cdot,\cdot]_{\mathcal{W}}$ is defined as

$$\mathcal{G}^{[\perp]} = \{w \in \mathcal{W} \mid [w,g]_{\mathcal{W}} = 0 \text{ for all } g \in \mathcal{G}\}.$$

If $\mathcal{W}$ is a Hilbert space, then we write $\mathcal{G}^{\perp}$ instead of $\mathcal{G}^{[\perp]}$. This is always a closed subspace of $\mathcal{W}$. Note that, by definition, a subspace $\mathcal{G}$ is neutral if and only if $\mathcal{G} \subset \mathcal{G}^{[\perp]}$. A stronger notion than a neutral subspace is that of a Lagrangian subspace: we say that a subspace $\mathcal{G} \subset \mathcal{W}$ is *Lagrangian* if $\mathcal{G} = \mathcal{G}^{[\perp]}$. It follows from the Cauchy inequality that every Lagrangian subspace is both maximal nonnegative and maximal nonpositive, and the converse is also true.

The fundamental decompositions that we have considered above are a special case of *orthogonal decompositions* $\mathcal{W} = -\mathcal{Y} \,[\dot{+}]\, \mathcal{U}$ of $\mathcal{W}$, where $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal with respect to $[\cdot,\cdot]_{\mathcal{W}}$, and both $\mathcal{Y}$ and $\mathcal{U}$ are Kreĭn spaces with the inner products inherited from $-\mathcal{W}$ and $\mathcal{W}$, respectively. Thus, if $w = y + u$ with $y \in \mathcal{Y}$ and $u \in \mathcal{U}$, then

$$[w,w]_{\mathcal{W}} = [y,y]_{\mathcal{W}} + [u,u]_{\mathcal{W}} = -[y,y]_{\mathcal{Y}} + [u,u]_{\mathcal{U}}. \tag{2.2}$$

This orthogonal decomposition is fundamental if and only if $\mathcal{Y}$ and $\mathcal{U}$ are Hilbert spaces.

In this work we shall also need direct sum decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of $\mathcal{W}$ which are not orthogonal with respect to the original inner product $[\cdot,\cdot]_{\mathcal{W}}$. In this case we shall treat $\mathcal{U}$ and $\mathcal{Y}$ as *Hilbert* spaces, and require that the inner products in $\mathcal{U}$ and $\mathcal{Y}$ are inherited from some Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}}$ in $\mathcal{W}$. We require the norm induced by $(\cdot,\cdot)_{\mathcal{W}}$ to be equivalent to an admissible one, and we also require $\mathcal{U}$ and $\mathcal{Y}$ to be orthogonal with respect to $(\cdot,\cdot)_{\mathcal{W}}$, so that $(\cdot,\cdot)_{\mathcal{W}} = (\cdot,\cdot)_{\mathcal{Y}\oplus\mathcal{U}}$. Thus,

$$\left(w,w'\right)_{\mathcal{W}} = \left(w,w'\right)_{\mathcal{Y}\oplus\mathcal{U}} = (P_{\mathcal{Y}}^{\mathcal{U}} w, P_{\mathcal{Y}}^{\mathcal{U}} w')_{\mathcal{Y}} + (P_{\mathcal{U}}^{\mathcal{Y}} w, P_{\mathcal{U}}^{\mathcal{Y}} w')_{\mathcal{U}}\,. \tag{2.3}$$

The equivalence of the two norms means that

$$C_1\|w\|_{\mathcal{W}_-\oplus\mathcal{W}_+} \leqslant \|w\|_{\mathcal{W}} \leqslant C_2\|w\|_{\mathcal{W}_-\oplus\mathcal{W}_+}, \quad w \in \mathcal{W},$$

for some constants $0 < C_1 \leqslant C_2 < \infty$ and some fundamental decomposition $\mathcal{W} = -\mathcal{W}_-\ [\dotplus]\ \mathcal{W}_+$ of $\mathcal{W}$. This is equivalent to the requirement that the strong and weak topologies in the Kreĭn space $\mathcal{W}$ coincides with the strong and weak topologies in $\mathcal{W}$ induced by the norm $\|\cdot\|_{\mathcal{W}}$.

LEMMA 2.1. *Let $\mathcal{W}$ be a Kreĭn space $\mathcal{W}$ with the inner product $[\cdot,\cdot]_{\mathcal{W}}$, and let $(\cdot,\cdot)$ be a positive inner product in $\mathcal{W}$.*

(1) *The norm induced by the inner product $(\cdot,\cdot)_{\mathcal{W}}$ is equivalent to an admissible norm on $\mathcal{W}$ if and only if there is a bounded linear operator $\mathcal{J}$ with a bounded inverse in the Kreĭn space $\mathcal{W}$ such for all w, $w' \in \mathcal{W}$,*

$$\left(w,w'\right)_{\mathcal{W}} = \left[w,\mathcal{J}^{-1}w'\right]_{\mathcal{W}}, \tag{2.4}$$

or equivalently, for all w, $w' \in \mathcal{W}$,

$$\left[w,w'\right]_{\mathcal{W}} = \left(w,\mathcal{J}w'\right)_{\mathcal{W}}. \tag{2.5}$$

The operator $\mathcal{J}$ is self-adjoint with respect to both $[\cdot,\cdot]_{\mathcal{W}}$ and $(\cdot,\cdot)_{\mathcal{W}}$, both $\mathcal{J}$ and $\mathcal{J}^{-1}$ are uniformly positive with respect to $[\cdot,\cdot]_{\mathcal{W}}$, and $\mathcal{J}$ is determined uniquely by the two inner products $[\cdot,\cdot]_{\mathcal{W}}$ and $(\cdot,\cdot)_{\mathcal{W}}$.

(2) *The inner product $(\cdot,\cdot)_{\mathcal{W}}$ is itself admissible if and only if the operator $\mathcal{J}$ in* (1) *is unitary with respect to $[\cdot,\cdot]_{\mathcal{W}}$, or equivalently, unitary with respect to $(\cdot,\cdot)_{\mathcal{W}}$.*

An operator which is both self-adjoint and unitary is usually called a *signature operator*.

Proof of Lemma 2.1. (Proof of (2)): Let $\mathcal{W} = -\mathcal{W}_-\ [\dotplus]\ \mathcal{W}_+$ be a fundamental decomposition of the Kreĭn space $\mathcal{W}$, and let $\mathcal{J}_1 = \begin{bmatrix} -1_{\mathcal{W}_-} & 0 \\ 0 & 1_{\mathcal{W}_+} \end{bmatrix}$. Then $\mathcal{J}_1$ is self-adjoint and unitary both in the Kreĭn space $\mathcal{W}$ and in the Hilbert space $\mathcal{W}_- \oplus \mathcal{W}_+$, and for all $w, w' \in \mathcal{W}$,

$$[w, w']_{\mathcal{W}} = (w, \mathcal{J}_1 w')_{\mathcal{W}_- \oplus \mathcal{W}_+}. \tag{2.6}$$

This proves one part of assertion (2): if $(\cdot,\cdot)_{\mathcal{W}}$ is admissible, i.e., if $(\cdot,\cdot)_{\mathcal{W}} = (\cdot,\cdot)_{\mathcal{W}_- \oplus \mathcal{W}_+}$ for some fundamental decomposition $\mathcal{W} = -\mathcal{W}_- [\dot{+}] \mathcal{W}_+$ of $\mathcal{W}$, then (2.5) holds for some operator $\mathcal{J}$ which is self-adjoint and unitary with respect to both $[\cdot,\cdot]_{\mathcal{W}}$ and $(\cdot,\cdot)_{\mathcal{W}}$.

Conversely, suppose that (2.5) holds for some operator $\mathcal{J}$ which is self-adjoint and unitary with respect to $[\cdot,\cdot]_{\mathcal{W}}$. Thus, $\mathcal{J}^2 = 1_{\mathcal{W}}$. Define $P_\pm = \frac{1}{2}(1_{\mathcal{W}} \pm \mathcal{J})$. Then $\mathcal{J} = P_+ - P_-$, $P_+ + P_- = 1_{\mathcal{W}}$, $P_\pm^2 = P_\pm$, and $P_+P_- = P_-P_+ = 0$. Let $\pm\mathcal{W}_\pm = P_\pm \mathcal{W}$. Then $\mathcal{W} = -\mathcal{W}_- [\dot{+}] \mathcal{W}_+$ is a fundamental decomposition of $\mathcal{W}$, and $(\cdot,\cdot)_{\mathcal{W}} = (\cdot,\cdot)_{\mathcal{W}_- \oplus \mathcal{W}_+}$, i.e., the inner product $[\cdot,\cdot]_{\mathcal{W}}$ is admissible.

As we shall see in the proof of assertion (1), $\mathcal{J}$ in (2.5) is self-adjoint with respect to both $[\cdot,\cdot]_{\mathcal{W}}$ and $(\cdot,\cdot)_{\mathcal{W}}$. By using this fact it is easy to see that $\mathcal{J}$ is unitary with respect to $[\cdot,\cdot]_{\mathcal{W}}$ if and only if $\mathcal{J}$ is unitary with respect to $(\cdot,\cdot)_{\mathcal{W}}$: both of these conditions are namely equivalent to the requirement that $\mathcal{J}^2 = 1_{\mathcal{W}}$.

Proof of (1): We begin with a proof of the existence of the operator $\mathcal{J}$. Suppose that the norm $\|\cdot\|_{\mathcal{W}}$ induced by the inner product $(\cdot,\cdot)_{\mathcal{W}}$ is equivalent to an admissible norm $\|\cdot\|_{\mathcal{W}_- \oplus \mathcal{W}_+}$ induced by the inner product $(\cdot,\cdot)_{\mathcal{W}_- \oplus \mathcal{W}_+}$, where $\mathcal{W} = -\mathcal{W}_- \oplus \mathcal{W}_+$ is a fundamental decomposition of $\mathcal{W}$. Then both $(\cdot,\cdot)_{\mathcal{W}}$ and $(\cdot,\cdot)_{\mathcal{W}_- \oplus \mathcal{W}_+}$ are Hilbert space inner products on $\mathcal{W}$. Moreover, $(\cdot,\cdot)_{\mathcal{W}}$ is a bounded sesqui-linear form with respect to $(\cdot,\cdot)_{\mathcal{W}_- \oplus \mathcal{W}_+}$, and $(\cdot,\cdot)_{\mathcal{W}_- \oplus \mathcal{W}_+}$ is a bounded sesqui-linear form with respect to $(\cdot,\cdot)_{\mathcal{W}}$. This implies the existence of unique operators $K_1 \in \mathcal{B}(\mathcal{W})$ (with respect to $\|\cdot\|_{\mathcal{W}_- \oplus \mathcal{W}_+}$) and $K_2 \in \mathcal{B}(\mathcal{W})$ (with respect to the equivalent norm $\|\cdot\|_{\mathcal{W}}$) such that, for all $w, w' \in \mathcal{W}$,

$$\begin{aligned} (w, w')_{\mathcal{W}} &= (w, K_1 w')_{\mathcal{W}_- \oplus \mathcal{W}_+}, \\ (w, w')_{\mathcal{W}_- \oplus \mathcal{W}_+} &= (w, K_2 w')_{\mathcal{W}}. \end{aligned}$$

Clearly, $K_2 = K_1^{-1}$, so K_2 has a bounded inverse. Combining the second of two equations above with (2.6) we get (2.5) with the boundedly invertible operator $\mathcal{J} = K_2 \mathcal{J}_1$.

Conversely, let $(\cdot,\cdot)_{\mathcal{W}}$ be a positive inner product on $\mathcal{W}$, and suppose that (2.5) holds for some operator $\mathcal{J}$ which is bounded and has a bounded inverse in the Kreĭn space $\mathcal{W}$. Let $\mathcal{W} = -\mathcal{W}_- [\dot{+}] \mathcal{W}_+$ be a fundamental decomposition of the Kreĭn space $\mathcal{W}$, and let $\mathcal{J}_1$ be the operator in (2.6). Then both (2.5) and (2.6) hold, and consequently,

$$(w, w')_{\mathcal{W}} = (w, K w')_{\mathcal{W}_- \oplus \mathcal{W}_+}, \tag{2.7}$$

where $K = \mathcal{J}_1 \mathcal{J}^{-1}$ is bounded and has a bounded inverse with respect to the norm $\|\cdot\|_{\mathcal{W}_- \oplus \mathcal{W}_+}$. It follows from (2.7) that the operator K is positive and self-adjoint since $(\cdot,\cdot)_W$ is a positive inner product (i.e., a positive sesqui-linear form). Being boundedly invertible, K is therefore bounded both from above and away from zero, meaning that there exist constants $0 < c_1 \leqslant c_2 < \infty$ such that for all $w \in \mathcal{W}$,

$$c_1(w,w)_{\mathcal{W}_-\oplus\mathcal{W}_+} \leqslant (w,Kw)_{\mathcal{W}_-\oplus\mathcal{W}_+} \leqslant c_2(w,w)_{\mathcal{W}_-\oplus\mathcal{W}_+},$$

or equivalently,

$$c_1\|w\|^2_{\mathcal{W}_-\oplus\mathcal{W}_+} \leqslant \|w\|^2_{\mathcal{W}} \leqslant c_2\|w\|^2_{\mathcal{W}_-\oplus\mathcal{W}_+}.$$

Thus, the norm $\|\cdot\|_{\mathcal{W}}$ induced by $(\cdot,\cdot)_{\mathcal{W}}$ is equivalent to the admissible norm $\|\cdot\|_{\mathcal{W}_-\oplus\mathcal{W}_+}$ induced by $(\cdot,\cdot)_{\mathcal{W}_-\oplus\mathcal{W}_+}$.

The self-adjointness of the operator $\mathcal{J}$ in (2.5) with respect to $(\cdot,\cdot)_{\mathcal{W}}$ follows immediately from (2.5): for all w, $w' \in \mathcal{W}$ we have

$$\big(\mathcal{J}w,w'\big)_{\mathcal{W}} = \overline{\big(w',\mathcal{J}w\big)}_{\mathcal{W}} = \overline{\big[w',w\big]}_{\mathcal{W}} = \big[w,w'\big]_{\mathcal{W}} = \big(w,\mathcal{J}w'\big)_{\mathcal{W}}.$$

This further implies the self-adjointness of $\mathcal{J}$ with respect to $[\cdot,\cdot]_{\mathcal{W}}$: for all w, $w' \in \mathcal{W}$,

$$\big[\mathcal{J}w,w'\big]_{\mathcal{W}} = \big(\mathcal{J}w,\mathcal{J}w'\big)_{\mathcal{W}} = \big(w,\mathcal{J}^2w'\big)_{\mathcal{W}} = \big[w,\mathcal{J}w'\big]_{\mathcal{W}}.$$

The uniform positivity of $\mathcal{J}^{-1}$ follows directly from (2.4) and the fact that $\|\cdot\|_{\mathcal{W}}$ is equivalent to an admissible norm, whereas the uniform positivity of $\mathcal{J}$ follows from the fact that, for all $w \in \mathcal{W}$,

$$\big[w,\mathcal{J}w\big]_{\mathcal{W}} = \big(w,\mathcal{J}^2w\big)_{\mathcal{W}} = \big(\mathcal{J}w,\mathcal{J}w\big)_{\mathcal{W}},$$

combined with the facts that the norm $\|\cdot\|_{\mathcal{W}}$ is equivalent to an admissible norm and that $\mathcal{J}$ it has a bounded inverse. ∎

LEMMA 2.2. *Let $\mathcal{W}$ be a Kreĭn space. Given any direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ there exists a Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}}$ in $\mathcal{W}$ such that the norm induced by $(\cdot,\cdot)_{\mathcal{W}}$ is equivalent to an admissible norm, and such that $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal with respect to $(\cdot,\cdot)_{\mathcal{W}}$.*

Proof. We begin by choosing an arbitrary admissible Hilbert space inner product $(\cdot,\cdot)_1$ in $\mathcal{W}$ (without requiring $\mathcal{Y}$ and $\mathcal{U}$ to be orthogonal), and denote the Hilbert space inner products in $\mathcal{Y}$ and $\mathcal{U}$ inherited from $(\cdot,\cdot)_1$ by $(\cdot,\cdot)_{\mathcal{Y}}$ and $(\cdot,\cdot)_{\mathcal{U}}$, respectively. We then define a new positive inner product $(\cdot,\cdot)_{\mathcal{W}}$ in $\mathcal{W}$ by

$$\big(w,w'\big)_{\mathcal{W}} = (P^{\mathcal{U}}_{\mathcal{Y}}w, P^{\mathcal{U}}_{\mathcal{Y}}w')_{\mathcal{Y}} + (P^{\mathcal{Y}}_{\mathcal{U}}w, P^{\mathcal{Y}}_{\mathcal{U}}w')_{\mathcal{U}}.$$

Clearly, $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal with respect to $(\cdot,\cdot)_{\mathcal{W}}$. The fact that $\mathcal{W}$ is a direct sum decomposition of $\mathcal{Y}$ and $\mathcal{U}$ implies that the norm induced by $(\cdot,\cdot)_1$ is equivalent to the norm induced by $(\cdot,\cdot)_{\mathcal{W}}$ (a sequence w_n tends to zero in $\mathcal{W}$ if and only if both $y_n = P^{\mathcal{U}}_{\mathcal{Y}}w_n$ and $u_n = P^{\mathcal{Y}}_{\mathcal{U}}w_n$ tend to zero in $\mathcal{Y}$, respectively $\mathcal{U}$). Thus, $(\cdot,\cdot)_{\mathcal{W}}$ satisfies the requirements listed in (1). ∎

LEMMA 2.3. *Let $\mathcal{W}$ be a Kreĭn space with the direct sum decomposition $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$, and let $(\cdot,\cdot)_{\mathcal{W}}$ be a Hilbert space inner product in $\mathcal{W}$ such that the norm induced by $(\cdot,\cdot)_{\mathcal{W}}$ is equivalent to an admissible norm, and such that $\mathcal{F}$ and $\mathcal{E}$ are orthogonal with respect to $(\cdot,\cdot)_{\mathcal{W}}$ (by Lemma 2.2, such an inner product always exists). Thus,*

$(\cdot,\cdot)_{\mathcal{W}} = (\cdot,\cdot)_{\mathcal{F}\oplus\mathcal{E}}$, *where we equip* $\mathcal{F}$ *and* $\mathcal{E}$ *with the inner products inherited from* $(\cdot,\cdot)_{\mathcal{W}}$. *Let* $\mathcal{J} \in \mathcal{B}(\mathcal{W})$ *be the operator in* (2.5) *obtained from part* (1) *of Lemma* 2.1, *and decompose* $\mathcal{J}$ *into* $\mathcal{J} = \left[\begin{smallmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{smallmatrix}\right]$ *is accordance with the decomposition* $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$ *(so that* $\mathcal{J}_{11} = P^{\mathcal{E}}_{\mathcal{F}}\mathcal{J}|_{\mathcal{F}}$, *etc.).*

(1) *The subspaces* $\mathcal{E}$ *and* $\mathcal{F}$ *are Lagrangian if and only if* $\mathcal{J}_{11} = 0$ *and* $\mathcal{J}_{22} = 0$. *In this case* $\mathcal{J}_{12}$ *and* $\mathcal{J}_{21}$ *are boundedly invertible,* $\mathcal{J}_{21} = \mathcal{J}_{12}^*$, *and for all* $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} f' \\ e' \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{F} \\ \mathcal{E} \end{smallmatrix}\right]$ *we have*

$$\left[\begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix}\right]_{\mathcal{W}} = (f, \Psi e')_{\mathcal{F}} + (\Psi e, f')_{\mathcal{F}}, \tag{2.8}$$

where $\Psi = \mathcal{J}_{12} \in \mathcal{B}(\mathcal{E};\mathcal{F})$ *has a bounded inverse* $\Psi^{-1} \in \mathcal{B}(\mathcal{F};\mathcal{E})$. *Moreover, the formula*

$$\left(\begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix}\right)_0 = (f, f')_{\mathcal{F}} + (\Psi e, \Psi e')_{\mathcal{F}} \tag{2.9}$$

defines an admissible inner product on $\mathcal{W}$, *and* $\mathcal{F}$ *and* $\mathcal{E}$ *are orthogonal also with respect to this inner product.*

(2) *Assuming that the subspaces* $\mathcal{E}$ *and* $\mathcal{F}$ *are Lagrangian, the inner product* $(\cdot,\cdot)_{\mathcal{W}}$ *is itself admissible if and only if the operator* Ψ *in* (2.8) *and* (2.9) *is unitary.*

Proof. It is easy to see that $\mathcal{F}$ is neutral if and only if $\mathcal{J}_{11} = 0$, and that $\mathcal{E}$ is neutral if and only if $\mathcal{J}_{22} = 0$. If both these conditions are satisfied, then the invertibility of $\mathcal{J}$ implies that $\mathcal{J}_{12}$ and $\mathcal{J}_{21}$ are invertible. The invertibility of $\mathcal{J}_{12}$ and $\mathcal{J}_{21}$ imply that $\mathcal{F}$ and $\mathcal{E}$ must, in fact, be Lagrangian. The self-adjointness of $\mathcal{J}$ with respect to $(\cdot,\cdot)_{\mathcal{W}}$ together with the orthogonality of $\mathcal{F}$ and $\mathcal{E}$ implies that $\mathcal{J}_{21} = \mathcal{J}_{12}^*$, after which (2.5) can be written in the form (2.8). By Lemma 2.1, $(\cdot,\cdot)_{\mathcal{W}}$ is itself admissible if and only if $\mathcal{J}$ is unitary, and this is true if and only if Ψ is unitary.

The inner product $(\cdot,\cdot)_0$ defined in (2.9) is obtained from $(\cdot,\cdot)_{\mathcal{W}}$ by a rescaling of the norm in $\mathcal{E}$ in the sense that for all $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} f' \\ e' \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{F} \\ \mathcal{E} \end{smallmatrix}\right]$,

$$\left(\begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix}\right)_0 = (f, f')_{\mathcal{F}} + (e, \Psi^*\Psi e')_{\mathcal{E}}.$$

Clearly, $\mathcal{F}$ and $\mathcal{E}$ are still orthogonal with respect to $(\cdot,\cdot)_0$, and the norm induced by $(\cdot,\cdot)_0$ is equivalent to the norm induced by $(\cdot,\cdot)_{\mathcal{W}}$, hence equivalent to an admissible norm. Let $\mathcal{E}'$ stand for $\mathcal{E}$ equipped with the inner product

$$\left(e, e'\right)_{\mathcal{E}'} = \left(e, \Psi^*\Psi e'\right)_{\mathcal{E}}, \quad e,\ e' \in \mathcal{E}$$

induced by $(\cdot,\cdot)_0$. Then $(\cdot,\cdot)_0 = (\cdot,\cdot)_{\mathcal{F}\oplus\mathcal{E}'}$. With respect to this inner product the operator Ψ is unitary, and hence, by assertion (2), $(\cdot,\cdot)_0$ is admissible. ∎

COROLLARY 2.4. *Let* $\mathcal{W}$ *be a Kreĭn space with the direct sum decomposition* $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$. *If both* $\mathcal{F}$ *and* $\mathcal{E}$ *are Lagrangian, then there exists an admissible Hilbert space* *inner product* $(\cdot,\cdot)_{\mathcal{W}}$ *in* $\mathcal{W}$ *such that* $\mathcal{F}$ *and* $\mathcal{E}$ *are orthogonal with respect to* $(\cdot,\cdot)_{\mathcal{W}}$*, and such that* (2.8) *holds for some bounded linear operator* $\Psi\colon \mathcal{E} \to \mathcal{F}$ *which is unitary with respect to the inner products in* $\mathcal{F}$ *and* $\mathcal{E}$ *induced by* $(\cdot,\cdot)_{\mathcal{W}}$*.*

Proof. The inner product $(\cdot,\cdot)_0$ constructed in Lemma 2.3 is of this type.

We shall in the sequel write $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ instead of $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$ whenever $\mathcal{W}$ is the direct sum of two Lagrangian subspaces $\mathcal{F}$ and $\mathcal{E}$ and (2.8) holds for some unitary operator $\Psi\colon \mathcal{E} \to \mathcal{F}$. We call this a *Lagrangian decomposition* of $\mathcal{W}$, with *impedance weighting operator* Ψ. ∎

3. PASSIVE NODES AND SYSTEMS

Let Σ be a s/s system of the type described in the introduction, determined by an equation of the type (1.1). The forward passivity inequality (1.2) together with (1.1) says that the graph V of the operator F in (1.1) is nonnegative with respect to the Kreĭn space inner product

$$\left[\begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix}, \begin{bmatrix} \dot{x}' \\ x' \\ w' \end{bmatrix}\right]_{\mathfrak{K}} = -(\dot{x},\dot{x}')_{\mathcal{X}} + (x,x')_{\mathcal{X}} + [w,w']_{\mathcal{W}}, \quad \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix}, \begin{bmatrix} \dot{x}' \\ x' \\ w' \end{bmatrix} \in \mathfrak{K} \tag{3.1}$$

in the *node space* $\mathfrak{K} := -\mathcal{X} [\dotplus] \mathcal{X} [\dotplus] \mathcal{W} = \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$. As shown in Part II, even more is true: Σ is forward passive if and only if V is nonnegative, Σ is passive if and only if V is maximal nonnegative, Σ is forward conservative if and only if V is neutral, and Σ is conservative if and only if V is Lagrangian.

The properties that we required in the introduction from the operator F in (1.1) are equivalent to the following properties of the graph V of F:

(i) V is closed in the node space $\mathfrak{K}$.

(ii) For every $x \in \mathcal{X}$ there is some $\left[\begin{smallmatrix} \dot{x} \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in V$.

(iii) If $\begin{bmatrix} \dot{x} \\ 0 \\ 0 \end{bmatrix} \in V$, then $\dot{x} = 0$.

(iv) The set $\left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in V \text{ for some } \dot{x} \in \mathcal{X} \right\}$ is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

A colligation $\Sigma := (V; \mathcal{X}, \mathcal{Y})$ where the state space $\mathcal{X}$ is a Hilbert space, the signal space $\mathcal{W}$ is a Kreĭn space, and V is a subspace of the node space $\mathfrak{K}$ with properties (i)–(iv) is called a *s/s node*, and a sequence $(x(\cdot), w(\cdot))$ of vectors $x(n) \in \mathcal{X}$ and $w(n) \in \mathcal{W}$, $n \in \mathbb{Z}^+$, satisfying

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^+, \quad x(0) \quad = x_0, \tag{3.2}$$

is called a trajectory of Σ with initial state x_0. The subspace V in (3.2) is called the *generating* subspace of Σ. Under conditions (i)–(iv), if we let V be the graph of F in (1.1), then the set of equations (3.2) are equivalent to (1.1). By the s/s system we mean this node together with the set of all its trajectories, and we use the same notation $\Sigma = (V; \mathcal{X}, \mathcal{Y})$ for the system as for the original node.

Each s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has an *adjoint s/s system* $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$, which is defined as follows. The state space of Σ_* is the same as the state space of Σ, and the signal space $\mathcal{W}_*$ of Σ_* is equal to $\mathcal{W}_* = -\mathcal{W}$. Thus, the node space $\mathcal{K}_*$ of Σ_* is given by $\mathfrak{K}_* := -\mathcal{X} [\dot{+}] \mathcal{X} [\dot{+}] \mathcal{W}_* = \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W}_* \end{bmatrix}$. The generating subspace V_* of Σ_* will be defined shortly, after we recall from the introduction and from Part II how the adjoints of the signal space $\mathcal{W}$ and the node space $\mathfrak{K}$ should be interpreted.

As we mentioned in the introduction, instead of identifying the dual of the Kreĭn signal space $\mathcal{W}$ with itself we identify it with $\mathcal{W}_* = -\mathcal{W}$. This is done in such a way that all bounded linear functionals on $\mathcal{W}$ has the form given in (1.5) for some $w_* \in \mathcal{W}_*$, where $\mathcal{I}$ is the identity operator from $\mathcal{W}_*$ to $\mathcal{W}$, with $\mathcal{I}^* = -\mathcal{I}^{-1}$. In a similar way we identify the dual of the node space $\mathfrak{K}$ with $\mathfrak{K}_*$ via the duality pairing

$$\begin{aligned}(3.3)\qquad \left\langle \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix}, \begin{bmatrix} \dot{x}_* \\ x_* \\ w_* \end{bmatrix} \right\rangle_{\langle \mathfrak{K}, \mathfrak{K}_* \rangle} &= \left[\begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix}, \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} \dot{x}_* \\ x_* \\ w_* \end{bmatrix} \right]_{\mathfrak{K}} \\ &= -(\dot{x}, x_*)_{\mathcal{X}} + (x, \dot{x}_*)_{\mathcal{X}} + \langle w, w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle}.\end{aligned}$$

After these preliminaries we are ready to define the generating subspace V_* of the adjoint system to be the annihilator of V in $\mathfrak{K}_*$ with respect to the above duality pairing (3.3), i.e.,

$$(3.4)\quad V_* = V^{\langle \perp \rangle} = \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & -\mathcal{I}^* \end{bmatrix} V^{[\perp]} = \left\{ k_* \in \mathfrak{K}_* \mid \langle k, k_* \rangle_{\langle \mathfrak{K}, \mathfrak{K}_* \rangle} = 0 \text{ for all } k \in V \right\},$$

where $V^{[\perp]}$ stands for the orthogonal complement of V in $\mathfrak{K}$. As shown in Proposition II.4.1, if $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a s/s node, i.e., if V has properties (i)–(iv) listed above, then $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$ is also a s/s node, i.e., (i)–(iv) hold with V and $\mathcal{W}$ replaced by V_* and $\mathcal{W}_*$, respectively. Moreover, by Proposition II.4.6, a sequence $(x_*(\cdot), w_*(\cdot))$, where each $x_*(n) \in \mathcal{X}$ and $w_*(n) \in \mathcal{W}_*$, is a trajectory of Σ_* if and only if (1.6) holds for every trajectory $(x(\cdot), w(\cdot))$ of Σ.

The *reachable subspace* $\mathfrak{R}$ of a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is the closure in $\mathcal{X}$ of the set of all possible values of the state components $x(n)$, $n \in \mathbb{Z}^+$, of all *externally generated trajectories* of Σ, i.e., trajectories satisfying $x(0) = 0$. We call Σ *controllable* if $\mathfrak{R} = \mathcal{X}$. The *unobservable* subspace $\mathfrak{U}$ of Σ consists of all initial values $x(0)$ of all *unobservable trajectories*, i.e., trajectories $(x(\cdot), w(\cdot))$ where $w(n) = 0$, $n \in \mathbb{Z}^+$. We call Σ *observable* if $\mathfrak{U} = \{0\}$. As shown in Part I, a s/s system is *minimal* if and only if it is controllable and observable, where minimality means that the system does not have any nontrivial compression (the notions of dilations and compressions of s/s systems are defined in Section I.8). Finally, we call Σ *simple* if $\mathfrak{U} \cap \mathfrak{R}^\perp = \{0\}$, or equivalently, if $\mathfrak{U}^\perp \vee \mathfrak{R} = \mathcal{X}$. As shown in Proposition II.4.11, $\mathfrak{R}^\perp$ is the unobservable subspace of the adjoint system $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$, and $\mathfrak{U}^\perp$

is the reachable subspace of Σ_*. In particular, Σ is controllable if and only if Σ_* is observable, Σ is observable if and only if Σ_* is controllable, Σ is minimal if and only if Σ_* is minimal, and Σ is simple if and only if Σ_* is simple.

The *behavior* induced by a s/s system Σ is the set of all the signal components $w(\cdot)$ of all externally generated trajectories of Σ. Two s/s systems are *externally equivalent* if they induce the same behaviors. The behavior does not change if we dilate or compress a system, so that the resulting s/s systems are externally equivalent to the original one. As shown in Theorem II.3.7 and Corollary II.3.8, if $\Sigma = (V;\mathcal{X};\mathcal{W})$ is a passive system with reachable subspace $\mathfrak{R}$ and orthogonally observable subspace $\mathfrak{O} = \mathfrak{U}^\perp$, and if we define $\mathcal{X}_\circ = \overline{P_{\mathcal{O}}\mathfrak{R}}$ and $\mathfrak{X}_\bullet = \overline{P_{\mathfrak{R}}\mathfrak{O}}$, then the orthogonal compressions $\Sigma_\circ = (V_\circ;\mathcal{X}_\circ,\mathcal{W})$ and $\Sigma_\bullet = (V_\bullet;\mathcal{X}_\bullet,\mathcal{W})$ of Σ onto $\mathcal{X}_\circ$ and $\mathcal{X}_\bullet$, respectively, are minimal passive s/s systems (which are externally equivalent to Σ).

As we mentioned in the introduction, a s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ with a Hilbert state space $\mathcal{X}$ and a Kreĭn space $\mathcal{W}$ is *forward passive* if all trajectories $(x(\cdot),w(\cdot))$ of Σ satisfy the 'energy' inequality (1.2), and it is *forward conservative* if this inequality holds in the form of the equality (1.3). The system Σ is *backward passive* if the adjoint s/s system Σ_* is forward passive, and *backward conservative* if Σ_* is forward conservative. Finally, Σ is *passive* if it is both forward and backward passive, and it is *conservative* if it is both forward and backward conservative. From these definitions follow that a s/s node $\Sigma = (v;\mathcal{X},\mathcal{W})$ is

(1) forward passive (forward conservative) if and only if V is nonnegative (or $V \subset V^{[\perp]}$, respectively, i.e., V is neutral) in the Kreĭn space $\mathfrak{K}$;

(2) backward passive (backward conservative) if and only if $V^{[\perp]}$ is nonpositive (or $V^{[\perp]} \subset V$, respectively, i.e., V is co-neutral) in $\mathfrak{K}$;

(3) passive (conservative) if and only if V is maximal nonnegative (or V is Lagrangian, respectively) in $\mathfrak{K}$.

The proof of claim (3) is given in Theorem II.5.6. Even a finite-dimensional system can be forward passive without being passive (see Example II.5.5).

The following theorem and corollary can be derived from Theorem II.5.7 and Corollary II.5.10.

THEOREM 3.1 (Theorem II.5.7). *Let the s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ be forward passive. Then the following conditions are equivalent:*

(1) *Σ is passive.*

(2) *At least one fundamental decomposition $\mathcal{W} = -\mathcal{W}_- [\dotplus] \mathcal{W}_+$ of $\mathcal{W}$ is admissible for Σ.*

(3) *Every fundamental decomposition $\mathcal{W} = -\mathcal{W}_- [\dotplus] \mathcal{W}_+$ of $\mathcal{W}$ is admissible for Σ.*

We recall that an i/s/o representation $\Sigma_{i/s/o}$ of a s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ corresponding to some admissible fundamental decomposition $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ is called a *scattering representation*, and its transfer function is called a *scattering matrix* of $\Sigma_{i/s/o}$ (or of Σ).

COROLLARY 3.2. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s signal system which has an admissible fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of the signal space. Let $\Sigma^{\mathrm{sca}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be the corresponding admissible scattering representation of Σ. Then the following claims are true:*

(1) *Σ is passive if and only if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a contraction from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$.*

(2) *Σ is passive and forward conservative if and only if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is an isometry from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$.*

(3) *Σ is conservative if and only if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a unitary operator from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$.*

By definition, the Schur class $S(\mathbb{D}; \mathcal{U}, \mathcal{Y})$ appearing in the following theorem is the set of all $\mathcal{B}(\mathcal{U}; \mathcal{Y})$-valued contractive holomorphic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

THEOREM 3.3. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and let $\mathcal{W} = -\mathcal{W}_-\,[\dotplus]\,\mathcal{W}_+$ be a fundamental decomposition of $\mathcal{W}$. Then the restriction of the corresponding scattering matrix $\mathfrak{D}$ to $\mathbb{D}$ belongs to $S(\mathbb{D}; \mathcal{W}_+, \mathcal{W}_-)$.*

This follows from Theorem 3.1, Corollary 3.2, and well-known facts about the scattering matrix of a passive i/s/o system (see Section II.6).

PROPOSITION 3.4 (Proposition II.6.2). *Let φ belong to the Schur class $S(\mathbb{D}; \mathcal{U}, \mathcal{Y})$ for some Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. Then there exists a simple conservative s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ such that $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$, and such that the corresponding scattering matrix $\mathfrak{D}$ of Σ satisfies $\mathfrak{D}|_{\mathbb{D}} = \varphi$. This system is determined uniquely by φ up to a unitary similarity transformation in the state space.*

In Sections 3–5 of Part I we developed the following three different kinds of representations of a s/s system.

PROPOSITION 3.5. *Let V be a subspace of the node space $\mathfrak{K}$. Then the following assertions are equivalent:*

(1) *V has properties* (i)–(iv), *i.e., $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a s/s node.*

(2) *V has the* driving variable *representation*

$$V = \mathcal{R}\left(\begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix}\right) = \left\{ \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in \mathfrak{K} \;\middle|\; \begin{matrix} \dot{x} = A'x + B'\ell, \\ w = C'x + D'\ell, \end{matrix}\; \ell \in \mathcal{L} \right\}, \tag{3.5}$$

for some bounded linear operator $\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{L} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right])$ with the additional requirement that D' is injective and has closed range. Here $\mathcal{L}$ is an auxiliary Hilbert space, called the driving variable *space.*

(3) *V has the output nulling representation*

$$V = \mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & A'' & B'' \\ 0 & C'' & D'' \end{bmatrix}\right) = \left\{ \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in \mathfrak{K} \;\middle|\; \begin{matrix} \dot{x} = A''x + B''w \\ 0 = C''x + D''w \end{matrix} \right\} \tag{3.6}$$

for some bounded linear operator $\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{K} \end{smallmatrix}\right])$ *with the additional requirement that* D'' *is surjective. Here* $\mathcal{K}$ *is an auxiliary Hilbert space, called the* error *space.*

(4) *V has the* i/s/o representation *(i.e., input/state/output) representation*

$$V = \mathcal{R}\left(\begin{bmatrix} A & B \\ 1_{\mathcal{X}} & 0 \\ C & D \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} -1_{\mathcal{X}} & A & 0 & B \\ 0 & C & -1_{\mathcal{Y}} & D \end{bmatrix}\right) = \left\{ \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in \mathfrak{K} \;\middle|\; \begin{aligned} \dot{x} &= Ax + Bu, \\ w &= Cx + Du + u, \end{aligned} \right\} \tag{3.7}$$

for some bounded linear operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right])$*. Here* $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ *is a direct sum decomposition of* $\mathcal{W}$*.*

This follows from Lemmas I.3.1 and I.4.1 and Theorem I.5.1.

In Parts I and II we used the following notations: a driving variable representation of Σ was typically denoted by $\Sigma_{dv/s/s} := \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$, an output nulling representation by $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}\right)$, and an i/s/o representation was denoted by $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$. In the case of an i/s/o representation with orthogonal input and output spaces we throughout use the Kreĭn space inner products in $\mathcal{Y}$ and $\mathcal{U}$ inherited from the orthogonal decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$, and this makes it possible to reconstruct $\mathcal{W}$ once we know the inner products in $\mathcal{Y}$ and $\mathcal{U}$. In this connection the corresponding i/s/o representation is called a *transmission representation*, and it is denoted by $\Sigma^{\mathrm{tra}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$. In the case of a Lagrangian decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ we use inner products in $\mathcal{F}$ and $\mathcal{E}$ which are induced by an admissible Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}}$ in $\mathcal{W}$ of the type mentioned in Corollary 2.4. This means that the original inner product in $\mathcal{W}$ is given by (2.8) for some signature operator $\mathcal{J} := \mathcal{J}_\Psi := \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$, where Ψ is a unitary operator from $\mathcal{E}$ to $\mathcal{F}$. In this case we supplement the notation for the corresponding impedance representation by also mentioning the impedance weighting operator Ψ, and denote it by $\Sigma^{\mathrm{imp}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \Psi\right)$.

We end this section with some results on the positivity of kernels deduced from driving variable or output nulling representations of a passive s/s system.

THEOREM 3.6. *Let* $\mathfrak{D}'(z)$ *be the driving-variable-to-signal transfer function of a driving variable representation* $\Sigma_{dv/s/s} := \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ *of a forward passive s/s system* $\Sigma = (V; \mathcal{X}, \mathcal{W})$*. Then* $\mathfrak{D}'(z)$ *is holomorphic on* $\Lambda_{A'}$*, and the kernel*

$$K_{\mathfrak{D}'}(z,\zeta) = \frac{\mathfrak{D}'(z)^*\mathfrak{D}'(\zeta)}{1 - \bar{z}\zeta}, \quad z, \zeta, \in \Omega'_+ \tag{3.8}$$

is positive definite on the set $\Omega'_+ \times \Omega'_+$, *where* $\Omega'_+ = \Lambda_{A'} \cap \mathbb{D}$. *(This means explicitly that* $\sum_{i=0}^{n}\sum_{j=0}^{n} \frac{1}{1-z_i\bar{z}_j}[\mathfrak{D}'(z_i)\ell_i, \mathfrak{D}'(z_j)\ell_j]_{\mathcal{W}} \geqslant 0$ *for all* $n \in \mathbb{Z}^+$ *and all sequences* $z_1, z_2, \dots, z_n \in \Omega'_+$ *and* $\ell_1, \ell_2, \dots, \ell_n \in \mathcal{L}$.*)*

Proof. The forward passivity of Σ implies that (see Lemma II.5.14)

$$\begin{bmatrix} (A')^* & 1_{\mathcal{X}} & (C')^* \\ (B')^* & 0 & (D')^* \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix} \geqslant 0.$$

The left-hand side of this inequality, being nonnegative, can be factored.

The forward passivity of Σ implies that the above inequality in Lemma II.5.14 holds. This implies that the left-hand side of this inequality, being nonnegative, can be factored as $\left[\begin{smallmatrix} M^* \\ N^* \end{smallmatrix}\right] \begin{bmatrix} M & N \end{bmatrix}$ for some operators $M \in \mathcal{B}(\mathcal{X};\mathcal{Z})$ and $N \in \mathcal{B}(\mathcal{L};\mathcal{Z})$ for some auxiliary space $\mathcal{Z}$ (for example, we may take $\mathcal{Z}$ to be the closure of the range of the square root of the left-hand side, and define M and N to be the restrictions to $\mathcal{X}$ and $\mathcal{L}$, respectively, of this square root). Thus, the inequality (1)(b) becomes an equality when we replace the right-hand side by $\left[\begin{smallmatrix} M^* \\ N^* \end{smallmatrix}\right] \begin{bmatrix} M & N \end{bmatrix}$. This leads to the following identities:

$$\begin{aligned} -A'^*A' + 1_{\mathcal{X}} + C'^*C' &= M^*M, & -A'^*B' + C'^*D' &= M^*N, \\ -B'^*A' + D'^*C' &= N^*M, & -B'^*B' + D'^*D' &= N^*N. \end{aligned}$$

Using these identities we get for all $z, \zeta \in \Lambda_{A'}$ (after a short computation)

$$\begin{aligned} \mathfrak{D}'(z)^*\mathfrak{D}'(\zeta) &= [D'^* + \bar{z}B'^*(1_{\mathcal{X}} - \bar{z}A'^*)^{-1}C'^*][D' + \zeta C'(1_{\mathcal{X}} - \zeta A')^{-1}B'] \\ &= \mathfrak{D}_{N,M}(z)^*\mathfrak{D}_{N,M}(\zeta) \\ &\quad + B'^*(1_{\mathcal{X}} - \bar{z}A'^*)^{-1}(1 - \bar{z}\zeta)(1_{\mathcal{X}} - \zeta A')^{-1}B', \end{aligned}$$

where

$$\mathfrak{D}_{N,M}(\zeta) = N + \zeta M(1_{\mathcal{X}} - \zeta A')^{-1}B', \quad \zeta \in \Lambda_{A'}. \tag{3.9}$$

All the four operators in (3.9) act between Hilbert spaces, and $\mathfrak{D}_{N,M}$ is a holomorphic $\mathcal{B}(\mathcal{L};\mathcal{Z})$-valued function on $\Lambda_{A'}$. At each point $\zeta \in \Lambda_{A'}$ the adjoint $\mathfrak{D}_{N,M}(\zeta)^*$ is computed with respect to the Hilbert spaces $\mathcal{L}$ and $\mathcal{Z}$. Since the kernel $\frac{1}{1-\bar{z}\zeta}$ is positive definite on $\mathbb{D} \times \mathbb{D}$, also the kernel $\frac{\mathfrak{D}_{N,M}(z)^*\mathfrak{D}_{N,M}(\zeta)}{1-\bar{z}\zeta}$ is positive definite on $\Omega'_+ \times \Omega'_+$. By the same argument with $\frac{1}{1-\bar{z}\zeta}$ replaced by 1, $B'^*(1_{\mathcal{X}} - \bar{z}A'^*)^{-1}(1_{\mathcal{X}} - \zeta A')^{-1}B'$ is positive definite on $\Lambda_{A'} \times \Lambda_{A'}$. Thus, $\frac{D'(z)^*D'(\zeta)}{1-\bar{z}\zeta}$ is the sum of two positive definite kernels on $\Omega'_+ \times \Omega'_+$, and hence itself positive definite. ∎

THEOREM 3.7. *Let* $\mathfrak{D}''(z)$ *be the signal-to-error transfer function of an output nulling representation* $\Sigma_{s/s/on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K}\right)$ *of a backward passive s/s system*

$\Sigma = (V; \mathcal{X}, \mathcal{W})$. *Then* $\mathfrak{D}''(\zeta)$ *is holomorphic on* $\Lambda_{A''}$, *and the kernel*

$$\widetilde{K_{\mathfrak{D}''}}(z,\zeta) = \frac{\mathfrak{D}''(z)\mathfrak{D}''(\zeta)^*}{1 - z\overline{\zeta}}, \quad z,\zeta \in \Omega''_+ \tag{3.10}$$

is positive definite on the set $\Omega''_+ \times \Omega''_+$, *where* $\Omega''_+ = \Lambda_{A''} \cap \mathbb{D}$.

Proof. This theorem may be proved in the same way as the previous theorem, with the inequality (1)(b) in Lemma II.5.14 replaced by the inequality (2)(b) in the same lemma. ∎

Alternatively, Theorem 3.7 may be derived from Theorem 3.6 and the following two facts: the adjoint of an output nulling representation is a driving variable representation of the adjoint s/s system (see Proposition II.4.10), and the adjoint s/s system is forward passive if and only if the original s/s system is backward passive.

THEOREM 3.8. *Let* $\mathfrak{D}'(z)$ *be the driving-variable-to-signal transfer function of a driving variable representation with main operator* A' *of a passive s/s system* Σ, *and let* $\mathfrak{D}''(z)$ *be the signal-to-error transfer function of an output nulling representation with main operator* A'' *of the same system. Let* $\Omega'_+ = \Lambda_{A'} \cap \mathbb{D}$, *and let* $\Omega''_+ = \Lambda_{A''} \cap \mathbb{D}$. *Then the kernels* $K_{\mathfrak{D}'}(z,\zeta)$ *and* $\widetilde{K_{\mathfrak{D}''}}(z,\zeta)$ *defined by* (3.8) *and* (3.10), *respectively, are positive definite on* $\Omega'_+ \times \Omega'_+$ *and* $\Omega''_+ \times \Omega''_+$, *respectively.*

Proof. This follows from the preceding two theorems. ∎

COROLLARY 3.9. *Let* $\mathfrak{D}^{\mathrm{tra}}$ *be the transmission matrix of a transmission representation* $\Sigma^{\mathrm{tra}} = \left(\left[\begin{smallmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *of a passive s/s system* Σ *corresponding to some admissible orthogonal decomposition* $\mathcal{W} = -\mathcal{Y} \,[\dotplus]\, \mathcal{U}$ *of the signal space* $\mathcal{W}$. *Let* $\Omega_+ = \Lambda_{A^{\mathrm{tra}}} \cap \mathbb{D}$. *Then the two kernels*

$$K_{\mathfrak{D}^{\mathrm{tra}}}(z,\zeta) = \frac{1_{\mathcal{U}} - \mathfrak{D}^{\mathrm{tra}}(z)^*\mathfrak{D}^{\mathrm{tra}}(\zeta)}{1 - \overline{z}\zeta}, \quad z,\zeta \in \Omega_+$$

and

$$\widetilde{K_{\mathfrak{D}^{\mathrm{tra}}}}(z,\zeta) = \frac{1_{\mathcal{Y}} - \mathfrak{D}^{\mathrm{tra}}(z)\mathfrak{D}^{\mathrm{tra}}(\zeta)^*}{1 - z\overline{\zeta}}, \quad z,\zeta \in \Omega_+,$$

are positive definite on $\Omega_+ \times \Omega_+$.

Proof. This follows from Theorem 3.8 since a transmission representation $\Sigma^{\mathrm{tra}} = \left(\left[\begin{smallmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ of Σ can be interpreted both as a driving variable representation with parameter space $\mathcal{U}$ and transfer function $\mathfrak{D}'(z) = \left[\begin{smallmatrix} \mathfrak{D}^{\mathrm{tra}}(z) \\ 1_{\mathcal{U}} \end{smallmatrix}\right]$, and as an output nulling representation with error space $\mathcal{Y}$ and transfer function $\mathfrak{D}''(z) = \begin{bmatrix} -1_{\mathcal{Y}} & \mathfrak{D}^{\mathrm{tra}}(z) \end{bmatrix}$. ∎

COROLLARY 3.10. *Let* $\mathfrak{D}^{\mathrm{sca}}$ *be the scattering matrix of a passive s/s system* Σ *corresponding to some fundamental decomposition* $\mathcal{W} = -\mathcal{W}_- \,[\dotplus]\, \mathcal{W}_+$ *of the signal* *space $\mathcal{W}$. Then the two kernels*

$$K_{\mathfrak{D}^{\mathrm{sca}}}(z,\zeta)=\frac{1_{\mathcal{W}_+}-\mathfrak{D}^{\mathrm{sca}}(z)^*\mathfrak{D}^{\mathrm{sca}}(\zeta)}{1-\bar{z}\zeta}$$

and

$$\widetilde{K_{\mathfrak{D}^{\mathrm{sca}}}}(z,\zeta)=\frac{1_{\mathcal{W}_-}-\mathfrak{D}^{\mathrm{sca}}(z)\mathfrak{D}^{\mathrm{sca}}(\zeta)^*}{1-z\bar{\zeta}},$$

are positive definite on $\mathbb{D}\times\mathbb{D}$.

Proof. This is a special case of Corollary 3.9. ∎

4. TRANSMISSION REPRESENTATIONS AND MATRICES

In this section we shall consider i/s/o representations of a s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ which are induced by admissible *orthogonal* decompositions $\mathcal{W}=-\mathcal{Y}[\dot{+}]\,\mathcal{U}$ of the signal space $\mathcal{W}$. Thus, if we denote $y = P_{\mathcal{Y}}w$ and $u = P_{\mathcal{U}}w$ (where the projections are orthogonal with respect to the original inner product in $\mathcal{W}$), then (2.2) holds. Recall that such a representation of Σ is called a *transmission representation*, and its transfer function is called the *transmission matrix* of $\Sigma_{i/s/o}$. In the transmission case the forward passivity condition (1.2) and the forward conservativity condition (1.3) can be rewritten in the form (1.16) and (1.17), respectively.

LEMMA 4.1. *Let $\Sigma = (V,\mathcal{X},\mathcal{W})$ be a s/s system. Suppose that $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ is an admissible orthogonal decomposition of $\mathcal{W}$, and let $\Sigma^{\mathrm{tra}} = (\left[\begin{smallmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}}\end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y})$ be the corresponding transmission representation of Σ.*

(1) *The following conditions are equivalent:*

(a) *Σ is forward passive (or forward conservative).*

(b) *The four block operator $\left[\begin{smallmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}}\end{smallmatrix}\right]$ is a contraction (or isometry), i.e.,*

$$\begin{bmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}}\end{bmatrix}^*\begin{bmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}}\end{bmatrix}\leqslant 1_{\mathcal{X}[\dot{+}]\mathcal{U}}\ (or = 1_{\mathcal{X}[\dot{+}]\mathcal{U}}) \tag{4.1}$$

in the Kreĭn space $\mathcal{X}\,[\dot{+}]\,\mathcal{U}$.

(2) *The following conditions are equivalent:*

(a) *Σ is backward passive (or backward conservative).*

(b) *The four block operator $\left[\begin{smallmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}}\end{smallmatrix}\right]^*$ is a contraction (or isometry), i.e.,*

$$\begin{bmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}}\end{bmatrix}\begin{bmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}}\end{bmatrix}^*\leqslant 1_{\mathcal{X}[\dot{+}]\mathcal{Y}}\ (or = 1_{\mathcal{X}[\dot{+}]\mathcal{Y}}) \tag{4.2}$$

in the Kreĭn space $\mathcal{X}\,[\dot{+}]\,\mathcal{Y}$.

(3) *The following conditions are equivalent:*

(a) *Σ is passive (or conservative).*

(b) *The four block operator $\left[\begin{smallmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}}\end{smallmatrix}\right]$ is a bi-contraction (or unitary), i.e.,* *both* (1)(b) *and* (2)(b) *hold.*

Proof. The inequality (4.1) is equivalent to the (1.16). By Proposition II.4.11, (4.2) is equivalent to the inequality (1.16) for the adjoint system. ∎

This result can be further refined by using the following well-known result.

LEMMA 4.2. *Let D be a contraction between the two Kreĭn spaces $\mathcal{U}$ and $\mathcal{Y}$. Let $\mathcal{U} = -\mathcal{U}_- [+] \mathcal{U}_+$ and $\mathcal{Y} = -\mathcal{Y}_- [+] \mathcal{Y}_+$ be fundamental decompositions of $\mathcal{U}$ and $\mathcal{Y}$, respectively, and decompose D accordingly into $D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$. Then D_{11} has a bounded left-inverse. Moreover, D^* is a contraction (i.e., D is a bi-contraction) if and only if D_{11} has a bounded inverse.*

This lemma is (essentially) contained in [14, Theorem 1.3.4] and also in [15, Theorem 2.8′, p. 169] (earlier more implicit versions are found in [10, Theorem 4.19, p. 124] and [11, Theorem 6.1, p. 158]).

For the convenience of the reader we outline a proof.

Outline of Proof. The left-invertibility of D_{11} follows from the contractivity of D which implies that, if we compute the adjoints of D_{ij}, $i, j = 1, 2$, with respect to the Hilbert spaces $\mathcal{U}_-, \mathcal{U}_+, \mathcal{U}_-$, and $\mathcal{U}_+$, then $D_{11}^* D_{11} \geqslant 1_{\mathcal{U}_-} + D_{21}^* D_{21} \geqslant 1_{\mathcal{U}_-}$. If also D^* is contractive, then $D_{11} D_{11}^* \geqslant 1_{\mathcal{Y}_-} + D_{12} D_{12}^* \geqslant 1_{\mathcal{Y}_-}$, so in this case D_{11} is invertible. For the converse proof it suffices to observe that if D_{11} is invertible, then

$$E := \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} D_{11}^{-1} & -D_{11}^{-1} D_{12} \\ D_{21} D_{11}^{-1} & D_{22} - D_{21} D_{11}^{-1} D_{12} \end{bmatrix}$$

is a contraction between the Hilbert spaces $\mathcal{Y}_- \oplus \mathcal{U}_+$ and $\mathcal{U}_- \oplus \mathcal{Y}_+$, hence its adjoint is contractive. Reverting the above transformation we find that D^* is contractive. ∎

PROPOSITION 4.3. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a forward passive system, and let $\mathcal{W} = -\mathcal{Y} [+] \mathcal{U}$ be an admissible orthogonal decomposition of $\mathcal{W}$, with the corresponding transmission representation $\Sigma^{\mathrm{tra}} = \left(\begin{bmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}} \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$. Let $\mathcal{U} = -\mathcal{U}_- [\dot{+}] \mathcal{U}_+ = \begin{bmatrix} -\mathcal{U}_- \\ \mathcal{U}_+ \end{bmatrix}$ and $\mathcal{Y} = -\mathcal{Y}_- [\dot{+}] \mathcal{Y}_+ = \begin{bmatrix} -\mathcal{Y}_- \\ \mathcal{Y}_+ \end{bmatrix}$ be fundamental decompositions of $\mathcal{U}$ and $\mathcal{Y}$, respectively, and decompose D^{tra} accordingly into $D^{\mathrm{tra}} = \begin{bmatrix} D_{11}^{\mathrm{tra}} & D_{12}^{\mathrm{tra}} \\ D_{21}^{\mathrm{tra}} & D_{22}^{\mathrm{tra}} \end{bmatrix}$. Then D^{tra} is a contraction between the Kreĭn spaces $\mathcal{U}$ and $\mathcal{Y}$, and the following conditions are equivalent:*

(1) *Σ is passive.*
(2) *$(D^{\mathrm{tra}})^*$ is a contraction between the Kreĭn spaces $\mathcal{Y}$ and $\mathcal{U}$.*
(3) *D_{11}^{tra} has a bounded inverse.*

Proof. That D^{tra} is a contraction from $\mathcal{U}$ to $\mathcal{Y}$ follows from part (1) of Lemma 4.1.

Suppose that Σ is passive. Then it follows from Lemma 4.1 that both $D^{\rm tra}$ and $(D^{\rm tra})^*$ are contractive. Thus, (1) $\Rightarrow$ (2). If (2) holds, then by Lemma 4.2, $D^{\rm tra}_{11}$ has a bounded inverse. Thus, (2) $\Rightarrow$ (3). If $D^{\rm tra}_{11}$ has a bounded inverse, then by the same lemma, $\left[\begin{smallmatrix} A^{\rm tra} & B^{\rm tra} \\ C^{\rm tra} & D^{\rm tra}\end{smallmatrix}\right]$ is a bi-contraction since $-\mathcal{Y}_- \,[\dot{+}]\, \left[\begin{smallmatrix} \mathcal{Y}_+ \\ \mathcal{X}\end{smallmatrix}\right]$ is a fundamental decomposition of $\mathcal{Y} \,[\dot{+}]\, \mathcal{X}$ and $-\mathcal{U}_- \,[\dot{+}]\, \left[\begin{smallmatrix} \mathcal{U}_+ \\ \mathcal{X}\end{smallmatrix}\right]$ is a fundamental decomposition of $\mathcal{U} \,[\dot{+}]\, \mathcal{X}$. Thus (3) $\Rightarrow$ (1). ∎

An i/s/o system $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ with a Hilbert state space and Kreĭn input and output spaces $\mathcal{U}$ and $\mathcal{Y}$, respectively, satisfying (4.1) and (4.2) is called *transmission passive*, or *transmission conservative* in the case where (4.1) and (4.2) hold as equalities. Transmission passive and conservative i/s/o systems have been studied in many places, see, e.g., the discussion on [1, p. 205] and [4]. These works use a 'standard' transformation that converts a transmission passive i/s/o system into a scattering i/s/o system. The most common name for this transformation is probably the *Potapov-Ginzburg transformation* (originating from the Odessa school). The name *chain scattering transformation* is also widely used in the west (see, e.g., [13] and [16]). In [20] it is called *partial flow inversion*, since it interchanges a part of the input with a part of the output. See Remark 4.6 below.

We remind the reader that for a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ every fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ is admissible. Let us denote the corresponding scattering representation of Σ by

$$\Sigma^{\rm sca} = \left(\left[\begin{smallmatrix} A^{\rm sca} & B^{\rm sca} \\ C^{\rm sca} & D^{\rm sca}\end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-\right).$$

The is/so transfer function of $\Sigma^{\rm sca}$ is defined and holomorphic on the unit disk $\mathbb{D}$, and it is given by

$$\begin{bmatrix} \mathfrak{A}^{\rm sca}(z) & \mathfrak{B}^{\rm sca}(z) \\ \mathfrak{C}^{\rm sca}(z) & \mathfrak{D}^{\rm sca}(z)\end{bmatrix} = \begin{bmatrix} (1_{\mathcal{X}} - zA^{\rm sca})^{-1} & (1_{\mathcal{X}} - zA^{\rm sca})^{-1}B^{\rm sca} \\ (1_{\mathcal{X}} - zA^{\rm sca})^{-1}C & C^{\rm sca}(1_{\mathcal{X}} - zA^{\rm sca})^{-1}B^{\rm sca} + D^{\rm sca}\end{bmatrix} \tag{4.3}$$

(in Parts I–II we called this function the 'four block transfer function'). In particular, the i/o transfer function $\mathfrak{D}^{\rm sca}$ is contractive in the unit disk $\mathbb{D}$, i.e., $\mathfrak{D}^{\rm sca}|_{\mathbb{D}} \in S(\mathbb{D}; \mathcal{W}_+, \mathcal{W}_-)$, where $S(\mathbb{D}; \mathcal{W}_+, \mathcal{W}_-)$ is the Schur class of $\mathcal{B}(\mathcal{W}_+; \mathcal{W}_-)$-valued functions on $\mathbb{D}$. However, it need not be true that every orthogonal decomposition $\mathcal{W} = -\mathcal{Y} \,[\dot{+}]\, \mathcal{U}$ of $\mathcal{W}$ is admissible. To study this case we introduce the following four block decompositions of the identity operator:

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22}\end{bmatrix} = \begin{bmatrix} P_{\mathcal{Y}}|_{\mathcal{W}_-} & P_{\mathcal{Y}}|_{\mathcal{W}_+} \\ P_{\mathcal{U}}|_{\mathcal{W}_-} & P_{\mathcal{U}}|_{\mathcal{W}_+}\end{bmatrix}, \tag{4.4}$$

$$\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22}\end{bmatrix} = \begin{bmatrix} P_{\mathcal{W}_-}|_{\mathcal{Y}} & P_{\mathcal{W}_-}|_{\mathcal{U}} \\ P_{\mathcal{W}_+}|_{\mathcal{Y}} & P_{\mathcal{W}_+}|_{\mathcal{U}}\end{bmatrix}. \tag{4.5}$$

Here the projections in (4.4) are orthogonal with respect to the original Kreĭn space inner product of $\mathcal{W}$ and also with respect to Hilbert space inner product $(\cdot,\cdot)_{\mathcal{Y}\oplus\mathcal{U}}$, but not, in general, with respect to the Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}_-\oplus\mathcal{W}_+}$. A similar comment applies to the projections in (4.5).

THEOREM 4.4. *Let* $\Sigma = (V;\mathcal{X},\mathcal{W})$ *be a passive s/s system with a scattering representation* $\Sigma^{\rm sca} = \big(\big[\begin{smallmatrix} A^{\rm sca} & B^{\rm sca} \\ C^{\rm sca} & D^{\rm sca}\end{smallmatrix}\big];\mathcal{X},\mathcal{W}_+,\mathcal{W}_-\big)$, *and define the is/so transfer function of* $\Sigma^{\rm sca}$ *by* (4.3). *Let* $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ *be an orthogonal decomposition of* $\mathcal{W}$, *and define* Θ *and* $\widetilde{\Theta}$ *by* (4.4)–(4.5).

(1) *The following conditions are equivalent:*

(a) *The decomposition* $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ *is admissible.*

(b) *The operator* $\Theta_{21}D^{\rm sca} + \Theta_{22}$ *is boundedly invertible.*

(c) *The operator* $\widetilde{\Theta}_{11} - D^{\rm sca}\widetilde{\Theta}_{21}$ *is boundedly invertible.*

(2) *Let the equivalent conditions* (a), (b), *and* (c) *in* (1) *hold. Then the corresponding (passive) transmission representation*

$$\Sigma^{\rm tra} = \left(\begin{bmatrix} A^{\rm tra} & B^{\rm tra} \\ C^{\rm tra} & D^{\rm tra}\end{bmatrix};\mathcal{X},\mathcal{U},\mathcal{Y}\right)$$

of Σ *is given by*

$$\begin{aligned}\begin{bmatrix} A^{\rm tra} & B^{\rm tra} \\ C^{\rm tra} & D^{\rm tra}\end{bmatrix} &= \begin{bmatrix} A^{\rm sca} & B^{\rm sca} \\ \Theta_{11}C^{\rm sca} & \Theta_{11}D^{\rm sca}+\Theta_{12}\end{bmatrix}\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21}C^{\rm sca} & \Theta_{21}D^{\rm sca}+\Theta_{22}\end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1_{\mathcal{X}} & -B^{\rm sca}\widetilde{\Theta}_{21} \\ 0 & \widetilde{\Theta}_{11} - D^{\rm sca}\widetilde{\Theta}_{21}\end{bmatrix}^{-1}\begin{bmatrix} A^{\rm sca} & B^{\rm sca}\widetilde{\Theta}_{22} \\ C^{\rm sca} & -\widetilde{\Theta}_{12} + D^{\rm sca}\widetilde{\Theta}_{22}\end{bmatrix}.\end{aligned}\tag{4.6}$$

(3) *Let the equivalent conditions* (a), (b), *and* (c) *in* (1) *hold, and let* $z \in \Lambda_{A^{\rm sca}}$. *Then the following conditions are equivalent:*

(a) $z \in \Lambda_{A^{\rm tra}}$.

(b) *The operator* $\Theta_{21}\mathfrak{D}^{\rm sca}(z) + \Theta_{22}$ *has a bounded inverse.*

(c) *The operator* $\left[\begin{smallmatrix} 1_{\mathcal{X}}-zA^{\rm sca} & -zB^{\rm sca} \\ \Theta_{21}C^{\rm sca} & \Theta_{21}D^{\rm sca}+\Theta_{22}\end{smallmatrix}\right]$ *has a bounded inverse.*

(d) *The operator* $\widetilde{\Theta}_{11} - \mathfrak{D}^{\rm sca}(z)\widetilde{\Theta}_{21}$ *has a bounded inverse.*

(e) *The operator* $\left[\begin{smallmatrix} 1_{\mathcal{X}} & -\mathfrak{B}^{\rm sca}(z)\widetilde{\Theta}_{21} \\ 0 & \widetilde{\Theta}_{11}-\mathfrak{D}^{\rm sca}(z)\widetilde{\Theta}_{21}\end{smallmatrix}\right]$ *has a bounded inverse.*

(4) *If the equivalent conditions* (a)–(e) *in* (1) *hold, then for all* $z \in \Lambda_{A^{\rm sca}} \cap \Lambda_{A^{\rm tra}}$ *(in particular, for all* $z \in \Lambda_{A^{\rm tra}} \cap \mathbb{D}$*), the is/so transfer function of* $\Sigma^{\rm tra}$ *is given by*

$$\begin{aligned}\begin{bmatrix} \mathfrak{A}^{\rm tra}(z) & \mathfrak{B}^{\rm tra}(z) \\ \mathfrak{C}^{\rm tra}(z) & \mathfrak{D}^{\rm tra}(z)\end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{11}C^{\rm sca} & \Theta_{11}D^{\rm sca}+\Theta_{12}\end{bmatrix}\begin{bmatrix} 1_{\mathcal{X}}-zA^{\rm sca} & -zB^{\rm sca} \\ \Theta_{21}C^{\rm sca} & \Theta_{21}D^{\rm sca}+\Theta_{22}\end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathfrak{A}^{\rm sca}(z) & \mathfrak{B}^{\rm sca}(z) \\ \Theta_{11}\mathfrak{C}^{\rm sca}(z) & \Theta_{11}\mathfrak{D}^{\rm sca}(z)+\Theta_{12}\end{bmatrix}\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21}\mathfrak{C}^{\rm sca}(z) & \Theta_{21}\mathfrak{D}^{\rm sca}(z)+\Theta_{22}\end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1_{\mathcal{X}}-zA^{\rm sca} & -zB^{\rm sca}\widetilde{\Theta}_{21} \\ -C^{\rm sca} & \widetilde{\Theta}_{11}-D^{\rm sca}\widetilde{\Theta}_{21}\end{bmatrix}^{-1}\begin{bmatrix} 1_{\mathcal{X}} & zB^{\rm sca}\widetilde{\Theta}_{22} \\ 0 & -\widetilde{\Theta}_{12}+D^{\rm sca}\widetilde{\Theta}_{22}\end{bmatrix} \\ &= \begin{bmatrix} 1_{\mathcal{X}} & -\mathfrak{B}^{\rm sca}(z)\widetilde{\Theta}_{21} \\ 0 & \widetilde{\Theta}_{11}-\mathfrak{D}^{\rm sca}(z)\widetilde{\Theta}_{21}\end{bmatrix}^{-1}\begin{bmatrix} \mathfrak{A}^{\rm sca}(z) & \mathfrak{B}^{\rm sca}(z)\widetilde{\Theta}_{22} \\ \mathfrak{C}^{\rm sca}(z) & -\widetilde{\Theta}_{12}+\mathfrak{D}^{\rm sca}(z)\widetilde{\Theta}_{22}\end{bmatrix}.\end{aligned}\tag{4.7}$$

Thus, in particular, for all $z \in \Lambda_{A^{\rm sca}} \cap \Lambda_{A^{\rm tra}}$ *we have*

$$\begin{aligned}\mathfrak{D}^{\rm tra}(z) &= \big(\Theta_{11}\mathfrak{D}^{\rm sca}(z) + \Theta_{12}\big)\big(\Theta_{21}\mathfrak{D}^{\rm sca}(z) + \Theta_{22}\big)^{-1} \\ &= -\big(\widetilde{\Theta}_{11} - \mathfrak{D}^{\rm sca}(z)\widetilde{\Theta}_{21}\big)^{-1}\big(\widetilde{\Theta}_{12} - \mathfrak{D}^{\rm sca}(z)\widetilde{\Theta}_{22}\big).\end{aligned}\tag{4.8}$$

Proof. This is a transmission version of Theorem II.6.3: the only difference between the two theorems is that in Theorem II.6.3 also the decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ is supposed to be fundamental, hence admissible for Σ. The only addition that we need to the proof of Theorem II.6.3 is to observe that Theorems I.5.11 and I.5.12 tell us that the conditions (a), (b), and (c) in part (1) are equivalent. In particular, formulas (4.6)–(4.8) are the same as the corresponding formulas in Theorem II.6.3. ∎

Note that the equivalent conditions listed in part (3) always hold for $z = 0$, and that they in this case reduce to the conditions (a), (b), and (c) in (1).

REMARK 4.5. Suppose that we have two different externally equivalent passive s/s systems Σ_1 and Σ_2, and that $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ is an admissible orthogonal decomposition of one of the systems, and hence for the other (externally equivalent s/s systems have the same admissible decompositions). We denote the main operators of the corresponding transmission representations $\Sigma_i^{\rm tra}$ of Σ_i by $A_i^{\rm tra}$, and their transfer functions by $\mathfrak{D}_i^{\rm tra}$, $i = 1, 2$. Then it follows from part (3) of Theorem 4.4 that $\Lambda_{A_1^{\rm tra}} \cap \mathbb{D} = \Lambda_{A_2^{\rm tra}} \cap \mathbb{D}$. Moreover, if we denote $\Omega_+ := \Lambda_{A_1^{\rm tra}} \cap \mathbb{D} = \Lambda_{A_2^{\rm tra}} \cap \mathbb{D}$, then $\mathfrak{D}_1^{\rm tra}|_{\Omega_+} = \mathfrak{D}_2^{\rm tra}|_{\Omega_+}$, since the scattering matrices of externally equivalent s/s systems coincide on $\mathbb{D}$, and since the transmission matrices can be obtained from the scattering matrices via (4.8). See also Proposition 4.7.

REMARK 4.6. By choosing the fundamental decomposition $\mathcal{W} = -\mathcal{W}_-\,[\dot{+}]\,\mathcal{W}_+$ in Theorem 4.4 in a suitable way we can recover the Potapov–Ginzburg transformation mentioned earlier. Let $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ be the orthogonal decomposition in Theorem 4.4. We fix some arbitrary fundamental decompositions $\mathcal{Y} = -\mathcal{Y}_-\,[\dot{+}]\,\mathcal{Y}_+ = \begin{bmatrix} -\mathcal{Y}_- \\ \mathcal{Y}_+ \end{bmatrix}$ and $\mathcal{U} = -\mathcal{U}_-\,[\dot{+}]\,\mathcal{U}_+ = \begin{bmatrix} -\mathcal{U}_- \\ \mathcal{U}_+ \end{bmatrix}$ of $\mathcal{Y}$ and $\mathcal{U}$. This induces the following fundamental decomposition of $\mathcal{W}$:

$$\mathcal{W} = -\mathcal{W}_-\,[\dot{+}]\,\mathcal{W}_+ := -\begin{bmatrix} \mathcal{U}_- \\ \mathcal{Y}_+ \end{bmatrix} [\dot{+}] \begin{bmatrix} \mathcal{Y}_- \\ \mathcal{U}_+ \end{bmatrix}. \tag{4.9}$$

The two four block decompositions of the identity operator in $\mathcal{W}$ with respect to the two decompositions $\mathcal{W} = -\mathcal{W}_-\,[\dot{+}]\,\mathcal{W}_+$ and $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ are given by

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{Y}}|_{\mathcal{W}_-} & P_{\mathcal{Y}}|_{\mathcal{W}_+} \\ P_{\mathcal{U}}|_{\mathcal{W}_-} & P_{\mathcal{U}}|_{\mathcal{W}_+} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{Y}_+}|_{\mathcal{W}_-} & P_{\mathcal{Y}_-}|_{\mathcal{W}_+} \\ P_{\mathcal{U}_-}|_{\mathcal{W}_-} & P_{\mathcal{U}_+}|_{\mathcal{W}_+} \end{bmatrix}, \tag{4.10}$$

$$\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{W}_-}|_{\mathcal{Y}} & P_{\mathcal{W}_-}|_{\mathcal{U}} \\ P_{\mathcal{W}_+}|_{\mathcal{Y}} & P_{\mathcal{W}_+}|_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{Y}_+}|_{\mathcal{Y}} & P_{\mathcal{U}_-}|_{\mathcal{U}} \\ P_{\mathcal{Y}_-}|_{\mathcal{Y}} & P_{\mathcal{U}_+}|_{\mathcal{U}} \end{bmatrix}. \tag{4.11}$$

We obtain a more easily visualized description of these operators by splitting them further into sixteen blocks according to the decompositions $\mathcal{W} = -\begin{bmatrix} \mathcal{U}_- \\ \mathcal{Y}_+ \end{bmatrix} [\dot{+}] \begin{bmatrix} \mathcal{Y}_- \\ \mathcal{U}_+ \end{bmatrix}$ and $\mathcal{W} = -\begin{bmatrix} -\mathcal{Y}_- \\ \mathcal{Y}_+ \end{bmatrix} [\dot{+}] \begin{bmatrix} -\mathcal{U}_- \\ \mathcal{U}_+ \end{bmatrix}$. This reveals their permutation structure, namely, no action is taken in $\mathcal{Y}_+$ and $\mathcal{U}_+$, whereas $\mathcal{Y}_-$ is interchanged with $\mathcal{U}_-$ as follows:

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1_{\mathcal{Y}_-} & 0 \\ 0 & 1_{\mathcal{Y}_+} & 0 & 0 \\ \hline 1_{\mathcal{U}_-} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}_+} \end{array}\right], \tag{4.12}$$

$$\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1_{\mathcal{U}_-} & 0 \\ 0 & 1_{\mathcal{Y}_+} & 0 & 0 \\ \hline 1_{\mathcal{Y}_-} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}_+} \end{array}\right]. \tag{4.13}$$

This decomposition induces a corresponding decomposition of the scattering representation Σ^{sca} in Theorem 4.4 and its is/so transfer function, namely

$$\Sigma^{\mathrm{sca}} = \left(\left[\begin{array}{c|cc} A^{\mathrm{sca}} & B_1^{\mathrm{sca}} & B_2^{\mathrm{sca}} \\ \hline C_1^{\mathrm{sca}} & D_{11}^{\mathrm{sca}} & D_{12}^{\mathrm{sca}} \\ C_2^{\mathrm{sca}} & D_{21}^{\mathrm{sca}} & D_{22}^{\mathrm{sca}} \end{array}\right]; \mathcal{X}, \begin{bmatrix} \mathcal{Y}_- \\ \mathcal{U}_+ \end{bmatrix}, \begin{bmatrix} \mathcal{U}_- \\ \mathcal{Y}_+ \end{bmatrix} \right), \tag{4.14}$$

$$\begin{bmatrix} \mathfrak{A}^{\mathrm{sca}} & \mathfrak{B}^{\mathrm{sca}} \\ \mathfrak{C}^{\mathrm{sca}} & \mathfrak{D}^{\mathrm{sca}} \end{bmatrix} = \left[\begin{array}{c|cc} \mathfrak{A}^{\mathrm{sca}} & \mathfrak{B}_1^{\mathrm{sca}} & \mathfrak{B}_2^{\mathrm{sca}} \\ \hline \mathfrak{C}_1^{\mathrm{sca}} & \mathfrak{D}_{11}^{\mathrm{sca}} & \mathfrak{D}_{12}^{\mathrm{sca}} \\ \mathfrak{C}_2^{\mathrm{sca}} & \mathfrak{D}_{21}^{\mathrm{sca}} & \mathfrak{D}_{22}^{\mathrm{sca}} \end{array}\right]. \tag{4.15}$$

Then the equivalent conditions (a), (b), and (c) in part (1) of Theorem 4.4 hold (i.e., the decomposition $\mathcal{W} = -\mathcal{Y} \,[\dotplus]\, \mathcal{U}$ is admissible for Σ) if and only if

$$D_{11}^{\mathrm{sca}} := P_{\mathcal{U}_-} D^{\mathrm{sca}}|_{\mathcal{Y}_-} \text{ has a bounded inverse.} \tag{4.16}$$

Supposing the decomposition $\mathcal{W} = -\mathcal{Y} \,[\dotplus]\, \mathcal{U}$ to be admissible, the corresponding transmission representation Σ^{tra} and its is/so transfer function splits into

$$\Sigma^{\mathrm{tra}} = \left(\left[\begin{array}{c|cc} A^{\mathrm{tra}} & B_1^{\mathrm{tra}} & B_2^{\mathrm{tra}} \\ \hline C_1^{\mathrm{tra}} & D_{11}^{\mathrm{tra}} & D_{12}^{\mathrm{tra}} \\ C_2^{\mathrm{tra}} & D_{21}^{\mathrm{tra}} & D_{22}^{\mathrm{tra}} \end{array}\right]; \mathcal{X}, \begin{bmatrix} -\mathcal{U}_- \\ \mathcal{U}_+ \end{bmatrix}, \begin{bmatrix} -\mathcal{Y}_- \\ \mathcal{Y}_+ \end{bmatrix} \right), \quad \begin{bmatrix} \mathfrak{A}^{\mathrm{tra}} & \mathfrak{B}^{\mathrm{tra}} \\ \mathfrak{C}^{\mathrm{tra}} & \mathfrak{D}^{\mathrm{tra}} \end{bmatrix} = \left[\begin{array}{c|cc} \mathfrak{A}^{\mathrm{tra}} & \mathfrak{B}_1^{\mathrm{tra}} & \mathfrak{B}_2^{\mathrm{tra}} \\ \hline \mathfrak{C}_1^{\mathrm{tra}} & \mathfrak{D}_{11}^{\mathrm{tra}} & \mathfrak{D}_{12}^{\mathrm{tra}} \\ \mathfrak{C}_2^{\mathrm{tra}} & \mathfrak{D}_{21}^{\mathrm{tra}} & \mathfrak{D}_{22}^{\mathrm{tra}} \end{array}\right].$$

Formula (4.6) becomes

$$\begin{aligned} \begin{bmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}} \end{bmatrix} &= \left[\begin{array}{c|cc} A^{\mathrm{sca}} & B_1^{\mathrm{sca}} & B_2^{\mathrm{sca}} \\ \hline 0 & 1_{\mathcal{Y}_-} & 0 \\ C_2^{\mathrm{sca}} & D_{21}^{\mathrm{sca}} & D_{22}^{\mathrm{sca}} \end{array}\right] \left[\begin{array}{c|cc} 1_{\mathcal{X}} & 0 & 0 \\ \hline C_1^{\mathrm{sca}} & D_{11}^{\mathrm{sca}} & D_{12}^{\mathrm{sca}} \\ 0 & 0 & 1_{\mathcal{U}_+} \end{array}\right]^{-1} \\ &= \left[\begin{array}{c|cc} 1_{\mathcal{X}} & -B_1^{\mathrm{sca}} & 0 \\ \hline 0 & -D_{11}^{\mathrm{sca}} & 0 \\ 0 & -D_{21}^{\mathrm{sca}} & 1_{\mathcal{Y}_+} \end{array}\right]^{-1} \left[\begin{array}{c|cc} A^{\mathrm{sca}} & 0 & B_2^{\mathrm{sca}} \\ \hline C_1^{\mathrm{sca}} & -1_{\mathcal{U}_-} & D_{12}^{\mathrm{sca}} \\ C_2^{\mathrm{sca}} & 0 & D_{22}^{\mathrm{sca}} \end{array}\right], \end{aligned} \tag{4.17}$$

and for all $z \in \Lambda_{A^{\mathrm{sca}}}$ the equivalent conditions (a)–(e) in part (3) of Theorem 4.4 hold (i.e., $z \in \Lambda_{A^{\mathrm{sca}}} \cap \Lambda_{A^{\mathrm{tra}}}$) if and only if

$$\text{the operator } \mathfrak{D}_{11}^{\mathrm{sca}}(z) \text{ has a bounded inverse,} \tag{4.18}$$

or equivalently, if and only if

$$\text{the operator } \begin{bmatrix} 1 - zA^{\mathrm{sca}} & -zB_1^{\mathrm{sca}} \\ C_1^{\mathrm{sca}} & D_{11}^{\mathrm{sca}} \end{bmatrix} \text{ has a bounded inverse.} \tag{4.19}$$

Finally, for all $z \in \Lambda_{A^{\mathrm{sca}}} \cap \Lambda_{A^{\mathrm{tra}}}$, the is/so transfer function of Σ^{tra} is given by

$$
\begin{aligned}
\begin{bmatrix} \mathfrak{A}^{\mathrm{tra}}(z) & \mathfrak{B}^{\mathrm{tra}}(z) \\ \mathfrak{C}^{\mathrm{tra}}(z) & \mathfrak{D}^{\mathrm{tra}}(z) \end{bmatrix}
&= \left[\begin{array}{c|cc} 1_{\mathcal{X}} & 0 & 0 \\ \hline 0 & 1_{\mathcal{Y}_-} & 0 \\ C_2^{\mathrm{sca}} & D_{21}^{\mathrm{sca}} & D_{22}^{\mathrm{sca}} \end{array}\right]
\left[\begin{array}{c|cc} 1-zA^{\mathrm{sca}} & -zB_1^{\mathrm{sca}} & -zB_2^{\mathrm{sca}} \\ \hline C_1^{\mathrm{sca}} & D_{11}^{\mathrm{sca}} & D_{12}^{\mathrm{sca}} \\ 0 & 0 & 1_{\mathcal{U}_+} \end{array}\right]^{-1} \\
&= \left[\begin{array}{c|cc} \mathfrak{A}^{\mathrm{sca}}(z) & \mathfrak{B}_1^{\mathrm{sca}}(z) & \mathfrak{B}_2^{\mathrm{sca}}(z) \\ \hline 0 & 1_{\mathcal{Y}_-} & 0 \\ \mathfrak{C}_2^{\mathrm{sca}}(z) & \mathfrak{D}_{21}^{\mathrm{sca}}(z) & \mathfrak{D}_{22}^{\mathrm{sca}}(z) \end{array}\right]
\left[\begin{array}{c|cc} 1_{\mathcal{X}} & 0 & 0 \\ \hline \mathfrak{C}_1^{\mathrm{sca}}(z) & \mathfrak{D}_{11}^{\mathrm{sca}}(z) & \mathfrak{D}_{12}^{\mathrm{sca}}(z) \\ 0 & 0 & 1_{\mathcal{U}_+} \end{array}\right]^{-1} \\
&= \left[\begin{array}{c|cc} 1_{\mathcal{X}}-zA^{\mathrm{sca}} & -zB_1^{\mathrm{sca}} & 0 \\ \hline -C_1^{\mathrm{sca}} & -D_{11}^{\mathrm{sca}} & 0 \\ -C_2^{\mathrm{sca}} & -D_{21}^{\mathrm{sca}} & 1_{\mathcal{Y}_+} \end{array}\right]^{-1}
\left[\begin{array}{c|cc} 1_{\mathcal{X}} & 0 & zB_2^{\mathrm{sca}} \\ \hline 0 & -1_{\mathcal{U}_-} & D_{12}^{\mathrm{sca}} \\ 0 & 0 & D_{22}^{\mathrm{sca}} \end{array}\right] \\
&= \left[\begin{array}{c|cc} 1_{\mathcal{X}} & -\mathfrak{B}_1^{\mathrm{sca}}(z) & 0 \\ \hline 0 & -\mathfrak{D}_{11}^{\mathrm{sca}}(z) & 0 \\ 0 & -\mathfrak{D}_{21}^{\mathrm{sca}}(z) & 1_{\mathcal{Y}_+} \end{array}\right]^{-1}
\left[\begin{array}{c|cc} \mathfrak{A}^{\mathrm{sca}}(z) & 0 & \mathfrak{B}_2^{\mathrm{sca}}(z) \\ \hline \mathfrak{C}_1^{\mathrm{sca}}(z) & -1_{\mathcal{U}_-} & \mathfrak{D}_{12}^{\mathrm{sca}}(z) \\ \mathfrak{C}_2^{\mathrm{sca}}(z) & 0 & \mathfrak{D}_{22}^{\mathrm{sca}}(z) \end{array}\right].
\end{aligned}
\tag{4.20}
$$

In particular, the transmission matrix is given by

$$
\begin{aligned}
\begin{bmatrix} \mathfrak{D}_{11}^{\mathrm{tra}}(z) & \mathfrak{D}_{12}^{\mathrm{tra}}(z) \\ \mathfrak{D}_{21}^{\mathrm{tra}}(z) & \mathfrak{D}_{22}^{\mathrm{tra}}(z) \end{bmatrix}
&= \begin{bmatrix} 1_{\mathcal{Y}_-} & 0 \\ \mathfrak{D}_{21}^{\mathrm{sca}}(z) & \mathfrak{D}_{22}^{\mathrm{sca}}(z) \end{bmatrix}
\begin{bmatrix} \mathfrak{D}_{11}^{\mathrm{sca}}(z) & \mathfrak{D}_{12}^{\mathrm{sca}}(z) \\ 0 & 1_{\mathcal{U}_+} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -\mathfrak{D}_{11}^{\mathrm{sca}}(z) & 0 \\ -\mathfrak{D}_{21}^{\mathrm{sca}}(z) & 1_{\mathcal{Y}_+} \end{bmatrix}^{-1}
\begin{bmatrix} -1_{\mathcal{U}_-} & \mathfrak{D}_{12}^{\mathrm{sca}}(z) \\ 0 & \mathfrak{D}_{22}^{\mathrm{sca}}(z) \end{bmatrix} \\
&= \begin{bmatrix} (\mathfrak{D}_{11}^{\mathrm{sca}}(z))^{-1} & -(\mathfrak{D}_{11}^{\mathrm{sca}}(z))^{-1}\mathfrak{D}_{12}^{\mathrm{sca}}(z) \\ \mathfrak{D}_{21}^{\mathrm{sca}}(z)(\mathfrak{D}_{11}^{\mathrm{sca}}(z))^{-1} & \mathfrak{D}_{22}^{\mathrm{sca}}(z)-\mathfrak{D}_{21}^{\mathrm{sca}}(z)(\mathfrak{D}_{11}^{\mathrm{sca}}(z))^{-1}\mathfrak{D}_{12}^{\mathrm{sca}}(z) \end{bmatrix}.
\end{aligned}
\tag{4.21}
$$

Note that Potapov-Ginzburg transformation described above is its own inverse in the sense that formulas (4.17)–(4.21) remain valid if we interchange the roles of Σ^{sca} and Σ^{tra}, so that they give Σ^{sca} and the is/so transfer function of Σ^{sca} in terms of Σ^{tra} and its is/so transfer function.

Let Ω be a nonempty subset of $\mathbb{D}$, and let $\mathcal{U}$ and $\mathcal{Y}$ be two Kreĭn spaces. By $P(\Omega;\mathcal{U},\mathcal{Y})$ we denote the (Potapov) class of functions on Ω, i.e., the set of holomorphic $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued functions θ on Ω with the property that the two kernels

$$
\begin{aligned}
K_\theta(z,\zeta) &= \frac{1_{\mathcal{U}} - \theta(z)^*\theta(\zeta)}{1-\bar{z}\zeta}, \quad z,\zeta \in \Omega, \\
\tilde{K_\theta}(z,\zeta) &= \frac{1_{\mathcal{Y}} - \theta(z)\theta(\zeta)^*}{1-z\bar{\zeta}}, \quad z,\zeta \in \Omega,
\end{aligned}
$$

are positive definite on $\Omega \times \Omega$ (the positivity and the adjoints are with respect to the Kreĭn space inner products in $\mathcal{U}$ and $\mathcal{Y}$; cf. the last parenthesis in the Theorem 3.6). Trivially, a function $\theta \in P(\Omega;\mathcal{U},\mathcal{Y})$ is bi-contractive on Ω, i.e.,

$$
\theta(z)^*\theta(z) \leqslant 1_{\mathcal{U}}, \quad \theta(z)\theta(z)^* \leqslant 1_{\mathcal{Y}}, \quad z \in \Omega.
$$

By Theorem 3.9, if $\mathfrak{D}^{\mathrm{tra}}$ is the transmission matrix of a passive s/s system, then $\mathfrak{D}^{\mathrm{tra}}|_\Omega \in P(\Omega_+;\mathcal{U},\mathcal{Y})$, where Ω_+ is the intersection of the domain of $\mathfrak{D}^{\mathrm{tra}}$ with $\mathbb{D}$.

Let $\mathcal{Y} = -\mathcal{Y}_- \,[\dot{+}]\, \mathcal{Y}_+ = \begin{bmatrix} -\mathcal{Y}_- \\ \mathcal{Y}_+ \end{bmatrix}$ and $\mathcal{U} = -\mathcal{U}_- \,[\dot{+}]\, \mathcal{U}_+ = \begin{bmatrix} -\mathcal{U}_- \\ \mathcal{U}_+ \end{bmatrix}$ be some fundamental decompositions of $\mathcal{Y}$ and $\mathcal{U}$, respectively, and define $\mathcal{W}_+$ and $\mathcal{W}_-$ as in (4.9). Then $\mathcal{W} = -\mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ is a fundamental decomposition of $\mathcal{W}$. We split θ into $\theta = \left[\begin{smallmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{smallmatrix}\right]$ in accordance with the decompositions $\mathcal{Y} = \left[\begin{smallmatrix} -\mathcal{Y}_- \\ \mathcal{Y}_+ \end{smallmatrix}\right]$ and $\mathcal{U} = \left[\begin{smallmatrix} -\mathcal{U}_- \\ \mathcal{U}_+ \end{smallmatrix}\right]$, and apply the Potapov-Ginzburg transform (4.21) to this function to get a new function φ given by

$$\begin{bmatrix} \varphi_{11}(z) & \varphi_{12}(z) \\ \varphi_{21}(z) & \varphi_{22}(z) \end{bmatrix} = \begin{bmatrix} \theta_{11}(z)^{-1} & -\theta_{11}(z)^{-1}\theta_{12}(z) \\ \theta_{21}(z)\theta_{11}(z)^{-1} & \theta_{22}(z) - \theta_{21}(z)\theta_{11}(z)^{-1}\theta_{12}(z) \end{bmatrix}, \tag{4.22}$$

where the four block decomposition of φ is given with respect to the orthogonal decompositions $\mathcal{W}_+ = \left[\begin{smallmatrix} \mathcal{Y}_- \\ \mathcal{U}_+ \end{smallmatrix}\right]$ and $\mathcal{W}_- = \left[\begin{smallmatrix} \mathcal{U}_- \\ \mathcal{Y}_+ \end{smallmatrix}\right]$. It is easy to see that this transformation takes the class $P(\Omega;\mathcal{U},\mathcal{Y})$ one-to-one onto the subclass of functions in $P(\Omega;\mathcal{W}_+,\mathcal{W}_-)$ for which the block $\varphi_{11}(z) = P_{\mathcal{U}_-}\varphi(z)|_{\mathcal{Y}_-}$ has a bounded inverse for every $z \in \Omega$. However, it is known (see [15]) that the functions in the class $P(\Omega;\mathcal{W}_+,\mathcal{W}_-)$ are simply the restrictions to Ω of functions in the Schur class $S(\mathbb{D};\mathcal{W},\mathcal{W}_+)$. This connection permits us to draw the following conclusions:

(1) A function $\theta \in P(\Omega;\mathcal{U},\mathcal{Y})$ is uniquely defined by its values on a subset $\Omega_0 \subset \Omega$ containing a cluster point. This is true in spite of the fact that Ω need not be connected.

(2) Each function $\theta \in P(\Omega;\mathcal{U},\mathcal{Y})$ has a unique extension to a maximal domain $\widetilde{\Omega} \subset \mathbb{D}$ when we require the extended function $\widetilde{\theta}$ to belong to $P(\widetilde{\Omega};\mathcal{U},\mathcal{Y})$. The set $\widetilde{\Omega}$ consists of those of points in $\mathbb{D}$ where the block $\widetilde{\varphi}_{11}(z) = P_{\mathcal{U}_-}\varphi(z)|_{\mathcal{Y}_-}$ of the corresponding Schur function $\widetilde{\varphi}$ has a bounded inverse. We call this set the *natural domain* of θ. On this domain the extended function $\widetilde{\theta}$ is defined by the formula that we get from (4.22) by replacing θ by $\widetilde{\varphi}$ and φ by $\widetilde{\theta}$.

PROPOSITION 4.7. *Let $\Sigma^{\mathrm{tra}} = \left(\left[\begin{smallmatrix} A^{\mathrm{tra}} & B^{\mathrm{tra}} \\ C^{\mathrm{tra}} & D^{\mathrm{tra}} \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ be a transmission representation of a passive s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$, let $\mathfrak{D}^{\mathrm{tra}}$ be the corresponding transmission matrix, and let $\Omega_+ = \Lambda_{A^{\mathrm{tra}}} \cap \mathbb{D}$. Then $\mathfrak{D}^{\mathrm{tra}}|_{\Omega_+} \in P(\Omega_+;\mathcal{U},\mathcal{Y})$, and Ω_+ is the natural domain of definition of $\mathfrak{D}^{\mathrm{tra}}|_{\Omega_+}$ in the sense explained above. Moreover, the intersection of the boundary of $\Lambda_{A^{\mathrm{tra}}}$ with $\mathbb{D}$ is the natural boundary of analyticity of $\mathfrak{D}^{\mathrm{tra}}|_{\Omega_+}$ in the sense that $\mathfrak{D}^{\mathrm{tra}}$ does not have an analytic extension to any boundary point of $\Lambda_{A^{\mathrm{tra}}}$ in $\mathbb{D}$.*

Proof. Most of this was proved in the explanation leading up to Proposition 4.7. That $\mathfrak{D}^{\mathrm{tra}}$ does not have an analytic extension to any boundary point of $\Lambda_{A^{\mathrm{tra}}}$ in $\mathbb{D}$ follows from (4.18), which show that the set $\mathbb{D} \setminus \Lambda_{A^{\mathrm{tra}}}$ coincides with the set of those points in $\mathbb{D}$ where $\mathfrak{D}^{\mathrm{sca}}_{11}(z)$ does not have a bounded inverse, and from (4.21), which shows that $\mathfrak{D}^{\mathrm{tra}}$ does not have an analytic continuation to a point where $\mathfrak{D}^{\mathrm{sca}}_{11}(z)$ does not have a bounded inverse. ∎

PROPOSITION 4.8. *Let Ω be an open subset of $\mathbb{D}$ containing zero, and let $\theta \in P(\Omega;\mathcal{U},\mathcal{Y})$. Then there exists a simple conservative s/s system $\Sigma = (V;\mathcal{X},\mathcal{W})$ with Kreĭn signal space $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ such that the decomposition $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ is admissible for Σ and the corresponding transmission matrix coincides with θ on Ω; in* *particular, $\Omega \subset \Lambda_{A^{\mathrm{tra}}}$, where A^{tra} is the main operator of the corresponding transmission representation Σ^{tra}. The system Σ is defined uniquely up to unitary similarity.*

Proof. Let $\mathcal{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$, define $\mathcal{W}_-$ and $\mathcal{W}_+$ as in (4.9), and define φ by (4.22). Then $\varphi \in S(\Omega; \mathcal{W}_+, \mathcal{W}_-)$, and hence φ can be extended to function $\widetilde{\varphi} \in S(\mathbb{D}; \mathcal{W}_+, \mathcal{W}_-)$. By Proposition II.6.2, there exists a simple conservative s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ such that $\mathcal{W} = -\mathcal{W}_- [\dot{+}] \mathcal{W}_+$ is a fundamental decomposition of $\mathcal{W}$, and such that the corresponding scattering matrix $\mathfrak{D}^{\mathrm{sca}}$ of Σ satisfies $\mathfrak{D}^{\mathrm{sca}}|_{\mathbb{D}} = \widetilde{\varphi}$. This system is determined uniquely by φ up to a unitary similarity transformation in the state space (see, e.g., [1, Theorem 2.1.3]). The fact that $0 \in \Omega$ implies that the decomposition $\mathcal{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$ is admissible, and it is also true that the corresponding transmission matrix of Σ coincides with θ on Ω. The system Σ is uniquely determined by θ up to unitary similarity since it is uniquely determined by φ up to unitary similarity. ∎

PROPOSITION 4.9. *Let Ω be an open subset of $\mathbb{D}$ containing* 0, *let $\theta \in P(\Omega; \mathcal{U}, \mathcal{Y})$, and let Σ be a s/s system of the type mentioned in Proposition* 4.8. *Let $\Sigma_\circ$ and $\Sigma_\bullet$ be the compressions of Σ constructed in Theorem* II.7.5. *Then $\Sigma_\circ$ and $\Sigma_\bullet$ are minimal passive s/s systems with Kreĭn signal space $\mathcal{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$ such that the decomposition $\mathcal{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$ is admissible for Σ and the corresponding transmission matrix coincides with θ on Ω; in particular, $\Omega \subset \Lambda_{A^{\mathrm{tra}}}$, where A^{tra} is the main operator of the corresponding transmission representation Σ^{tra}.*

Proof. This follows from Proposition 4.8 and Theorem II.7.5. ∎

The two systems in Proposition 4.9 are extremal in a certain sense. We shall return to this elsewhere.

5. IMPEDANCE REPRESENTATIONS AND IMPEDANCES

We recall that a decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{+} \mathcal{E}$ of $\mathcal{W}$ is called Lagrangian if both $\mathcal{F}$ and $\mathcal{E}$ are Lagrangian subspaces of $\mathcal{W}$ with some Hilbert space inner products $(\cdot,\cdot)_{\mathcal{E}}$ and $(\cdot,\cdot)_{\mathcal{F}}$, and if Ψ is a unitary operator $\mathcal{E} \to \mathcal{F}$ such that the inner product in $\mathcal{W}$ is given by (2.8). If $\mathcal{W} = \mathcal{F} \overset{\Psi}{+} \mathcal{E}$ is an admissible Lagrangian decomposition of the signal space of a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, then we call the corresponding i/s/o representation $\Sigma^{\mathrm{imp}} = \left(\begin{bmatrix} A^{\mathrm{imp}} & B^{\mathrm{imp}} \\ C^{\mathrm{imp}} & D^{\mathrm{imp}} \end{bmatrix}; \mathcal{X}, \mathcal{E}, \mathcal{F} \right)$ an *impedance representation* of Σ, and the corresponding transfer function $\mathfrak{D}^{\mathrm{imp}}$ is called an *impedance matrix* of Σ^{imp} (or of Σ). The name 'resistance' has also been used.

We claim that $\mathcal{W}$ has a Lagrangian decomposition (hence infinitely many such decompositions) if and only of $\mathrm{ind}_+\mathcal{W} = \mathrm{ind}_-\mathcal{W}$. This can be seen as follows. Fix some fundamental decomposition $\mathcal{W} = -\mathcal{W}_- [\dot{+}] \mathcal{W}_+$. If $\mathcal{W}$ has a Lagrangian subspace, then it follows from assertion (3) of Proposition II.2.2 that $\dim \mathcal{W}_+ = \dim \mathcal{W}_-$, i.e., $\mathrm{ind}_+\mathcal{W} = \mathrm{ind}_-\mathcal{W}$. Conversely, suppose that $\dim \mathcal{W}_+ =$

dim $\mathcal{W}_-$. Then it is possible to choose some unitary operator Φ mapping $\mathcal{W}_+$ one-to-one onto $\mathcal{W}_-$. By assertion (3) in Proposition II.2.2, $\mathcal{E} := \mathcal{R}\left(\left[\begin{smallmatrix}\Phi \\ 1_{\mathcal{W}_+}\end{smallmatrix}\right]\right)$ is a Lagrangian subspaces of $\mathcal{W}$. Let $\mathcal{F}$ be the orthogonal complement to $\mathcal{E}$ in the admissible inner product $(\cdot,\cdot)_{\mathcal{W}_-\oplus\mathcal{W}_+}$. Then $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$. Moreover, $\mathcal{F} := \mathcal{R}\left(\left[\begin{smallmatrix}1_{\mathcal{W}_-} \\ -\Phi^*\end{smallmatrix}\right]\right)$, so that by Proposition II.2.2, $\mathcal{F}$ is another Lagrangian subspace of $\mathcal{W}$. We make $\mathcal{E}$ and $\mathcal{F}$ into Hilbert spaces by letting them inherit the admissible inner product $(\cdot,\cdot)_{\mathcal{W}_-\oplus\mathcal{W}_+}$ from $\mathcal{W}$. Then, by Lemmas 2.1 and 2.3, there is a unitary operator $\Psi\colon \mathcal{E} \to \mathcal{F}$ such that the inner product in $\mathcal{W}$ is given by (2.8), and $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ is a Lagrangian decomposition of $\mathcal{W}$. It is easy to check that the complementary projections onto $\mathcal{E}$ and $\mathcal{F}$ are given by

$$P_{\mathcal{E}}^{\mathcal{F}} = \frac{1}{2}\begin{bmatrix}1_{\mathcal{W}_-} & \Phi \\ \Phi^* & 1_{\mathcal{W}_+}\end{bmatrix}, \quad P_{\mathcal{F}}^{\mathcal{E}} = \frac{1}{2}\begin{bmatrix}1_{\mathcal{W}_-} & -\Phi \\ -\Phi^* & 1_{\mathcal{W}_+}\end{bmatrix}. \tag{5.1}$$

Note that the Lagrangian decomposition constructed above is completely determined once we have fixed the fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \,[\dotplus]\, \mathcal{W}_+$ and the operator Φ. We shall refer to the operator Φ as the *angle operator* of this decomposition.

Conversely, it is also possible to proceed in the opposite direction and recover a unique fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \,[\dotplus]\, \mathcal{W}_+$ and a unique angle operator Φ from any given Lagrangian decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$. By the definition of the Lagrangian decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$, the inner product in $\mathcal{W}$ is given by (2.8), where $\Psi\colon \mathcal{E} \to \mathcal{F}$ is unitary. By Lemma 2.1, the corresponding admissible inner product in $\mathcal{W}$ is given by (2.5), with $\mathcal{J} = \mathcal{J}_\Psi = \left[\begin{smallmatrix}0 & \Psi \\ \Psi^* & 0\end{smallmatrix}\right]$. This determines the fundamental decomposition corresponding to this admissible inner product as follows: $\mathcal{W}_+$ is the positive eigenspace of $\mathcal{J}_\Psi$ corresponding to the eigenvalue $+1$, $\mathcal{W}_-$ is the negative eigenspaces of $\mathcal{J}_\Psi$ corresponding to the eigenvalue -1, and hence

$$\begin{aligned} P_{\mathcal{W}_+} &= \frac{1}{2}(1_{\mathcal{W}} + \mathcal{J}_\Psi) = \frac{1}{2}\begin{bmatrix}1_{\mathcal{F}} & \Psi \\ \Psi^* & 1_{\mathcal{E}}\end{bmatrix}, \\ P_{\mathcal{W}_-} &= \frac{1}{2}(1_{\mathcal{W}} - \mathcal{J}_\Psi) = \frac{1}{2}\begin{bmatrix}1_{\mathcal{F}} & -\Psi \\ -\Psi^* & 1_{\mathcal{E}}\end{bmatrix}. \end{aligned} \tag{5.2}$$

Once we know the fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \,[\dotplus]\, \mathcal{W}_+$ and the Lagrangian subspace $\mathcal{E}$ we can recover the angle operator Φ as the unique unitary operator $\mathcal{W}_+ \to \mathcal{W}_-$ such that $\mathcal{E} = \mathcal{R}\left(\left[\begin{smallmatrix}\Phi \\ 1_{\mathcal{W}_+}\end{smallmatrix}\right]\right)$. Note also that by (5.1) and (5.2),

$$\Psi = 2P_{\mathcal{F}}^{\mathcal{E}} P_{\mathcal{W}_+}|_{\mathcal{E}}, \quad \Phi = 2P_{\mathcal{W}_-} P_{\mathcal{E}}^{\mathcal{F}}|_{\mathcal{W}_+}. \tag{5.3}$$

We next analyze the decomposition $\widetilde{\Theta}$ of the identity operator in $\mathcal{W}$ with the respect to the two decompositions $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ and $\mathcal{W} = -\mathcal{W}_- \,[\dotplus]\, \mathcal{W}_+$. From (5.2) we see that $P_{\mathcal{W}_-}|_{\mathcal{F}} = -P_{\mathcal{W}_-}|_{\mathcal{E}}\Psi^*$ and $P_{\mathcal{W}_+}|_{\mathcal{E}} = P_{\mathcal{W}_+}|_{\mathcal{F}}\Psi$, and hence this four block operator can be written in the form

(5.4)
$$
\begin{aligned}
\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} &= \begin{bmatrix} P_{\mathcal{W}_-}|_{\mathcal{F}} & P_{\mathcal{W}_-}|_{\mathcal{E}} \\ P_{\mathcal{W}_+}|_{\mathcal{F}} & P_{\mathcal{W}_+}|_{\mathcal{E}} \end{bmatrix} \\
&= \begin{bmatrix} -\widetilde{\Theta}_{12}\Psi^* & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{21}\Psi \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2}\,\widetilde{\Theta}_{12} \\ \sqrt{2}\,\widetilde{\Theta}_{21} & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1_{\mathcal{F}} & \Psi \\ -\Psi^* & 1_{\mathcal{E}} \end{bmatrix},
\end{aligned}
$$

where

$$
\widetilde{\Theta}_{12} = P_{\mathcal{W}_-}|_{\mathcal{E}} = \tfrac{1}{2}\begin{bmatrix} \Psi \\ 1_{\mathcal{E}} \end{bmatrix}, \quad \widetilde{\Theta}_{21} = P_{\mathcal{W}_+}|_{\mathcal{F}} = \tfrac{1}{2}\begin{bmatrix} 1_{\mathcal{F}} \\ -\Psi^* \end{bmatrix}. \tag{5.5}
$$

This decomposition has the property that $\sqrt{2}\,\widetilde{\Theta}_{ij}$ is a unitary map from its domain onto its range space, for all i and j. It is also true that $\widetilde{\Theta}$ is a unitary map from $\mathcal{F} \oplus \mathcal{E}$ to $\mathcal{W}_- \oplus \mathcal{W}_+$, and that $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1_{\mathcal{F}} & \Psi \\ -\Psi^* & 1_{\mathcal{E}} \end{smallmatrix}\right]$ is a unitary map from $\mathcal{F} \oplus \mathcal{E}$ to itself. Moreover, since $\widetilde{\Theta}_{12} = -\widetilde{\Theta}_{11}\Psi$ and $\widetilde{\Theta}_{22} = \widetilde{\Theta}_{21}\Psi$, we have $\widetilde{\Theta}_{12}\widetilde{\Theta}_{22}^{-1} = -\widetilde{\Theta}_{11}\widetilde{\Theta}_{21}^{-1}$, and so

$$
\Phi := \widetilde{\Theta}_{12}\widetilde{\Theta}_{22}^{-1} = -\widetilde{\Theta}_{11}\widetilde{\Theta}_{21}^{-1} \tag{5.6}
$$

is a unitary map from $\mathcal{W}_+$ to $\mathcal{W}_-$. This identity can be rewritten as the two identities $\widetilde{\Theta}_{11} = -\Phi\widetilde{\Theta}_{21}$ and $\widetilde{\Theta}_{22} = \Phi^*\widetilde{\Theta}_{12}$, which gives the following alternative formula for $\widetilde{\Theta}$:

$$
\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} = \begin{bmatrix} -\Phi\widetilde{\Theta}_{21} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \Phi^*\widetilde{\Theta}_{12} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1_{\mathcal{W}_-} & -\Phi \\ \Phi^* & 1_{\mathcal{W}_+} \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2}\,\widetilde{\Theta}_{12} \\ \sqrt{2}\,\widetilde{\Theta}_{21} & 0 \end{bmatrix}. \tag{5.7}
$$

To compute the inverse decomposition $\Theta := \widetilde{\Theta}^{-1}$ we observe that $\Theta = \widetilde{\Theta}^{-1} = \widetilde{\Theta}^*$ since $\widetilde{\Theta}$ is unitary. Combining this with (5.4) and (5.7) we get

(5.8)
$$
\begin{aligned}
\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} &= \begin{bmatrix} P_{\mathcal{F}}^{\mathcal{E}}|_{\mathcal{W}_-} & P_{\mathcal{F}}^{\mathcal{E}}|_{\mathcal{W}_+} \\ P_{\mathcal{E}}^{\mathcal{F}}|_{\mathcal{W}_-} & P_{\mathcal{E}}^{\mathcal{F}}|_{\mathcal{W}_+} \end{bmatrix} \\
&= \begin{bmatrix} -\Theta_{12}\Phi^* & \Theta_{12} \\ \Theta_{21} & \Theta_{21}\Phi \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2}\,\Theta_{12} \\ \sqrt{2}\,\Theta_{21} & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1_{\mathcal{W}_-} & \Phi \\ -\Phi^* & 1_{\mathcal{W}_+} \end{bmatrix} \\
&= \begin{bmatrix} -\Psi\Theta_{21} & \Theta_{12} \\ \Theta_{21} & \Psi^*\Theta_{12} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1_{\mathcal{F}} & -\Psi \\ \Psi^* & 1_{\mathcal{E}} \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2}\,\Theta_{12} \\ \sqrt{2}\,\Theta_{21} & 0 \end{bmatrix},
\end{aligned}
$$

where

$$
\Theta_{12} = P_{\mathcal{F}}^{\mathcal{E}}|_{\mathcal{W}_+} = \tfrac{1}{2}\begin{bmatrix} \Phi \\ 1_{\mathcal{W}_+} \end{bmatrix}, \quad \Theta_{21} = P_{\mathcal{E}}^{\mathcal{F}}|_{\mathcal{W}_-} = \tfrac{1}{2}\begin{bmatrix} 1_{\mathcal{W}_-} \\ -\Phi^* \end{bmatrix}. \tag{5.9}
$$

Here $\sqrt{2}\,\Theta_{ij}$ is a unitary map from its domain onto its range space, for all i and j, Θ is a unitary map from $\mathcal{W}_- \oplus \mathcal{W}_+$ to $\mathcal{F} \oplus \mathcal{E}$, and $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1_{\mathcal{F}} & -\Psi \\ \Psi^* & 1_{\mathcal{E}} \end{smallmatrix}\right]$ is a unitary map from $\mathcal{F} \oplus \mathcal{E}$ to itself.

When we specialize Lemma II.5.14 to a Lagrangian (possibly non-admissible) decomposition we get the following result.

LEMMA 5.1. *Let $\Sigma = (V, \mathcal{X}, \mathcal{W})$ be a s/s system with driving variable representation $\Sigma_{dv/s/s} := \left(\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}; \mathcal{X}, \mathcal{L}, \mathcal{W}\right)$ and output nulling representation $\Sigma_{s/s/on} = \left(\begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}; \mathcal{X}, \mathcal{W}, \mathcal{K}\right)$. Let $\mathcal{W} = \mathcal{F} \overset{\Psi}{+} \mathcal{E}$ be a Lagrangian decomposition of $\mathcal{W}$, and decompose $\begin{bmatrix} C' & D' \end{bmatrix}$ and $\begin{bmatrix} B'' \\ D'' \end{bmatrix}$ accordingly into*

$$\begin{bmatrix} C' & D' \end{bmatrix} = \begin{bmatrix} C'_{\mathcal{F}} & D'_{\mathcal{F}} \\ C'_{\mathcal{E}} & D'_{\mathcal{E}} \end{bmatrix} \in \mathcal{B}\left(\begin{bmatrix} \mathcal{X} \\ \mathcal{L} \end{bmatrix}; \begin{bmatrix} \mathcal{F} \\ \mathcal{E} \end{bmatrix}\right),$$

$$\begin{bmatrix} B'' \\ D'' \end{bmatrix} = \begin{bmatrix} B''_{\mathcal{F}} & B''_{\mathcal{E}} \\ D'_{\mathcal{F}} & D''_{\mathcal{E}} \end{bmatrix} \in \mathcal{B}\left(\begin{bmatrix} \mathcal{F} \\ \mathcal{E} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{K} \end{bmatrix}\right).$$

(1) *The following conditions are equivalent (the left-hand side of* (b) *should be nonnegative in the Hilbert space $\begin{bmatrix} \mathcal{X} \\ \mathcal{L} \end{bmatrix}$; we here identify the duals of the Hilbert spaces $\mathcal{X}$, $\mathcal{L}$, $\mathcal{F}$, and $\mathcal{E}$ with themselves):*

(a) *Σ is forward passive (or forward conservative).*

(b) $$\begin{bmatrix} (A')^* & 1_{\mathcal{X}} \\ (B')^* & 0 \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & 0 \\ 0 & 1_{\mathcal{X}} \end{bmatrix} \begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \end{bmatrix} + \begin{bmatrix} (C'_{\mathcal{F}})^* & (C'_{\mathcal{E}})^* \\ (D'_{\mathcal{F}})^* & (D'_{\mathcal{E}})^* \end{bmatrix} \begin{bmatrix} 0 & \Psi \\ \Psi^* & 0 \end{bmatrix} \begin{bmatrix} C'_{\mathcal{F}} & D'_{\mathcal{F}} \\ C'_{\mathcal{E}} & D'_{\mathcal{E}} \end{bmatrix} \geqslant 0 \text{ (or } = 0).$$

(2) *The following conditions are equivalent (the left-hand side should be nonnegative in the Hilbert space $\begin{bmatrix} \mathcal{X} \\ \mathcal{K} \end{bmatrix}$.):*

(a) *Σ is backward passive (or backward conservative).*

(b) $$\begin{bmatrix} A'' & 1_{\mathcal{X}} \\ C'' & 0 \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & 0 \\ 0 & 1_{\mathcal{X}} \end{bmatrix} \begin{bmatrix} (A'')^* & (C'')^* \\ 1_{\mathcal{X}} & 0 \end{bmatrix} + \begin{bmatrix} B''_{\mathcal{F}} & B''_{\mathcal{E}} \\ D''_{\mathcal{F}} & D''_{\mathcal{E}} \end{bmatrix} \begin{bmatrix} 0 & \Psi \\ \Psi^* & 0 \end{bmatrix} \begin{bmatrix} (B''_{\mathcal{F}})^* & (D''_{\mathcal{F}})^* \\ (B''_{\mathcal{E}})^* & (D''_{\mathcal{E}})^* \end{bmatrix} \geqslant 0 \text{ (or } = 0).$$

(3) *The following conditions are equivalent:*

(a) *Σ is passive (or conservative).*

(b) *Both* (1)(b) *and* (2)(b) *hold.*

Proof. This follows from Lemma II.5.14. ∎

We now focus our attention on the case where a Lagrangian decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{+} \mathcal{E}$ is admissible for a s/s system with signal space $\mathcal{W}$.

Applying Lemma 5.1 to an admissible Lagrangian representation of the signal space we get the following result:

LEMMA 5.2. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system. Let $\mathcal{W} = \mathcal{F} \overset{\Psi}{+} \mathcal{E}$ be an admissible Lagrangian decomposition of $\mathcal{W}$, and denote the corresponding impedance representation of Σ by $\Sigma^{\text{imp}} = \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{E}, \mathcal{F}; \Psi\right)$. Then the following conditions are equivalent:*

(1) *Σ is forward passive.*

(2) *Σ is passive.*

(3) $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the inequality*

$$\begin{bmatrix} 1_{\mathcal{X}} - A^*A & C^*\Psi - A^*B \\ \Psi^*C - B^*A & D^*\Psi + \Psi^*D - B^*B \end{bmatrix} \geqslant 0. \tag{5.10}$$

Proof. Clearly (2) implies (1). It follows from part (1) of Lemma 5.1 (we interpret the i/s/o representation as a driving variable representation as explained in Remark I.5.2) that (1) is equivalent to (3). Thus, it remains to show that the equivalent conditions (1) and (3) imply (2).

Assume (1) and (3). We first show that the fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \, [\dot{+}] \, \mathcal{W}_+$ corresponding the partition of the unity given in (5.4) is admissible. By that identity and Theorem I.5.11, this is true if and only if the operator

$$\widetilde{\Theta}_{21} D + \widetilde{\Theta}_{22} = \widetilde{\Theta}_{21}(D + \Psi)$$

maps $\mathcal{E}$ one-to-one onto $\mathcal{W}_+$. But this is true since $\widetilde{\Theta}_{21}$ is invertible and since, due to condition (3) and the fact that Ψ is unitary,

$$\begin{aligned} (D+\Psi)^*(D+\Psi) &= D^*D + D^*\Psi + \Psi^*D + \Psi^*\Psi \geqslant 1_{\mathcal{E}}, \\ (D+\Psi)(D+\Psi)^* &= DD^* + D^*\Psi + \Psi^*D + \Psi\Psi^* \geqslant 1_{\mathcal{F}}. \end{aligned}$$

This proves that the fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \, [\dot{+}] \, \mathcal{W}_+$ is admissible. By Theorem 3.1, Σ is passive. ∎

THEOREM 5.3. *Let $\Sigma = (V, \mathcal{X}, \mathcal{W})$ be a s/s system, and let $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ be an admissible Lagrangian decomposition of $\mathcal{W}$, with the corresponding impedance representation $\Sigma^{\mathrm{imp}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{E}, \mathcal{F}; \Psi\right)$ of Σ. Then $\mathcal{W}_* = \mathcal{F} \overset{-\Psi}{\dotplus} \mathcal{E}$ is an admissible Lagrangian decomposition of $\mathcal{W}_*$, and*

$$(\Sigma_*)^{\mathrm{imp}} = \left(\left[\begin{smallmatrix} A^* & -C^*\Psi \\ \Psi B^* & -\Psi D^*\Psi \end{smallmatrix}\right]; \mathcal{X}, \mathcal{E}, \mathcal{F}; -\Psi\right)$$

is an impedance representation of the adjoint system Σ_* (the adjoints have been computed with respect to the Hilbert space inner products in $\mathcal{E}$ and $\mathcal{F}$).

Proof. Let $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$ be the adjoint of the s/s system Σ. Then, by definition of the adjoint system, $k_* = \left[\begin{smallmatrix} \dot{x}_* \\ x_* \\ w_* \end{smallmatrix}\right] \in V_*$, where $w_* = \left[\begin{smallmatrix} f_* \\ e_* \end{smallmatrix}\right] \in \mathcal{W}_*$, if and only if

$$-(\dot{x}, x_*)_{\mathcal{X}} + (x, \dot{x}_*)_{\mathcal{X}} + \left\langle \left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right], \left[\begin{smallmatrix} f_* \\ e_* \end{smallmatrix}\right] \right\rangle_{\langle W, W_* \rangle} = 0 \tag{5.11}$$

for all $k = \left[\begin{smallmatrix} \dot{x} \\ x \\ w \end{smallmatrix}\right] \in V$. Here

$$\left\langle \left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right], \left[\begin{smallmatrix} f_* \\ e_* \end{smallmatrix}\right] \right\rangle_{\langle W, W_* \rangle} = \left[\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right], \mathcal{I} \left[\begin{smallmatrix} f_* \\ e_* \end{smallmatrix}\right] \right]_{\mathcal{W}} = \left\langle \left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathcal{I}_{\mathcal{F}} f_* \\ \mathcal{I}_{\mathcal{E}} e_* \end{smallmatrix}\right] \right\rangle_{\mathcal{F} \oplus \mathcal{E}},$$

where $\mathcal{I}_{\mathcal{F}}$ and $\mathcal{I}_{\mathcal{E}}$ are the identity operators from $\mathcal{F} \subset \mathcal{W}_*$ onto $\mathcal{F} \subset \mathcal{W}$ and from $\mathcal{E} \subset \mathcal{W}_*$ onto $\mathcal{E} \subset \mathcal{W}$, respectively. Since Σ^{imp} is an i/s/o representation of Σ

we can substitute the values of $\dot{x}$ and f obtained from (3.7) into (5.11) we get the following condition, valid for all $x \in \mathcal{X}$ and $e \in \mathcal{E}$:

$$-(Ax + Be, x_*)_{\mathcal{X}} + (x, \dot{x}_*)_{\mathcal{X}} + (Cx + De, \Psi \mathcal{I}_{\mathcal{E}} e_*)_{\mathcal{F}} + (e, \Psi \mathcal{I}_{\mathcal{F}} f_*)_{\mathcal{F}} = 0.$$

This is equivalent to the system of equations

$$\begin{aligned} \dot{x}_* &= A^* x_* - C^* \Psi \mathcal{I}_{\mathcal{E}} e_*, \\ \mathcal{I}_{\mathcal{F}} f_* &= \Psi B^* x_* - \Psi D^* \Psi \mathcal{I}_{\mathcal{E}} e_*, \quad x_* \in \mathcal{X},\ e_* \in \mathcal{E}. \end{aligned}$$

If we here ignore the distinction between $\mathcal{F}$ and $\mathcal{E}$ as subspaces of $\mathcal{W}$ or subspaces of $\mathcal{W}_*$, then the operators $\mathcal{I}_{\mathcal{E}}$ and $\mathcal{I}_{\mathcal{F}}$ disappear, and we obtain the impedance representation of Σ_* described in Theorem 5.3 (the operator Ψ for Σ is replaced by the operator $-\Psi$ for Σ_*^{imp} since $\mathcal{W}_* = -\mathcal{W}$). ∎

REMARK 5.4. Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{E} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{F} \end{smallmatrix}\right])$, where $\mathcal{X}$, $\mathcal{E}$, and $\mathcal{F}$ are Hilbert spaces, and let Ψ be a unitary operator from $\mathcal{E}$ to $\mathcal{F}$ (thus, $\mathcal{E}$ and $\mathcal{F}$ must have the same dimension). Then the inequality (5.10) is equivalent to the inequality

$$\begin{bmatrix} 1_{\mathcal{X}} - AA^* & B\Psi^* - AC^* \\ \Psi B^* - CA^* & D^*\Psi + \Psi^* D - CC^* \end{bmatrix} \geqslant 0. \tag{5.12}$$

Indeed, if (5.10) holds, then $\Sigma^{\mathrm{imp}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{E}, \mathcal{F}; \Psi\right)$ is an impedance passive i/s/o system which can be interpreted as an impedance representation of a s/s system $\Sigma = (V, \mathcal{X}, \mathcal{W})$ for which $\mathcal{W} = \mathcal{F} \overset{\Psi}{+} \mathcal{E}$ is an admissible decomposition. The inequality (5.10) means that Σ is forward passive, whereas (5.12) means that Σ is backward passive. According to Lemma 5.2, forward passivity implies backward passivity if $\mathcal{W}$ has an admissible Lagrangian decomposition. By applying the same argument to the adjoint system we find that backward passivity implies forward passivity. Thus, (5.10) and (5.12) are equivalent. (The equivalence of (5.10) and (5.12) can also be proved directly by an algebraic argument without any reference to systems theory.)

THEOREM 5.5. *Let* $\Sigma^{\mathrm{imp}} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{E}, \mathcal{F}; \Psi\right)$ *be an impedance representation of a s/s system* $\Sigma = (V; \mathcal{X}, \mathcal{W})$. *Then* Σ *is conservative if and only if the following three conditions hold:*

(1) A *is a unitary operator in* $\mathcal{X}$.
(2) $C = \Psi B^* A$.
(3) $\Psi^* D + D^* \Psi = B^* B$.

Proof. This follows from the easily verified fact that (1)–(3) hold if and only if both (5.10) and (5.12) hold as equalities instead of inequalities. ∎

DEFINITION 5.6. An i/s/o system $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{E}, \mathcal{F}; \Psi\right)$, where $\mathcal{X}$, $\mathcal{F}$ and $\mathcal{E}$ are Hilbert spaces, and $\Psi \in \mathcal{B}(\mathcal{E}; \mathcal{F})$ is unitary is called a *passive impedance* *system* if all its trajectories $(x(\cdot), u(\cdot), y(\cdot))$ satisfy the inequalities (1.20). If its trajectories satisfy the (stronger) equations (1.21), then we call it a *forward conservative impedance system*, and if both the system and its dual are forward conservative, then we call it a *conservative impedance system*. The input/output transfer function of this system is called its *impedance matrix*.

REMARK 5.7. It is possible to parametrize the set of all possible impedance representation of a given passive s/s system by means of the same representation as in Theorem 4.4 with the following minor modifications. We fix one fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ of $\mathcal{W}$, replace the decomposition $\mathcal{W} = -\mathcal{Y} \,[\dot{+}]\, \mathcal{U}$ in Theorem 4.4 by an arbitrary Lagrangian decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dot{+}} \mathcal{E}$, and replace the orthogonal projections $P_{\mathcal{Y}}$ and $P_{\mathcal{U}}$ by the oblique projections $P_{\mathcal{F}}^{\mathcal{E}}$ and $P_{\mathcal{E}}^{\mathcal{F}}$, respectively. Then the conclusion of Theorem 4.4 remains valid with no changes (the orthogonality of the decomposition $\mathcal{W} = -\mathcal{Y} \,[\dot{+}]\, \mathcal{U}$ in Theorem 4.4 was irrelevant). However, we get a more specific result, resembling the one given for the transmission case in Remark 4.6, by applying these theorems to the specific decompositions $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dot{+}} \mathcal{E}$ and $\mathcal{W} = -\mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ considered earlier in this section. The result is described in our next theorem.

THEOREM 5.8. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system, and let $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dot{+}} \mathcal{E}$ be a Lagrangian decomposition of $\mathcal{W}$ with the corresponding fundamental decomposition $\mathcal{W} = -\mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ obtained from* (5.2). *Suppose that the latter decomposition is admissible, and let $\Sigma^{\mathrm{sca}} = \left(\left[\begin{smallmatrix} A^{\mathrm{sca}} & B^{\mathrm{sca}} \\ C^{\mathrm{sca}} & D^{\mathrm{sca}}\end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-\right)$ be corresponding scattering representation of Σ. We denote the is/so transfer function of Σ^{sca} by* (4.3) *(so that the scattering matrix is $\mathfrak{D}^{\mathrm{sca}}$). Finally we define Φ, $\widetilde{\Theta}$, and Θ by* (5.3) (5.4), *and* (5.8).

(1) *The following conditions are equivalent:*

(a) *The decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dot{+}} \mathcal{E}$ is admissible.*

(b) *The operator $D^{\mathrm{sca}} + \Phi$ is boundedly invertible.*

(2) *Let the equivalent conditions* (a) *and* (b) *in* (1) *hold. Then the corresponding impedance representation $\Sigma^{\mathrm{imp}} = \left(\left[\begin{smallmatrix} A^{\mathrm{imp}} & B^{\mathrm{imp}} \\ C^{\mathrm{imp}} & D^{\mathrm{imp}}\end{smallmatrix}\right]; \mathcal{X}, \mathcal{E}, \mathcal{F}; \Psi\right)$ of Σ is given by*

$$
\begin{aligned}
\begin{bmatrix} A^{\mathrm{imp}} & B^{\mathrm{imp}} \\ C^{\mathrm{imp}} & D^{\mathrm{imp}} \end{bmatrix} &= \begin{bmatrix} A^{\mathrm{sca}} & B^{\mathrm{sca}} \\ \Theta_{11} C^{\mathrm{sca}} & \Theta_{11}(D^{\mathrm{sca}} - \Phi) \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21} C^{\mathrm{sca}} & \Theta_{21}(D^{\mathrm{sca}} + \Phi) \end{bmatrix}^{-1} \\
&= -\begin{bmatrix} -1_{\mathcal{X}} & B^{\mathrm{sca}} \widetilde{\Theta}_{21} \\ 0 & (D^{\mathrm{sca}} + \Phi)\widetilde{\Theta}_{21} \end{bmatrix}^{-1} \begin{bmatrix} A^{\mathrm{sca}} & B^{\mathrm{sca}} \widetilde{\Theta}_{22} \\ C^{\mathrm{sca}} & (D^{\mathrm{sca}} - \Phi)\widetilde{\Theta}_{22} \end{bmatrix},
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
A^{\mathrm{imp}} &= A^{\mathrm{sca}} - B^{\mathrm{sca}}(D^{\mathrm{sca}} + \Phi)^{-1} C^{\mathrm{sca}}, \\
B^{\mathrm{imp}} &= \sqrt{2}\, B^{\mathrm{sca}}(D^{\mathrm{sca}} + \Phi)^{-1} \sqrt{2}\, \widetilde{\Theta}_{12},
\end{aligned}
\tag{5.13}
$$

$$\begin{aligned} C^{\mathrm{imp}} &= -\sqrt{2}\,\Theta_{12}(D^{\mathrm{sca}}+\Phi)^{-1}\sqrt{2}\,C^{\mathrm{sca}},\\ D^{\mathrm{imp}} &= \sqrt{2}\,\Theta_{11}(D^{\mathrm{sca}}-\Phi)(D^{\mathrm{sca}}+\Phi)^{-1}\sqrt{2}\,\widetilde{\Theta}_{12}\\ &= -\sqrt{2}\,\Theta_{12}(D^{\mathrm{sca}}+\Phi)^{-1}(D^{\mathrm{sca}}-\Phi)\sqrt{2}\,\widetilde{\Theta}_{22}. \end{aligned}$$

(3) *Let the equivalent conditions* (a) *and* (b) *in* (1) *hold, and let* $z \in \Lambda_{A^{\mathrm{sca}}}$. *Then the following conditions are equivalent:*

(a) $z \in \Lambda_{A^{\mathrm{imp}}}$.

(b) *The operator* $\mathfrak{D}^{\mathrm{sca}}(z)+\Phi$ *has a bounded inverse.*

(c) *The operator* $\begin{bmatrix} 1_{\mathcal{X}}-zA^{\mathrm{sca}} & -zB^{\mathrm{sca}} \\ C^{\mathrm{sca}} & D^{\mathrm{sca}}+\Phi \end{bmatrix}$ *has a bounded inverse.*

(4) *If the equivalent conditions* (a) *and* (b) *in* (1) *hold, and if we denote the is/so transfer function of* Σ^{imp} *as in* (4.3) *with* 'sca' *replaced by* 'imp', *then for all* $z \in \Lambda_{A\mathrm{sca}} \cap \Lambda_{A^{\mathrm{imp}}}$,

$$\begin{aligned} \begin{bmatrix} \mathfrak{A}^{\mathrm{imp}}(z) & \mathfrak{B}^{\mathrm{imp}}(z) \\ \mathfrak{C}^{\mathrm{imp}}(z) & \mathfrak{D}^{\mathrm{imp}}(z) \end{bmatrix} &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{11}C^{\mathrm{sca}} & \Theta_{11}(D^{\mathrm{sca}}-\Phi) \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}}-zA^{\mathrm{sca}} & -zB^{\mathrm{sca}} \\ \Theta_{21}C^{\mathrm{sca}} & \Theta_{21}(D^{\mathrm{sca}}+\Phi) \end{bmatrix}^{-1}\\ &= \begin{bmatrix} \mathfrak{A}^{\mathrm{sca}}(z) & \mathfrak{B}^{\mathrm{sca}}(z) \\ \Theta_{11}\mathfrak{C}^{\mathrm{sca}}(z) & \Theta_{11}(\mathfrak{D}^{\mathrm{sca}}(z)-\Phi) \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21}\mathfrak{C}^{\mathrm{sca}}(z) & \Theta_{21}(\mathfrak{D}^{\mathrm{sca}}(z)+\Phi) \end{bmatrix}^{-1}\\ &= \begin{bmatrix} 1_{\mathcal{X}}-zA^{\mathrm{sca}} & -zB^{\mathrm{sca}}\widetilde{\Theta}_{21} \\ -C^{\mathrm{sca}} & -(D^{\mathrm{sca}}+\Phi)\widetilde{\Theta}_{21} \end{bmatrix}^{-1} \begin{bmatrix} 1_{\mathcal{X}} & zB^{\mathrm{sca}}\widetilde{\Theta}_{22} \\ 0 & (D^{\mathrm{sca}}-\Phi)\widetilde{\Theta}_{22} \end{bmatrix}\\ &= \begin{bmatrix} 1_{\mathcal{X}} & -\mathfrak{B}^{\mathrm{sca}}(z)\widetilde{\Theta}_{21} \\ 0 & -(\mathfrak{D}^{\mathrm{sca}}(z)+\Phi)\widetilde{\Theta}_{21} \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{A}^{\mathrm{sca}}(z) & \mathfrak{B}^{\mathrm{sca}}(z)\widetilde{\Theta}_{22} \\ \mathfrak{C}^{\mathrm{sca}}(z) & (\mathfrak{D}^{\mathrm{sca}}(z)-\Phi)\widetilde{\Theta}_{22} \end{bmatrix}, \end{aligned} \tag{5.14}$$

or equivalently,

$$\begin{aligned} \mathfrak{A}^{\mathrm{imp}}(z) &= \mathfrak{A}^{\mathrm{sca}}(z) - \mathfrak{B}^{\mathrm{sca}}(z)(\mathfrak{D}^{\mathrm{sca}}(z)+\Phi)^{-1}\mathfrak{C}^{\mathrm{sca}}(z),\\ \mathfrak{B}^{\mathrm{imp}}(z) &= \sqrt{2}\,\mathfrak{B}^{\mathrm{sca}}(z)(\mathfrak{D}^{\mathrm{sca}}(z)+\Phi)^{-1}\sqrt{2}\,\widetilde{\Theta}_{12},\\ \mathfrak{C}^{\mathrm{imp}}(z) &= -\sqrt{2}\,\Theta_{12}(\mathfrak{D}^{\mathrm{sca}}(z)+\Phi)^{-1}\sqrt{2}\,\mathfrak{C}^{\mathrm{sca}}(z),\\ \mathfrak{D}^{\mathrm{imp}}(z) &= \sqrt{2}\,\Theta_{11}(\mathfrak{D}^{\mathrm{sca}}(z)-\Phi)(\mathfrak{D}^{\mathrm{sca}}(z)+\Phi)^{-1}\sqrt{2}\,\widetilde{\Theta}_{12}\\ &= -\sqrt{2}\,\Theta_{12}(\mathfrak{D}^{\mathrm{sca}}(z)+\Phi)^{-1}(\mathfrak{D}^{\mathrm{sca}}(z)-\Phi)\sqrt{2}\,\widetilde{\Theta}_{22}. \end{aligned} \tag{5.15}$$

(5) *If* Σ *is passive, and if the equivalent conditions* (a) *and* (b) *in* (1) *hold, then* $\mathfrak{D}^{\mathrm{sca}}(z)+\Phi$ *has a bounded inverse for every* $z \in \mathbb{D}$ *and the formulas in* (4) *hold for all* $z \in \mathbb{D}$.

Proof. Apply Theorems I.5.11, I.5.12, and I.6.5, taking into account the special form of the operator $\widetilde{\Theta}$ in (5.4) and Θ in (5.8). The last statement follows from the fact that if Σ is passive, then $\mathbb{D} \subset \Lambda_{A^{\mathrm{sca}}} \cap \Lambda_{A^{\mathrm{imp}}}$ since A^{sca} and A^{imp} are contractive. ∎

REMARK 5.9. The formulas in Theorem 5.8 describe how we can check the admissibility of the Lagrangian decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{+} \mathcal{E}$ when the corresponding fundamental decomposition $\mathcal{W} = -\mathcal{W}_{-} \,[\dotplus]\, \mathcal{W}_{+}$ is admissible, and it also gives formulas for the corresponding impedance representation and the impedance matrix in terms of the scattering representation and the scattering matrix. It is also possible to proceed in the opposite direction, even if Σ is not passive: Suppose that the Lagrangian decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{+} \mathcal{E}$ is admissible, and let $\mathcal{W} = -\mathcal{W}_- \,[\dotplus]\, \mathcal{W}_+$ be the corresponding fundamental decomposition. Then all the statement and conclusions of Theorem 5.8 remain valid if we throughout interchange $\mathcal{F} \overset{\Psi}{+} \mathcal{E}$ and $-\mathcal{W}_- \,[\dotplus]\, \mathcal{W}_+$, interchange $\widetilde{\Theta}$ and Θ, interchange Ψ and Φ, and interchange the indices 'sca' and 'imp'. In particular, in this setting formulas (5.15) are rewritten in the form, valid for all $z \in \Lambda_{A^{\rm sca}} \cap \Lambda_{A^{\rm imp}}$,

$$\begin{aligned}
\mathfrak{A}^{\rm sca}(z) &= \mathfrak{A}^{\rm imp}(z) - \mathfrak{B}^{\rm imp}(z)(\mathfrak{D}^{\rm imp}(z)+\Psi)^{-1}\mathfrak{C}^{\rm imp}(z),\\
\mathfrak{B}^{\rm sca}(z) &= \sqrt{2}\,\mathfrak{B}^{\rm imp}(z)(\mathfrak{D}^{\rm imp}(z)+\Psi)^{-1}\sqrt{2}\,\Theta_{12},\\
\mathfrak{C}^{\rm sca}(z) &= -\sqrt{2}\,\widetilde{\Theta}_{12}(\mathfrak{D}^{\rm imp}(z)+\Psi)^{-1}\sqrt{2}\,\mathfrak{C}^{\rm imp}(z),\\
\mathfrak{D}^{\rm sca}(z) &= \sqrt{2}\,\widetilde{\Theta}_{11}(\mathfrak{D}^{\rm imp}(z)-\Psi)(\mathfrak{D}^{\rm imp}(z)+\Psi)^{-1}\sqrt{2}\,\Theta_{12}\\
&= -\sqrt{2}\,\widetilde{\Theta}_{12}(\mathfrak{D}^{\rm imp}(z)+\Psi)^{-1}(\mathfrak{D}^{\rm imp}(z)-\Psi)\sqrt{2}\,\Theta_{22}.
\end{aligned} \tag{5.16}$$

Let $\Sigma^{\rm imp} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{F}, \mathcal{E}; \Psi\right)$ be an impedance representation of a passive s/s system Σ. It is easy to see that the corresponding impedance matrix $\mathfrak{D}^{\rm imp}$ has the property that the restriction of $\Psi^*\mathfrak{D}^{\rm imp}(z)$ to the unit disk $\mathbb{D}$ belongs to the Caratheodory class $\mathcal{C}(\mathbb{D};\mathcal{E})$ of holomorphic $\mathcal{B}(\mathcal{E})$-valued functions on $\mathbb{D}$ which have a nonnegative real part, i.e., $\mathfrak{D}^{\rm imp}$ is holomorphic on $\mathbb{D}$ and $\Psi^*\mathfrak{D}^{\rm imp}(z) + \mathfrak{D}^{\rm imp}(z)^*\Psi \geqslant 0$ for all $z \in \mathbb{D}$. It is well-known, every $\theta \in \mathcal{C}(\mathbb{D};\mathcal{E})$ has an impedance conservative (as well as a controllable passive and forward conservative, and an observable passive and backward conservative, and a minimal passive) realization in the sense that θ is the restriction to $\mathbb{D}$ of the transfer function of some impedance passive i/s/o system $\Sigma^{\rm imp}$ (with any of the additional properties listed above). See, e.g., [4], [12, Theorem 4.5, p. 23 and Theorem 1, p. 226], or [19].

The following proposition is an impedance version of Corollary 3.10.

PROPOSITION 5.10. *Let $\mathfrak{D}^{\rm imp}$ be the impedance of a passive s/s system Σ, corresponding to some admissible Lagrangian decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{+} \mathcal{E}$ of the signal space $\mathcal{W}$. Then the kernel*

$$K_{\mathfrak{D}^{\rm imp}}(z,\zeta) = \frac{\Psi^*\mathfrak{D}^{\rm imp}(z) + \mathfrak{D}^{\rm imp}(\zeta)^*\Psi}{1 - z\overline{\zeta}}$$

is positive definite on $\mathbb{D} \times \mathbb{D}$.

Proof. This follows immediately from Theorem 3.6 applied to an impedance representation interpreted as a driving variable representation. ∎

The result of Proposition 5.10 is, of course, well-known (see, e.g., [18] and the references cited there).

THEOREM 5.11. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, let $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ be a Lagrangian decomposition of $\mathcal{W}$ with the corresponding fundamental decomposition $\mathcal{W} = -\mathcal{W}_- [\dotplus] \mathcal{W}_+$, and let $\Sigma^{\text{sca}} = \left(\left[\begin{smallmatrix} A^{\text{sca}} & B^{\text{sca}} \\ C^{\text{sca}} & D^{\text{sca}} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-\right)$ be corresponding scattering representation of Σ. We denote the is/so transfer functions of Σ^{sca} by* (4.3) *(so that the scattering matrix is $\mathfrak{D}^{\text{sca}}$). Finally we define $\widetilde{\Theta}$ and Θ by* (5.4) *and* (5.8)*. Then the following conditions are equivalent:*

(1) *The decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ is admissible for Σ.*
(2) *$\Phi + D^{\text{sca}}$ has a bounded inverse.*
(3) *$\Phi + \mathfrak{D}^{\text{sca}}(\alpha)$ has a bounded inverse for at least one $\alpha \in \mathbb{D}$.*
(4) *$\Phi + \mathfrak{D}^{\text{sca}}(\alpha)$ has a bounded inverse for every $\alpha \in \mathbb{D}$.*

Proof. The equivalence of (1) and (2) and the implication (1) $\Rightarrow$ (4) are part of Theorem 5.8. Trivially (4) $\Rightarrow$ (1). Thus, it only remains to prove that (3) $\Rightarrow$ (4). This we do in Remark 5.14 below. ∎

REMARK 5.12. If $\mathcal{W}$ is finite-dimensional with $\operatorname{ind}_+\mathcal{W} = \operatorname{ind}_-\mathcal{W}$, then every passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with this signal space has infinitely many impedance representations. Indeed, let $\mathcal{W} = -\mathcal{W}_- [\dotplus] \mathcal{W}_+$ be a fundamental decomposition of $\mathcal{W}$, and let $\Sigma^{\text{sca}} = \left(\left[\begin{smallmatrix} A^{\text{sca}} & B^{\text{sca}} \\ C^{\text{sca}} & D^{\text{sca}} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-\right)$ be the corresponding scattering representation of Σ. Choose an arbitrary unitary angle operator $\Phi\colon \mathcal{W}_+ \to \mathcal{W}_-$, and let $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ be the corresponding Lagrangian decomposition (see the discussion at the beginning of this section). Then the decomposition $\mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ is admissible if and only if $D^{\text{sca}} + \Phi$ has a bounded inverse, or equivalently, if and only if $\Phi^* D^{\text{sca}} + 1_{\mathcal{W}_+}$ has a bounded inverse. The operator $\Phi^* D^{\text{sca}}$ is a finite-dimensional contraction, so it has only finitely many eigenvalues in the closed unit disk. By multiplying Φ by a complex constant, if necessary, we may assure that -1 is not an eigenvalue of $\Phi^* D^{\text{sca}}$. After this modification the Lagrangian decomposition $\mathcal{W} = \mathcal{F} \overset{\Psi}{\dotplus} \mathcal{E}$ will be admissible.

On the contrary, if $\mathcal{W}$ is infinite-dimensional, and if $\operatorname{ind}_+\mathcal{W}$ and $\operatorname{ind}_-\mathcal{W}$ are equal (but infinite), then it is possible that no Lagrangian decomposition of the signal space is admissible, as the following example shows.

EXAMPLE 5.13. Let $\mathcal{W}$ have the fundamental decomposition $\mathcal{W} = -\mathcal{W}_- [\dotplus] \mathcal{W}_+$, where $\operatorname{ind}_+\mathcal{W} = \operatorname{ind}_-\mathcal{W} = \infty$, and let $D \in \mathcal{B}(\mathcal{W}_+; \mathcal{W}_-)$ be an isometric operator which is not unitary. If Φ is an arbitrary unitary operator in $\mathcal{B}(\mathcal{W}_+; \mathcal{W}_-)$, then the spectrum of $\Phi^* D$ is the closed unit disk $\overline{\mathbb{D}}$, so that $\Phi^* D + 1_{\mathcal{W}_+}$ does not have a bounded inverse. Let φ be the constant function $\varphi(z) = D$, $z \in \mathbb{D}$. Then $\varphi \in S(\mathbb{D}; \mathcal{W}_+, \mathcal{W}_-)$, and the simple conservative s/s system Σ whose scattering matrix with respect to the decomposition $\mathcal{W} = -\mathcal{W}_- [\dotplus] \mathcal{W}_+$ is equal to φ has the property that no Lagrangian decomposition of $\mathcal{W}$ is admissible. We remark that the state space of Σ is infinite-dimensional, and that the corresponding set $\Lambda_{A^{\text{sca}}}$ is the closed unit disk $\overline{\mathbb{D}}$. The externally equivalent minimal s/s system has a state space of dimension zero. It is passive, the scattering matrix for this minimal system is equal to D in all of $\mathbb{C}$, and the decomposition $\mathcal{W} = -\mathcal{W}_- \, [\dot{+}] \, \mathcal{W}_+$ is not admissible for this system either (or for any other externally equivalent s/s system).

REMARK 5.14. From Theorem 5.8 we can recover the following known generalized maximum principle: If $\theta \in S(\mathbb{D}; \mathcal{W}_+; \mathcal{W}_-)$ for some Hilbert spaces $\mathcal{W}_+$ and $\mathcal{W}_-$, if $\Phi\colon \mathcal{W}_+ \to \mathcal{W}_-$ is unitary, and if $\theta(z) + \Phi$ is invertible at one point $z = \alpha \in \mathbb{D}$, then $\theta(z) + \Phi$ is invertible for all $z \in \mathbb{D}$. To prove this statement it suffices to consider the case where $\alpha = 0$, since the general case can be reduced to this one by composing θ with the linear fractional transformation $z \mapsto \frac{z-\alpha}{1-\overline{\alpha}z}$. Let Σ be an arbitrary passive s/s system with signal space $\mathcal{W} = -\mathcal{W}_- \, [\dot{+}] \, \mathcal{W}_+$, such that the scattering matrix corresponding to this fundamental decomposition of $\mathcal{W}$ coincides with θ in $\mathbb{D}$. Then part (5) of Theorem 5.8 implies that $\theta(z) + \Phi$ is invertible for all $z \in \mathbb{D}$.

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