The Linear Stationary State/Signal Systems Story

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- Background
- State/signal systems in the time domain
  - Stationary input/state/output systems
  - Conservative input/state/output systems
  - State/signal systems and their i/s/o representations
- State/signal systems in the frequency domain
  - Input/state/output systems in the frequency domain
  - State/signal systems in the frequency domain
- Conservative and non-conservative state/signal realizations
- Further state/signal systems theory
- The team behind the state/signal saga

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The Beginning

- In the fall of 2003, Dima (= Prof. Damir Z. Arov) came to work with me in Åbo for one month.
- We decided to join forces to study the relationship between the (external) reciprocal symmetry of a conservative linear system and the (internal) symmetry structure of the system in three different settings, namely the scattering, the impedance, and the transmission setting.
- Instead of writing three separate papers with three separate sets of results and proofs we wanted to rationalize and to find some "general setting" that would cover the "common part" of the theory. The basic plan was to first develop the theory in such a "general setting" as far as possible, before discussing the three related symmetry problems mentioned above in detail.

- After a couple of days we realized that the "behavioral approach" of (BS06) seemed to provide a suitable "general setting".
- As time went by the borderline between the "general theory" and the application to the original symmetry problem keept moving forward.
- Our first paper had to be split in two because it became too long. Then the second part had to be split in two because it became too long, then the third part had to be split in to, and so on.
- We have by now written 13 papers on state/signal systems with an average length of 50 pages (some of them together with Dr. Mikael Kurula), and are presently writing the first volume of a book on state/signal systems.
- The original application of our state/signal theory to the study of symmetries in the scattering, impedance, and transmission settings is still "work in progress".

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Stationary Discrete Time I/S/O System

A (well-posed) linear stationary discrete time i/s/o (input/state/output) system is of the form

$$\Sigma_{\text{iso}} : \begin{cases} x(n+1) = Ax(n) + Bu(n), \\ \quad y(n) = Cx(n) + Du(n), \end{cases} \qquad n \in \mathbb{Z}^+. \qquad (1)$$

A, B, C, D, are bounded linear operators and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.
the input $u(n) \in \mathcal{U}$ = the input space,
the state $x(n) \in \mathcal{X}$ = the state space,
the output $y(n) \in \mathcal{Y}$ = the output space (all Hilbert spaces).
A future trajectory = a triple of sequences (u, x, y) satisfying (1).

Stationary I/S/O System in Continuous Time

A uniformly continuous linear stationary continuous time i/s/o (input/state/output) system is of the form

$$\Sigma_{\mathrm{iso}} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad t \in \mathbb{R}^+. \tag{2}$$

A, B, C, D, are bounded linear operators and $\mathbb{R}^+ = [0, \infty)$.
the input $u(t) \in \mathcal{U}$ = the input space,
the state $x(t) \in \mathcal{X}$ = the state space,
the output $y(t) \in \mathcal{Y}$ = the output space (all Hilbert spaces).
A classical future trajectory = a triple of continuous functions (u, x, y) satisfying (2) (in particular, x is continuously differentiable).

- Typical stationary i/s/o systems modelled by partial differential equations are not uniformly continuous.
- Note that equation (2) can be rewritten in the form

$$\Sigma_{\text{iso}} : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \tag{3}$$

  where S is the bounded block matrix operator $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.
- We get a much more general class of equations by allowing S in (3) bo be unbounded (but still closed) and rewriting (3) in the form

$$\Sigma_{\text{iso}} : \begin{cases} \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{4}$$

  This class of systems covers "all" the standard models from mathematical physics. We call S the generator of Σ_{iso}. A classical future trajectory = a triple of continuous functions (u, x, y), with x continuously differentiable, satisfying (4).

Discrete Versus Continuous Time

- The two easiest cases to study are the (well-posed) discrete time case (with bounded A, B, C, and D) and the continuous time case with bounded S, i.e., S is of the form $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where A, B, C, and D are bounded.
- However, from a mathematical physics point of view the continuous time case with an unbounded generator S is the most interesting one.
- Our first joint papers with Dima were written in discrete time, and the more recent ones in continous time, with an unbounded generator S.
- The motivation for the introduction of the class of state/signal systems is the same in discrete and continuous time. To explain this motivation we next look at conservative i/s/o systems.

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The Adjoint I/S/O System

Recall the equation describing the dynamics of a linear stationary continuous time system.

$$\Sigma_{\rm iso}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{4}$$

I now assume that the generator S is closed and densely defined. The adjoint i/s/o system is the one whose dynamics is described by the same equation with S replaced by S^*:

$$\Sigma_{\rm iso}^*: \begin{cases} \begin{bmatrix} x^\ddagger(t) \\ y^\ddagger(t) \end{bmatrix} \in \operatorname{dom}(S^*), \\ \begin{bmatrix} \dot x^\ddagger(t) \\ u^\ddagger(t) \end{bmatrix} = S \begin{bmatrix} x^\ddagger(t) \\ y^\ddagger(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{5}$$

Assume Solvability

In the following discussion I assume that both $\Sigma_{\rm iso}$ and $\Sigma_{\rm iso}^*$ are forward solvable in the following weak sense:

- $\Sigma_{\rm iso}$ is forward solvable if for every $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S)$ there exists at least one classical trajectory (u, x, y) of $\Sigma_{\rm iso}$ with $\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]$.
- $\Sigma_{\rm iso}^*$ is forward solvable if for every $\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S^*)$ there exists at least one classical trajectory $(y^\ddagger, x^\ddagger, y^\ddagger)$ of $\Sigma_{\rm iso}^*$ with $\begin{bmatrix} x^\ddagger(0) \\ y^\ddagger(0) \end{bmatrix} = \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]$.

Conservative I/S/O System

- Σ_{iso} is scattering conservative if all its classical future trajectories (u, x, y) satisfy the balance equation

$$\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|y(s)\|^2_{\mathcal{Y}}\, ds = \|x(0)\|^2_{\mathcal{X}} + \int_0^t \|u(s)\|^2_{\mathcal{U}}\, ds, \quad t \in \mathbb{R}^+, \tag{6}$$

  (and the adjoint system Σ^*_{iso} has the same property).
- Σ_{iso} is Ψ-impedance conservative if all its classical future trajectories (u, x, y) satisfy the balance equation

$$\|x(t)\|^2_{\mathcal{X}} = \|x(0)\|^2_{\mathcal{X}} + 2\Re \int_0^t \langle u(s), \Psi y(s)\rangle_{\mathcal{U}}\, ds \quad t \in \mathbb{R}^+, \tag{7}$$

  (and the adjoint system Σ^*_{iso} has the same property with Ψ replaced by Ψ^*). Here $\Psi \colon \mathcal{Y} \to \mathcal{U}$ us a unitary operator.

- $\Sigma_{\rm iso}$ is $(J_{\mathcal U}, J_{\mathcal Y})$-transmission conservative if all its classical future trajectories (u,x,y) satisfy the balance equation

$$\begin{aligned} \|x(t)\|^2_{\mathcal X} &+ \int_0^t \langle y(s), J_{\mathcal Y} y(s)\rangle_{\mathcal Y}\, ds \\ &= \|x(0)\|^2_{\mathcal X} + \int_0^t \langle u(s), J_{\mathcal U} u(s)\rangle_{\mathcal U}\, ds \quad t \in \mathbb{R}^+, \end{aligned} \tag{8}$$

  and the adjoint system $\Sigma^*_{\rm iso}$ has the same property with $(J_{\mathcal U}, J_{\mathcal Y})$ replaced by $(J_{\mathcal Y}, J_{\mathcal U})$. Here $\mathcal{J}_{\mathcal U}$ and $\mathcal{J}_{\mathcal Y}$ are signature operators in $\mathcal U$ and $\mathcal Y$, respectively (usually with the same negative index).

$\mathcal{J}$-Conservative I/S/O System

The three different balance equations can be rewritten into the common form

$$\|x(t)\|^2_{\mathcal{X}} = \|x(0)\|^2_{\mathcal{X}} + \int_0^t \left\langle \begin{bmatrix} u(s) \\ y(s) \end{bmatrix}, \mathcal{J} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} \right\rangle_{\mathcal{U}} ds \quad t \in \mathbb{R}^+, \tag{9}$$

where $\mathcal{J}$ is a signature operator in the product space $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$:

- $\mathcal{J} = \mathcal{J}_{\mathrm{scat}} = \begin{bmatrix} 1_{\mathcal{U}} & 0 \\ 0 & -1_{\mathcal{Y}} \end{bmatrix}$ in the scattering case,
- $\mathcal{J} = \mathcal{J}_{\mathrm{imp}} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$ in the impedance case,
- $\mathcal{J} = \mathcal{J}_{\mathrm{tra}} = \begin{bmatrix} J_{\mathcal{U}} & 0 \\ 0 & -J_{\mathcal{Y}} \end{bmatrix}$ in the transmission case,

Formula (9) treats the input u and the output y in an equal way: the operator $\mathcal{J}$ is simply a signature operator in the signal space $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and it defines a Kreĭn space inner product in $\mathcal{W}$. From the point of (9) it does not matter if u is the input and y the output, or the other way around, or if neither u nor y is the input or output.

It is well-known that one can pass from a Ψ-impedance or $(J_{\mathcal{U}}, J_{\mathcal{Y}})$-transmission conservative i/s/o system to a scattering conservative i/s/o system by simply reinterpreting which part of the combined i/o signal $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ is the input, and which part is the input.

- If $\Sigma_{\rm iso}$ is Ψ-impedance conservative, and if we take the new input and output to be

$$\begin{bmatrix} u_{\rm scat} \\ y_{\rm scat} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1_{\mathcal{U}} & \Psi \\ \Psi^* & -1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} u_{\rm imp} \\ y_{\rm imp} \end{bmatrix},$$

  then the resulting i/s/o system is scattering conservative.
- If $\Sigma_{\rm iso}$ is $(J_{\mathcal{U}}, J_{\mathcal{Y}})$-transmission conservative, and if we take the new input and output to be

$$\begin{bmatrix} u_{\rm scat} \\ y_{\rm scat} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{U}^+} & P_{\mathcal{Y}^-} \\ P_{\mathcal{U}^-} & P_{\mathcal{Y}^+} \end{bmatrix} \begin{bmatrix} u_{\rm tra} \\ y_{\rm tra} \end{bmatrix},$$

  where $(P_{\mathcal{U}^+}, P_{\mathcal{U}^-})$ and $(P_{\mathcal{Y}^+}, P_{\mathcal{Y}^-})$ are complementary projections onto the positive and negative subspaces of $J_{\mathcal{U}}$ and $J_{\mathcal{Y}}$, respectively, then the resulting i/s/o system is again scattering conservative.

- Of course, the transformations described above lead to new dynamic equations with new generators S_{scat}, which can be explicitly derived from the original generators S_{imp} and S_{tra}, but the formulas for S_{scat} tend to be complicated, especially when S_{imp} and S_{tra} are unbounded.
- This motivated us to try to rewrite the original equation describing the dynamics of Σ_{iso} in such a way that it does not distinguish between inputs and outpus!

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How to Go from an I/S/O System to a S/S system?

Recall the equation describing the i/s/o dynamics in continuous time:

$$\Sigma_{\rm iso}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{4}$$

1. First s/s formulation: Write $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and move the output equation into the domain of a new generator F (whose domain is no longer dense in $\mathcal{W}$):

$$\Sigma: \begin{cases} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \mathrm{dom}\,(F), \\ \dot{x}(t) = F\left(\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}\right), \end{cases} \qquad t \in \mathbb{R}^+, \tag{10}$$

$$\mathrm{dom}\,(F) = \left\{ \left[\begin{smallmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{smallmatrix} \right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix} \right] \;\middle|\; \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix} \right] \in \mathrm{dom}\,(S),\; y_0 = P_{\mathcal{Y}} S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix} \right] \right\},$$

$$F \left[\begin{smallmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{smallmatrix} \right] = P_{\mathcal{X}} S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix} \right].$$

How to Go from an I/S/O System to a S/S system?

$$\Sigma_{\rm iso}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \dot x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{4}$$

2. Second s/s formulation: Use graph Representation of (4), $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$:

$$\Sigma\colon \begin{bmatrix} \dot x(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+. \tag{11}$$

where the generating subspace V is the (reordered) graph of S (or of F):

$$\begin{aligned} V &= \left\{ \begin{bmatrix} z_0 \\ x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \mathfrak{K} \,\middle|\, \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S),\ \left[\begin{smallmatrix} z_0 \\ y_0 \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \right\} \\ &= \left\{ \begin{bmatrix} z_0 \\ x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \mathfrak{K} \,\middle|\, \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \mathrm{dom}\,(F),\ z_0 = F \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \right\}. \end{aligned}$$

Classical Future Trajectories

$$\Sigma_{\text{iso}}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{4}$$

$$\Sigma: \begin{cases} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \operatorname{dom}(F), \\ \dot{x}(t) = F\left(\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}\right), \end{cases} \qquad t \in \mathbb{R}^+, \tag{10}$$

$$\Sigma: \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+. \tag{11}$$

- A classical future trajectory of (10) or (11) is a pair of continuous functions (x, w), with x continuously differentiable, which satisfies (10) or (11).
- If (u, x, y) is a classical future trajectory of the i/s/o system Σ_{iso}, then $\left(x, \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\right)$ is a classical future trajectory of the corresponding s/s system Σ.

Properties of S, F, and V

1. $S\colon \begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix} \to \begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$ is a closed operator
$\Leftrightarrow$ $F\colon \begin{bmatrix}\mathcal{X}\\ \mathcal{W}\end{bmatrix} \to \mathcal{X}$ is a closed operator
$\Leftrightarrow$ V is a closed subspace of $\mathfrak{K}$.
2. If the above conditions hold, then S is bounded if and only if $\mathrm{dom}\,(F)$ is closed. In this case S can be split into block matrix form $S = \begin{bmatrix} A & B\\ C & D\end{bmatrix}$ with bounded operators A, B, C, and D.
3. $\mathrm{dom}\,(F)$ is the projection of V onto the domain space of F:

$$\mathrm{dom}\,(F) = \left\{ \begin{bmatrix} x_0\\ w_0\end{bmatrix} \in \begin{bmatrix}\mathcal{X}\\ \mathcal{W}\end{bmatrix} \;\middle|\; \begin{bmatrix} z_0\\ w_0\\ w_0\end{bmatrix} \in V \text{ for some } z_0 \in \mathcal{X} \right\}.$$

4. If $\mathrm{dom}\,(F)$ is dense in $\begin{bmatrix}\mathcal{X}\\ \mathcal{W}\end{bmatrix}$, then $\mathcal{W} = \mathcal{U}$ and $\mathcal{Y} = \{0\}$ (there is no output, only an input).
5. If $\begin{bmatrix} 0\\ w_0\end{bmatrix} \in \mathrm{dom}\,(F) \Rightarrow w_0 = 0$, then $\mathcal{W} = \mathcal{Y}$ and $\mathcal{U} = \{0\}$ (there is no input, only an output).

To go in the opposite direction and construct an i/s/o representation $\Sigma_{\rm iso}$ of a s/s Σ we need to first decompose $\mathcal{W}$ into a direct sum $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, where we want to interpret $\mathcal{U}$ as the input space and $\mathcal{Y}$ as the output space. This is possible if and only if

- If $\left[\begin{smallmatrix} z \\ 0 \\ y \end{smallmatrix}\right] \in V$ where $y \in \mathcal{Y}$, then $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = 0$.

In other words, the z-component and the y-component of a vector $\left[\begin{smallmatrix} z \\ x \\ u+y \end{smallmatrix}\right] \in V$ must be uniquely determined by the x component and the u-component. If this condition holds, and if we denote the linear map from $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ by S (where $\left[\begin{smallmatrix} z \\ x \\ u+y \end{smallmatrix}\right] \in V$), then S is the generator of an i/s/o system $\Sigma_{\rm iso}$, and V has the graph representation

$$V := \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \in \mathrm{dom}\,(S) \text{ and } \begin{bmatrix} z \\ P_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} = S \begin{bmatrix} x \\ P_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \right\}. \tag{12}$$

$$\Sigma\colon \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^{+}. \tag{11}$$

Theorem

The state/signal system Σ in (11) has an i/s/o representation Σ_{iso} with a bounded generator $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if and only if V satisfies the following four conditions:

1. *V is a closed subspace of $\mathfrak{K} = \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$.*
2. *$\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0$ (if $x(t) = 0$ and $w(t) = 0$ then $\dot{x}(t) = 0$),*
3. *For every $x_0 \in \mathcal{X}$ there exists some $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$ (every initial state x_0 is possible).*
4. *$\mathrm{dom}\,(F)$ is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ (this gives the boundedness).*

Thus, this theorem tells when it is possible to decompose $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ in such a way that S is bounded and "V is the graph of S"

Discrete Time State/Singal Systems

The same result is is true in the discrete time setting. In all our discrete time papers we assume that the generating subspace V satisfies conditions (1)–(4) above. The discrete time dynamics is analogous to the continuous time dynamics:

$$\Sigma_{\rm iso}\colon \begin{bmatrix} x(n+1) \\ y(n) \end{bmatrix} = S \begin{bmatrix} x(n) \\ u(n) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(n) \\ u(n) \end{bmatrix}, \qquad n \in \mathbb{Z}^+. \tag{13}$$

$$\Sigma\colon \begin{cases} \begin{bmatrix} x(n) \\ w(n) \end{bmatrix} \in \mathrm{dom}\,(F), \\ x(n+1) = F\left(\begin{bmatrix} x(n) \\ w(n) \end{bmatrix}\right), \end{cases} \qquad n \in \mathbb{Z}^+, \tag{14}$$

$$\Sigma\colon \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+. \tag{15}$$

In the continuous time setting conditions (1)–(4) above are too strong. Dima and I found it to be useful to define several different classes of continuous time s/s systems:

- In the most general case we assume only that V is closed. In this case we use the name state/signal pre-system.
- If a s/s pre-system has a nonempty resolvent set, then it is possible and natural to work also in the frequency domain.
- We reserve the name state/signal system for the case where V satisfies the following three conditions
    1. V is closed,
    2. $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0$ (if $x(t) = 0$ and $w(t) = 0$ then $\dot{x}(t) = 0$),
    3. The projection of V onto its middle component is dense in $\mathcal{X}$ (the set of all possible classical initial states is dense in $\mathcal{X}$).
- Again, if a s/s system has a nonempty resolvent set, then it is again possible and natural to work in the frequency domain. (This is, in particular, true for conservative s/s systems.)

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The Resolvent of a Semigroup Generator

- Every generator A of a C_0 semigroup defines a linear autonomous dynamical system in continuous time:

$$\Sigma\colon \begin{cases} x(t) \in \operatorname{dom}(A), \\ \dot{x}(t) = Ax(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{16}$$

  We call x a classical trajectory of (16) on $\mathbb{R}^+$ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$ and (16) holds.

- By taking Laplace transforms in (16) we see that the Laplace transform $\hat{x}$ of x satisfies the resolvent equation

$$\lambda\hat{x}(\lambda) - x_0 = A\hat{x}(\lambda), \qquad \hat{x}(\lambda) \in \operatorname{dom}(A), \tag{17}$$

  for sufficiently large $\Re\lambda$ (proof: multiply by $e^{-\lambda t}$ and integrate by parts.)

- By definition, λ belongs to the resolvent set $\rho(A)$ of A if it is true for every $x_0 \in \mathcal{X}$ that the resolvent equation (17) has a unique solution $\hat{x}(\lambda)$.
- $\hat{x}(\lambda) = (\lambda - A)^{-1}x_0$, $\lambda \in \rho(A)$. Here $(\lambda - A)^{-1} \in \mathcal{L}(\mathcal{X})$.

$$\Sigma_{\rm iso}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \dot x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \qquad (4)$$

If x, $\dot x$, u, and y in (4) are Laplace transformable, then it follows from (4) (since we assume S to be closed) that the Laplace transforms $\hat x$, $\hat u$, and $\hat y$ of x, u, and y satisfy the i/s/o resolvent equation

$$\widehat\Sigma_{\rm iso}: \begin{cases} \begin{bmatrix} \hat x(\lambda) \\ \hat u(\lambda) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \lambda\hat x(\lambda) - x_0 \\ \hat y(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat x(\lambda) \\ \hat u(\lambda) \end{bmatrix} \end{cases} \qquad (18)$$

(proof: multiply by $\mathrm{e}^{-\lambda t}$ and integrate by parts in the $\dot x$-component.)

$$\widehat{\Sigma}_{\rm iso}: \begin{cases} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}. \end{cases} \tag{18}$$

Definition

1. $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(\Sigma_{\rm iso})$ of $\Sigma_{\rm iso}$ if for every $x_0 \in \mathcal{X}$ and for every $\hat{u}(\lambda) \in \mathcal{U}$ there is a unique pair of vectors $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ satisfying the i/s/o resolvent equation (18).
2. For each $\lambda \in \rho(\Sigma_{\rm iso})$ we define the i/s/o resolvent matrix $\widehat{\mathfrak{S}}(\lambda)$ of $\Sigma_{\rm iso}$ at λ to be the linear operator $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$.

The I/S/O Resolvent Matrix

- It follows from the closed graph theorem that the i/s/o resolvent matrix $\widehat{\mathfrak{S}}(\lambda)$ is a bounded linear operator for each $\lambda \in \rho(\Sigma)$.
- In particular, this implies that $\widehat{\mathfrak{S}}(\lambda)$ has a block matrix representation

$$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}, \quad \lambda \in \rho(\Sigma),$$

  where each of the components $\widehat{\mathfrak{A}}(\lambda)$, $\widehat{\mathfrak{B}}(\lambda)$, $\widehat{\mathfrak{C}}(\lambda)$, $\widehat{\mathfrak{D}}(\lambda)$ is a bounded linear operator.
- $\widehat{\mathfrak{S}}(\lambda)$ is actually even an analytic function of λ, hence so are $\widehat{\mathfrak{A}}(\lambda)$, $\widehat{\mathfrak{B}}(\lambda)$, $\widehat{\mathfrak{C}}(\lambda)$, and $\widehat{\mathfrak{D}}(\lambda)$.
- $\widehat{\mathfrak{S}}$ satisfies the i/s/o resolvent identity

$$\widehat{\mathfrak{S}}(\lambda) - \widehat{\mathfrak{S}}(\mu) = (\mu - \lambda) \begin{bmatrix} \widehat{\mathfrak{A}}(\mu) \\ \widehat{\mathfrak{C}}(\mu) \end{bmatrix} \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \end{bmatrix} \qquad (19)$$

  for all $\mu, \lambda \in \rho_{\rm iso}(S)$.

Components of the I/S/O Resolvent Matrix

Definition

The components $\widehat{\mathfrak{A}}$, $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{C}}$, and $\widehat{\mathfrak{D}}$ of the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ are called as follows:

1. $\widehat{\mathfrak{A}}$ is the *s/s (state/state) resolvent function* of Σ,
2. $\widehat{\mathfrak{B}}$ is the *i/s (input/state) resolvent function* of Σ,
3. $\widehat{\mathfrak{C}}$ is the *s/o (state/output) resolvent function* of Σ,
4. $\widehat{\mathfrak{D}}$ is the *i/o (input/output) resolvent function* of Σ,

- $\widehat{\mathfrak{A}}$ is the(standard) resolvent of the main operator A of S (both in the single-valued case and the multi-valued case).
- The i/o resolvent function $\widehat{\mathfrak{D}}$ is know as the under different names, such as "transfer function", or "characteristic function", or "Weyl function".
- In operator theory the i/s resolvent function $\widehat{\mathfrak{B}}$ is sometimes called the Γ-field.

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$$\Sigma\colon \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \tag{11}$$

If x, $\dot{x}$, and $w$in (11) are Laplace transformable, then it follows from (11) (since we assume V to be closed) that the Laplace transforms $\hat{x}$ and $\hat{w}$ of x and w y satisfy

$$\widehat{\Sigma}_{\rm iso}\colon \begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V \tag{20}$$

(proof: multiply by $\mathrm{e}^{-\lambda t}$ and integrate by parts in the $\dot{x}$-component.)

The Characteristic Node Bundle

$$\widehat{\Sigma}_{\rm iso}: \begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V. \tag{21}$$

This formula can be rewritten in the form

$$\begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) := \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V. \tag{22}$$

Definition

The family of subspaces $\widehat{\mathfrak{E}}: \{\widehat{\mathfrak{E}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ is called the characteristic node bundle. We refer to each of the subspaces $\widehat{\mathfrak{E}}(\lambda)$ as the fiber of $\widehat{\mathfrak{E}}$ att the point $\lambda \in \mathbb{C}$.

Thus, $\widehat{\mathfrak{E}}$ is an "analytic subspace-valued function" defined on $\mathbb{C}$.
Claim: $\widehat{\mathfrak{E}}$ can be interpreted as the graph of the i/s/o resolvent matrix of an arbitrary i/s/o representation $\Sigma_{\rm iso}$ of Σ.)

I/S/O Interpretation of the Characteristic Node Bundle

Let $\Sigma_{\rm iso}$ be an i/s/o representation of Σ, and split the i/o signal w into $w = u + y$, where $u \in \mathcal{U}$ is the input and $y \in \mathcal{Y}$ is the output. Then

$$\widehat{\Sigma}_{\rm iso}: \begin{cases} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}. \end{cases} \tag{18}$$

If $\lambda \in \rho(\Sigma_{\rm iso})$, then

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \widehat{\mathfrak{S}}(\lambda) \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix},$$

which can be rewritten in the form

$$\begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} = \begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \begin{bmatrix} \hat{u}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \begin{bmatrix} 0 \\ \widehat{\mathfrak{C}}(\lambda) \end{bmatrix} & \begin{bmatrix} 1_{\mathcal{U}} \\ \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix}.$$

Here $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ can be arbitrary, and we get (see next slide)

Lemma

Let $\Sigma_{\rm iso}$ be an i/s/o representation of the s/s system Σ, and suppose that $\lambda \in \rho(\Sigma_{\rm iso})$. Then the fiber $\widehat{\mathfrak{E}}(\lambda)$ of the characteristic node bundle $\widehat{\mathfrak{E}}$ at λ has the representation

$$\widehat{\mathfrak{E}}(\lambda) = \operatorname{im}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \begin{bmatrix} 0 \\ \widehat{\mathfrak{C}}(\lambda) \end{bmatrix} & \begin{bmatrix} 1_{\mathcal{U}} \\ \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \end{bmatrix}\right) \tag{23}$$

Note that this can be interpreted as a graph representation of $\widehat{\mathfrak{E}}(\lambda)$ over the first copy of $\mathcal{X}$ and the input space $\mathcal{U}$.

Frequency Domain Input/Output Behavior

- In i/s/o systems theory one is often intersted in the "pure i/o behavior", which one gets by "ignoring the state". More precisely, one takes the inital state $x_0 = 0$, and only looks at the relationship between the input u and the output y, ignoring the state x.
- If we in the frequency domain setting take $x_0 = 0$ and ignore $\hat{x}$, then the full frequency domain relation

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix}$$

is replaced by the i/o relation $\hat{y}(\lambda) = \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda)$, where $\widehat{\mathfrak{D}}(\lambda)$ is the i/o resolvent function.

- The same procedure can be carried out in the case of a s/s system: We take $x_0 = 0$ and ignore the values of $\hat{x}(\lambda)$. (next slide)

Frequency Domain Signal Behavior

Recall the full frequency domain s/s signal behavior:

$$\begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) := \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V. \qquad (22)$$

Taking $x_0 = 0$ and ignoring the value of $\hat{x}(\lambda)$ we see that $\hat{w}(\lambda) \in \widehat{\mathfrak{F}}(\lambda)$, where

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \;\middle|\; \begin{bmatrix} 0 \\ z \\ w \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \text{ for some } z \in \mathcal{X} \right\}. \qquad (24)$$

The Characteristic Signal Bundle

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \;\middle|\; \begin{bmatrix} 0 \\ z \\ w \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \text{ for some } z \in \mathcal{X} \right\}. \qquad (24)$$

Definition

The family of subspaces $\widehat{\mathfrak{F}} : \{\widehat{\mathfrak{F}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathcal{W}$ is called the characteristic signal bundle. We refer to each of the subspaces $\widehat{\mathfrak{F}}(\lambda)$ as the fiber of $\widehat{\mathfrak{F}}$ att the point $\lambda \in \mathbb{C}$.

- Whereas the characteristic node bundle $\widehat{\mathfrak{E}}$ is analytic everywhere in $\mathbb{C}$ (i.e., the fibers depend on λ in an analytic way), the same is not true for the singal bundle $\widehat{\mathfrak{F}}$. Even the dimension of the fibers $\widehat{\mathfrak{F}}(\lambda)$ may change from one point to another.
- However, if $\lambda \in \rho(\Sigma_{\mathrm{iso}})$ for some i/s/o representation Σ_{iso} of Σ, then $\widehat{\mathfrak{F}}$ is analytic at λ (see next slide).

The Resolvent Set of a State/Signal System

Lemma

If $\Sigma_{\rm iso}$ is an i/s/o representation of Σ with $\rho(\Sigma_{\rm iso}) \neq \emptyset$, then for each $\lambda \in \rho(\Sigma_{\rm iso})$ the fibers of the characteristic signal bundle have the graph representation $\widehat{\mathfrak{F}}(\lambda) = \operatorname{im}\left(\begin{bmatrix} \widehat{\mathfrak{D}}(\lambda) \\ 1_{\mathcal{U}} \end{bmatrix}\right)$, $\lambda \in \rho(\Sigma_{\rm iso})$.

Thus, we may interpret $\widehat{\mathfrak{F}}(\lambda)$ as the graph of the i/o resolvent function of an arbitrary i/s/o representation of Σ.

Definition

1. By the resolvent set $\rho(\Sigma)$ of a s/s system Σ we mean the union of the resolvent sets of all i/s/o representations of Σ.
2. By the spectrum of a s/s system Σ we mean the intersection of the spectra of all i/s/o representations $\Sigma_{\rm iso}$ of Σ.

Lemma

The characteristic signal bundle $\widehat{\mathfrak{F}}$ is analytic in $\rho(\Sigma)$.

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Realization Theory

In i/s/o systems theory one is ofter interested in the "converse problem" of finding a "realization" of a given analytic "transfer function" φ. By this we mean an i/s/o system whose i/o resolvent function coincides with φ is some specified open subset Ω of $\mathbb{C}$. For example,

1. φ is a Schur function over $\mathbb{C}^+$, and one wants to construct a scattering conservative realization $\Sigma_{\rm iso}$ of φ,
2. φ is a positive real function over $\mathbb{C}^+$, and one wants to construct an impedance conservative realization $\Sigma_{\rm iso}$ of φ.
3. φ is a Potapov function over $\mathbb{C}^+$, and one wants to construct a transmission conservative realization $\Sigma_{\rm iso}$ of φ.

In the state/signal setting all these three problems collapse into one and the same problem: We are given a passive signal bundle over $\mathbb{C}^+$, and want to construct a conservative s/s realization Σ of this signal bundle, i.e., Σ is a conservative s/s system with $\mathbb{C}^+ \subset \rho(\Sigma)$, and the given passive signal bundle coincides with $\widehat{\mathfrak{F}}$ in $\mathbb{C}^+$.

Recall the i/s/o $\mathcal{J}$-energy balance equation

$$\|x(t)\|_{\mathcal{X}}^2 = \|x(0)\|_{\mathcal{X}}^2 + \int_0^t \left\langle \begin{bmatrix} u(s) \\ y(s) \end{bmatrix}, \mathcal{J} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} \right\rangle_{\mathcal{U}} ds \quad t \in \mathbb{R}^+, \tag{9}$$

where $\mathcal{J}$ is a signature operator in the product space $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$. To get a s/s system we write $w(t) = \left[\begin{smallmatrix} u(t) \\ y(t) \end{smallmatrix}\right]$. We make $\mathcal{W}$ into a Kreĭn space with the inner product

$$[w_1, w_2]_{\mathcal{W}} = \left[\begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \mathcal{J} \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}\right]_{\mathcal{U} \oplus \mathcal{Y}},$$

after which (9) becomes

$$\|x(t)\|_{\mathcal{X}}^2 = \|x(0)\|_{\mathcal{X}}^2 + \int_0^t [w(s), w(s)]_{\mathcal{W}}\, ds, \quad t \in \mathbb{R}^+. \tag{25}$$

$$\|x(t)\|_{\mathcal{X}}^2 = \|x(0)\|_{\mathcal{X}}^2 + \int_0^t [w(s), w(s)]_{\mathcal{W}}\, ds, \quad t \in \mathbb{R}^+. \qquad (25)$$

Differentiating this equation with respect to t we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 = [w(t), w(t)]_{\mathcal{W}}, \qquad t \in \mathbb{R}^+,$$

or equivalently,

$$-\langle \dot{x}(t), x(t)\rangle_{\mathcal{X}} - \langle x(t), \dot{x}(t)\rangle_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} = 0, \qquad t \in \mathbb{R}^+. \qquad (26)$$

In particular, this equation is true for $t = 0$. Let us assume that Σ is forward solvable, i.e., to each $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ there exists at least one classical future trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ with $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]$. Assuming (26) to hold for all classical future trajectories of Σ, we get

$$-\langle z_0, x_0\rangle_{\mathcal{X}} - \langle x_0, z_0\rangle_{\mathcal{X}} + [w_0, w_0]_{\mathcal{W}} = 0, \qquad \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V. \qquad (27)$$

$$-\langle z_0, x_0\rangle_{\mathcal{X}} - \langle x_0, z_0\rangle_{\mathcal{X}} + [w_0, w_0]_{\mathcal{W}} = 0, \qquad \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V. \qquad (27)$$

We can make the node space $\mathfrak{K} = \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ into a Kreĭn space by introducing the node inner product

$$\left[\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]\right]_{\mathfrak{K}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}, \quad \begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \in \mathfrak{K}. \qquad (28)$$

Then (27) says that

$V \subset V^{[\perp]}$, i.e., the generating subspace V of the s/s system Σ is a neutral subpace of $\mathfrak{K}$ with respect to the node inner product!

In the original definitions that I gave of a scattering, impedance, or transmission i/s/o system I also asked the adjoint system to satisfy an analogous condition, and if we also take that adjoint condition into account we find that

$V = V^{[\perp]}$, i.e., V is a Lagrangian (or supermaximal neutral) subspace of $\mathfrak{K}$!

Definition

1. By a conservative s/s system Σ we mean a s/s system whose signal space $\mathcal{W}$ is a Kreĭn space, and whose generating subspace V is a Lagrangian subspace of the node space $\mathfrak{K}$ (with respect to the inner product (28)).
2. By a passive s/s system Σ we mean a s/s system whose signal space $\mathcal{W}$ is a Kreĭn space, and whose generating subspace V is a maximal nonnegative subspace of the node space $\mathfrak{K}$ (with respect to the inner product (28)).

Thus, in particular, every conservative s/s system is also passive.

Theorem

Let Σ be a passive s/s system with signal space space $\mathcal{W}$ and characteristic signal bundle $\widehat{\mathfrak{F}}$. Then

1. $\mathbb{C}^+ \subset \rho(\Sigma)$ *(and hence $\widehat{\mathfrak{F}}$ is analytic in $\mathbb{C}^+$),*
2. *for each $\lambda \in \mathbb{C}^+$ the fiber $\widehat{\mathfrak{F}}(\lambda)$ of $\widehat{\mathfrak{F}}$ is a maximal nonnegative subspace of $\mathcal{W}$.*

Definition

By a passive signal bundle in a Kreĭn (signal) space $\mathcal{W}$ we mean an analytic signal bundle Ψ in $\mathbb{C}^+$ with the property that for each $\lambda \in \mathbb{C}^+$ the fiber $\Psi(\lambda)$ is a maximal nonnegative subspace of $\mathcal{W}$.

The Conservative State/Signal Realization Problem: Given a passive signal bundle Ψ, find a conservative s/s system Σ such that the characteristic signal bundle of Σ coincides with Ψ in $\mathbb{C}^+$. This will be discussed by Mikael Kurula this afternoon.

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Things I Have Discussed

- I/s/o systems in discrete and continuous time
- $\mathcal{J}$-conservative i/s/o systems
- Redefining the inputs and outputs of an i/s/o system
- By conbining inputs and outputs we get a s/s system
- The equations describing the dynamics of a s/s system
- Relationship between a s/s system and its i/s/o representations
- Well-posed s/s systems in discrete time and uniformly continuous s/s systems in continuous time
- General s/s systems in continuous time and their classical trajectories
- State/signal systems in the frequency domain
- The characteristic node and signal bundles of a s/s system
- Passive and conservative state/signal systems
- Passive signal bundles
- Conservative realizations of passive signal bundles.

Things I Have Not Discussed (Page 1/4)

- Driving variable and output nulling representations of s/s systems
- Generalized time domain trajectories of s/s systems in continuous time
- Existence and uniqueness of classical and generalized trajectories of s/s systems
- Well-posed s/s systems in continuous time
- Boundary control s/s systems
- Stability, stabilizability, and detectability of s/s systems
- Relationships between classical and generalized trajectories of continuous time s/s systems
- Past, future, and two-sided time domain behaviors of s/s systems

Things I Have Not Discussed (Page 2/4)

- Frequency domain trajectories in discrete and continuous time
- Frequency domain behaviors of s/s systems
- Controllability and observability of s/s systems in time domain
- Controllability and observability of s/s systems in the frequency domain
- Strongly invariant and unobservably invariant subspaces of a s/s system
- External equivalence of s/s systems
- Intertwinements of s/s systems
- Similarities and pseudo-similarities between s/s systems
- Restrictions, projections, compressions, and dilations of s/s systems
- Minimal s/s systems

Things I Have Not Discussed (Page 3/4)

- The dual and the adjoint of a s/s system
- Passive past, future, and two-sided time domain behaviors
- Passive frequency domain behaviors
- Optimal and $*$-optimal s/s systems (available storage and required supply)
- Normalized driving variable and output nulling realizations of a signal bundle
- Passive balanced s/s systems
- Energy and co-energy preserving s/s systems
- Controllable energy-preserving and observable co-energy preserving realizations of passive signal bundles
- Quadratic optimal control and KYP-theory for s/s systems

Things I Have Not Discussed (Page 4/4)

- S/s systems with extra symmetries (reality, reciprocity, real-reciprocity)
- Relationships between the symmetries of a s/s system and the symmetries of its i/s/o representations
- S/s versions of the de Branges complementary spaces of type $\mathcal{H}$ and $\mathcal{D}$

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The Team Behind the State/Signal Saga

The state/signal saga that I have described above is based on joint work with Dima (and partly also with Mikael Kurula).

- What is it like to work with Dima?

1. Dima can be trusted to attack any mathematical problem

Dima is an internationally recognized expert in many different fields of mathematics, such as

- Ergodic theory
- AAK-theory (Adamjan, Arov, and Kreĭn)
- Infinite-dimensional passive systems theory in discrete and continuous time (many different aspects)
- Interpolation and extension problems for J-inner matrix functions (many variations)
- Linear stationary state/signal systems

He is even able to do work on several different problems at the same time (3 months in Finland with me, and then 4 months in Israel with Harry on a totally different subject, sometimes interrupted by heavy E-mail or Skype conversations with me)

2. Dima can not always be trusted to solve practical everyday problems

- Dima can also be trusted to attack any practical problem, but the result is not always the expected one.
- He gets easily lost. Fortunately, he is not afraid to ask for help, and eventually he always ends up in the right place.
- Somehow he always manages to get things right at the end. Like a cat, ha always "lands on his feet".
- His best support in practical matters is his wife Natalija. She usually knows what should be done (or what he should have done in the first place, when he forgets to ask her advice).

3. Dima is a born optimist

- Dima is a born optimist. He does not hesitate to attack (very) difficult mathematical problems. Sometimes this leads to new grand discoveries, but also to frustrations.
- If you ask Dima for the time table when our s/s book will be ready he will say: Maybe we finish the first volume this fall, and then we shall need maybe one or two more years for the second volume.
- If you ask me, I will say: Maybe 2 years for the first volume if we are lucky, and then maybe 3-4 years for the second volume. (I am also an optimist, but less so than Dima.)

4. Dima is a workaholic

- Once Dima starts thinking about a problem he does not stop until it gets resolved in one way or another. This includes evenings and week-ends, and sometimes even nights.
- On Monday he may show up with a 10 page handwritten manuscript, and is very surprised if I have not had the time to type it all up by Tuesday.
- This is simply the way Dima is built. He simply does not stop thinking about mathematics once he gets going.

5. Dima has strong opinions

- Dima is stubborn, and he has strong opinions!
- I am also stubborn, and I also has strong opinions!
- As a result, when we work together, the sound volumes rises. Usually the whole department can hear us, and may think we are quarreling. But we are just having loud discussions.
- We do not argue about the mathematical correctness of a particular result (there we do agree). But we do argue about
  - In which order should the results be presented? (Dima says "strictly logical.")
  - How general should the formulation be? (Dima says "more general" and I "less general".)
  - In which detail should a result be presented? (Mixed opinions.)
  - How carefully should we distinguish between closely related notions? (Mixed opinions.)
  - Does every new notion also need a new notation? (Dima say "yes" and I say "no".)
  - And so on.

6. Different background, common goal

One of the reasons that Dima and I fit so well together is that we both have a common strong mathematical interest, but quite different backgrounds.

- Dima's background is more in "operator theory" and "complex function theory", and mine is more in a "control theory" and "infinite-dimensional dynamics".
- Together we cover a much broader field than either of us does separately.

7. Dima has a positive attitude and is very friendly

- Most of the time Dima is smiling!
- He has a great sense of humor, and we are joking and laughing a lot when we work (except when we are shouting at each other)
- The first Facebook friend request that I ever got was from Dima
- The majority of all "wish you well"-greetings that I and Marjatta have received in our lives have come from Dima and Nata.

8. Final conclusion

In our joint work with Dima

- Dima is the genius
- I am the independent secretary
- Nata is the hidden support layer.

Damir Z. Arov, Mikael Kurula, and Olof J. Staffans, *Canonical state/signal shift realizations of passive continuous time behaviors*, Complex Anal. Oper. Theory **5** (2011), 331–402.

_______, *Boundary control state/signal systems and boundary triplets*, Operator Methods for Boundary Value Problems, Cambridge University Press, 2012.

_______, *Passive state/signal systems and conservative boundary relations*, Operator Methods for Boundary Value Problems, Cambridge University Press, 2012.

Damir Z. Arov and Olof J. Staffans, *Passive and conservative infinite-dimensional linear state/signal systems*, Proceedings of MTNS2004, 2004.

_______, *Reciprocal passive linear time-invariant systems*, Proceedings of MTNS2004, 2004.

_______, *State/signal linear time-invariant systems theory. Part I: Discrete time systems*, The State Space Method, Generalizations and Applications (Basel Boston Berlin), Operator Theory: Advances and Applications, vol. 161, Birkhäuser-Verlag, 2005, pp. 115–177.

______, *Affine input/state/output representations of state/signal systems*, Proceedings of MTNS2006, 2006.

______, *The infinite-dimensional continuous time Kalman–Yakubovich–Popov inequality*, Proceedings of CDC-ECC05, 2006.

______, *The infinite-dimensional continuous time Kalman–Yakubovich–Popov inequality*, The Extended Field of Operator Theory, Operator Theory: Advances and Applications, vol. 171, 2007, pp. 37–72.

______, *State/signal linear time-invariant systems theory. Part II: Passive discrete time systems*, Internat. J. Robust Nonlinear Control **17** (2007), 497–548.

______, *State/signal linear time-invariant systems theory. Part III: Transmission and impedance representations of discrete time systems*, Operator Theory, Structured Matrices, and Dilations, Tiberiu Constantinescu Memorial Volume, Theta Foundation, 2007, available from American Mathematical Society, pp. 101–140.

———, *State/signal linear time-invariant systems theory. Part IV: Affine representations of discrete time systems*, Complex Anal. Oper. Theory **1** (2007), 457–521.

———, *Bi-inner dilations and bi-stable passive scattering realizations of Schur class operator-valued functions*, Integral Equations Operator Theory (2008), 29–42.

———, *A Kreĭn space coordinate free version of the de Branges complementary space*, J. Funct. Anal. **256** (2009), 3892–3915.

———, *Two canonical passive state/signal shift realizations of passive discrete time behaviors*, J. Funct. Anal. **257** (2009), 2573–2634.

———, *Canonical conservative state/signal shift realizations of passive discrete time behaviors*, J. Funct. Anal. **259** (2010), 3265–3327.

———, *Symmetries in special classes of passive state/signal systems*, J. Funct. Anal. **262** (2012), 5021–5097.

———, *Linear stationary systems in continuous time*, 2014, Book manuscript, available at http://users.abo.fi/staffans/publ.html.

———, *Passive linear state/signal systems*, 2016, In preparation.

Joseph A. Ball and Olof J. Staffans, *Conservative state-space realizations of dissipative system behaviors*, Integral Equations Operator Theory **54** (2006), 151–213.