Three Canonical Passive State/Signal Shift Models in the Unit Disk

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Abstract: During the last years the authors have studied a class of discrete time system, called s/s (state/signal) systems, which has its roots in classical circuit theory, but which also contains infinite-dimensional systems. Our systems are time-independent, and in this talk, they will also be passive. Frequently circuits are studied from an i/s/o (input/state/output) point of view where the port variables are divided into inputs and outputs, and in this setting the system is characterized by its transfer function from the inputs to the outputs, which in the passive scattering formalism is a Schur function. On the other hand, in the s/s setting one does not distinguish between inputs and output, and the transfer function from inputs to outputs is replaced by a shift-invariant subspace of H^2 (equal to the graph of the scattering function), which in the passive case is maximal nonnegative with respect to a certain indefinite inner product in H^2. We refer to this subspace as the (frequency domain future) "behavior" of the system.

In the scattering version of the standard passive i/s/o realization problem (the "inverse" problem) a Schur function φ is given, and one searches for a i/s/o function whose transfer function is φ. Here we study the s/s version of the same problem: given a maximal nonnegative shift-invariant subspace $\widehat{\mathfrak{W}}_+$ of H^2, we want to construct a s/s system whose behavior coincides with $\widehat{\mathfrak{W}}_+$. We present three such passive s/s realizations with different additional properties: the first one is observable and backward conservative, the second is controllable and forward conservative, and the third is simple and conservative. Our three models are 'canonical' in the sense that they are determined uniquely by the given shift-invariant subspace $\widehat{\mathfrak{W}}_+$. By decomposing the signal space in different ways into inputs and outputs it is possible to use our models to derive scattering, impedance, and transmission models for different i/s/o settings.

1. INTRODUCTION

The i/s/o (input/state/output) system

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n) \\ y(n) &= Cx(n) + Du(n), \quad n \in \mathbb{Z}^+; \\ x(0) &= x_0, \end{aligned} \tag{1}$$

is called a *passive realization of the Schur function* $\mathfrak{D}$ if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is contractive and $\mathfrak{D}(z) = zC(1-zA)^{-1}B + D$, $z \in \mathbb{D}^+ := \{z \in \mathbb{C} \mid |z| < 1\}$. We denote this system by $\Sigma_{i/s/o} := \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$. If $u \in \ell^2_+(\mathcal{U})$, then the Z-transforms of x, u, and y satisfy

$$\begin{bmatrix} (\hat{x}(z) - x_0)/z \\ \hat{y}(z) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x}(z) \\ \hat{u}(z) \end{bmatrix}, \quad z \in \mathbb{D}^+. \tag{2}$$

From this equation we can solve $\hat{x}(z)$ and $\hat{y}(z)$ in terms of x_0 and $\hat{u}(z)$ as follows:

$$\begin{bmatrix} \hat{x}(z) \\ \hat{y}(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(z) \end{bmatrix}, \quad z \in \mathbb{D}^+, \tag{3}$$

where

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} z(1-zA)^{-1} & z(1-zA)^{-1}B \\ C(1-zA)^{-1} & zC(1-zA)^{-1}B + D \end{bmatrix} \tag{4}$$

for all $z \in \mathbb{D}^+$. The 2×2 block operator in (4) is the *input-state/state-output transfer function*, and its lower right corner $\mathfrak{D}$ is the *i/o (input/output) transfer function.* Thus, $\Sigma_{i/s/o}$ is a realization of a given Schur function φ if its i/o transfer function $\mathfrak{D}$ coincides with φ in the unit disk.

We get the corresponding passive s/s (state/signal) system $\Sigma_{s/s} = (V; \mathcal{X}, \mathcal{W})$ by replacing $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ by its graph:

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+, \quad x(0) = x_0, \tag{5}$$

where $w(n) = \left[\begin{smallmatrix} y(n) \\ u(n) \end{smallmatrix}\right]$ and

$$V = \left\{ \begin{bmatrix} Ax_0 + Bu_0 \\ x_0 \\ Cx_0 + Du_0 \\ u_0 \end{bmatrix} \;\middle|\; \begin{bmatrix} x_0 \\ w_0 \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \right\}, \; \mathcal{W} = \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}. \tag{6}$$

The corresponding graph representation of (2) is given by

$$\begin{bmatrix} (\hat{x}(z) - x_0)/z \\ \hat{x}(z) \\ \hat{w}(z) \end{bmatrix} \in V, \qquad z \in \mathbb{D}_+. \tag{7}$$

Let $\mathcal{W}$ be the Kreĭn space $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$. The graph $\widehat{\mathfrak{W}}_+ := \left\{ \left[\begin{smallmatrix} \mathfrak{D}\hat{u} \\ \hat{u} \end{smallmatrix} \right] \mid \hat{u} \in H^2(\mathbb{D}^+;\mathcal{U}) \right\}$ of the Schur function $\mathfrak{D}$ in (3) is equal to the set of all $\hat{w} \in H^2(\mathbb{D}^+;\mathcal{W})$ for which there exists some $\hat{x}$ such that (7) holds with $x_0 = 0$. This set is a *future frequency domain behavior*, i.e., a *closed shift-invariant subspace of* $K^2(\mathbb{D}^+;\mathcal{W})$, where $K^2(\mathbb{D}^+;\mathcal{W})$ stands for $H^2(\mathbb{D}^+;\mathcal{W})$ equipped with the indefinite Kreĭn space inner product inherited from $\mathcal{W}$. The system $\Sigma_{s/s}$ is called a s/s realization of this behavior. (Time domain behaviors are discussed at length in Polderman and Willems [1998].)

The (passivity) requirement that the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]$ in (2) is a contraction is equivalent to the requirement that the subspace V defined in (6) is a maximal nonnegative subspace of the node space $\mathfrak{K} = -\mathcal{X} [\dotplus] \mathcal{X} [\dotplus] \mathcal{W}$. It also implies that the future frequency domain behavior $\widehat{\mathfrak{W}}_+$ induced by $\Sigma_{s/s}$ is *passive*, i.e., that $\widehat{\mathfrak{W}}_+$ is a maximal nonnegative shift-invariant subspace of the Kreĭn space $K^2(\mathbb{D}^+;\mathcal{W})$.

In the sequel we study the state signal system (5) without requiring a priory that V has a representation of the type (6). The standard assumption is that $\mathcal{X}$ is a Hilbert space, $\mathcal{W}$ is a Kreĭn space, and that V is a maximal nonnegative subspace of the (Kreĭn) node space $\mathfrak{K} = -\mathcal{X} [\dotplus] \mathcal{X} [\dotplus] \mathcal{W}$. (This implies that, given any fundamental decomposition $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ of $\mathcal{W}$, V has a representation of the type (6) for some contractive operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]$.) Naturally, since we are here restricting our attention to *passive s/s systems*, we are also restricting our attention to *passive future frequency domain behaviors*, i.e., maximal nonnegative shift-invariant subspaces $\widehat{\mathfrak{W}}_+$ of the Kreĭn space $K^2(\mathbb{D}^+;\mathcal{W})$. For more details, see Arov and Staffans [2009a,b]. Continuous time s/s systems have been studied in Kurula and Staffans [2009].

2. PASSIVE AND CONSERVATIVE I/S/O REALIZATIONS

One of the first researchers to study realization theory was Kalman [1963a,b, 1965]. Among others, he showed that every rational matrix-valued function which does not have a pole at the origin has an i/s/o realization (i.e, it is the i/o transfer function of some i/s/o system), and that any two realizations with minimal state dimension are similar to each other. Approximately at the same time Yakubovich [1962] and Popov [1961] begun to study the passivity of the system, which in the discrete time scattering setting manifests itself in the contractivity of the matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]$ in (1). The transfer function of such a system is necessarily a Schur function, i.e., it is analytic and contractive in the unit disk. It was also clear more or less from the beginning that lossless rational matrix-valued functions (i.e., Schur functions which take unitary values on the unit circle) have realizations where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]$ is a unitary operator, and any two such minimal realizations differ from each other by a unitary similarity transformation. These realizations are called *conservative*.

The next step was to construct conservative i/s/o realizations of Schur functions that are not lossless. Every such realization must have an infinite-dimensional state space, and it is typically not minimal in the sense that it should be controllable and observable. However, if we instead require it to be simple, i.e., that the closed linear span of the reachable subspace and the orthogonal complement to the unobservable subspace is equal to the full state space, then any two such realizations are unitarily similar to each other.

Over time several constructions of simple conservative realizations of Schur functions have appeared. One of the oldest construction is the one due to Sz.-Nagy and Foiaş [1970]. This realization is not symmetric in the sense that the adjoint of the Sz.-Nagy–Foiaş model for a Schur function φ does not coincide with the Sz.-Nagy–Foiaş model of the dual Schur function $\tilde{\varphi}(z) := \varphi(\overline{z})^*$. Another model is the Pavlov model which is maybe most easily found in Nikolskiĭ and Vasyunin [1989]. This model is symmetric with respect to duality. Yet another symmetric model was discovered by de Branges and Rovnyak [1966a,b] (this model can also be found in, e.g., Alpay et al. [1997]). Finally, Nikolskiĭ and Vasyunin [1989, 1998] developed a whole family of models that contains all the above mentioned ones. In all these models the operator A is a compressed shift operator acting on some space of functions with values in $\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix} \right]$, where $\mathcal{U}$ and $\mathcal{Y}$ are the input and output spaces. Each of the three models mentioned above, the Sz.-Nagy–Foiaş model, the Pavlov model, and the deBranges–Rovnyak model are canonical in the sense that they are determined uniquely by the given Schur function φ, and as we mentioned above, they are all unitarily similar to each other.

3. PASSIVE AND CONSERVATIVE S/S REALIZATIONS

The purpose of this note is to present some canonical passive and conservative realizations of a passive future frequency domain behavior $\widehat{\mathfrak{W}}_+$. They can be interpreted as s/s versions of the the three deBranges–Rovnyak model, namely their backward conservative and observable model, their forward conservative and controllable model, and the symmetric simple conservative model mentioned above. The requirement that our models should be *canonical* means that they should be uniquely determined by the given future behavior $\widehat{\mathfrak{W}}_+$. In particular, they are therefore not allowed to depend on any "coordinate representation" of the $\widehat{\mathfrak{W}}_+$ as the graph of some analytic operator-valued function $\mathcal{U} \to \mathcal{Y}$, where $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is some (more or less arbitrarily chosen) direct sum decomposition of $\mathcal{W}$. This requirement makes it impossible to directly convert the existing i/s/o models into s/s models, since the i/s/o models used in such a construction would have to depend on the choice of the the particular decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$, and hence the result would not be canonical. However, it is possible to go in the other direction, and to derive i/s/o models from our s/s models. Depending on how the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is chosen it is possible to interpret $\widehat{\mathfrak{W}}_+$ as the graph of a scattering matrix, or as the graph of an impedance matrix, or as the graph of a transmission matrix. This interpretation has no direct influence on our three s/s models. However, the use of such decompositions makes it possible to convert our s/s models into canonical i/s/o realizations of a given scattering matrix, or a given impedance matrix, or a given transmission matrix. In the scattering case the resulting i/s/o models coincide with the corresponding deBranges–Rovnyak models. For more details, see Arov and Staffans [2005, 2007a,b,c].

The three models that we describe below are functional models in the unit disk, i.e., they can be interpreted as discrete time frequency domain models. The time domain versions of the first two models appear in Arov and Staffans [2009a], but the third model has not been published before.

4. THE PASSIVE BACKWARD CONSERVATIVE OBSERVABLE REALIZATION

We begin by presenting the backward conservative observable s/s realization. The state space $\mathcal{X}_+$ of this realization is a subspace of the quotient space $K^2(\mathbb{D}^+;\mathcal{W})/\widehat{\mathfrak{W}}_+$. Let us denote the quotient map $K^2(\mathbb{D}^+;\mathcal{W}) \to K^2(\mathbb{D}^+;\mathcal{W})/\widehat{\mathfrak{W}}_+$ by P_+. Thus, for each $w_+ \in K^2(\mathbb{D}^+;\mathcal{W})$, $P_+w_+ = w_+ + \widehat{\mathfrak{W}}_+ := \{w_+ + z_+ \mid z_+ \in \widehat{\mathfrak{W}}_+\}$. The space $\mathcal{X}_+$ is the subspace of equivalence classes in $K^2(\mathbb{D}^+;\mathcal{W})/\widehat{\mathfrak{W}}_+$which have finite $\mathcal{X}_+$-norm:

$$\mathcal{X}_+ = \{P_+\hat{w}_+ \in K^2(\mathbb{D}^+;\mathcal{W})/\widehat{\mathfrak{W}}_+ \mid \|P_+\hat{w}_+\|_{\mathcal{X}_+} < \infty\}, \tag{8}$$

where

$$\|P_+\hat{w}_+\|^2_{\mathcal{X}_+} = \sup_{\hat{v}_+ - \hat{w}_+ \in \widehat{\mathfrak{W}}_+} -[\hat{v}_+, \hat{v}_+]_{K^2(\mathbb{D}^+;\mathcal{W})}. \tag{9}$$

Let us denote the inverse image under P_+ of $\mathcal{X}_+$ by $\mathcal{K}_+(\mathcal{W})$, i.e., $\mathcal{K}_+ := \{w_+ \in K^2(\mathbb{D}^+;\mathcal{W}) \mid P_+w_+ \in \mathcal{X}_+\}$. The generating subspace $V_+ \subset \begin{bmatrix} \mathcal{X}_+ \\ \mathcal{X}_+ \\ \mathcal{W} \end{bmatrix}$ of the passive backward conservative observable realization has the image representation

$$V_+ = \left\{ \begin{bmatrix} P_+(\hat{w}_+(z) - \hat{w}_+(0))/z \\ P_+\hat{w}_+(z) \\ \hat{w}_+(0) \end{bmatrix} \middle| \hat{w}_+(\cdot) \in \mathcal{K}_+(\mathcal{W}) \right\}. \tag{10}$$

Explicitly, this means the following: Given any initial state $x_0 \in \mathcal{X}_+$ we choose some representative $\hat{w}_+$ of x_0 (i.e., $P_+\hat{w}_+ = x_0$). Then $\hat{w}_+(0)$ represents an arbitrary admissible signal value at time $t = 0$, and the state at time one corresponding to the state x_0 and the signal $\hat{w}_+(0)$ at time zero is $P_+((\hat{w}_+(z) - \hat{w}_+(0))/z)$, where $(\hat{w}_+(z) - \hat{w}_+(0))/z$ is the function that one gets by applying the incoming shift to $\hat{w}_+$.

5. THE PAST AND FULL FREQUENCY DOMAIN BEHAVIORS

Before presenting the passive forward conservative controllable realization and the simple conservative realization we define the *passive full frequency domain behavior* $\widehat{\mathfrak{W}}$ and the orthogonal companion $\widehat{\mathfrak{W}}_-^{[\perp]}$ of the *passive past frequency domain behavior* $\widehat{\mathfrak{W}}_-$ induced by a given passive future frequency domain behavior $\widehat{\mathfrak{W}}_+$. These two behaviors can be defined, e.g., in the following way. Let $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle, let $\mathbb{D}^- := \{z \in \mathbb{C} \mid |z| > 1\} \cup \{\infty\}$, and let $\mathcal{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. Then $\widehat{\mathfrak{W}}_+$ has the graph representation $\widehat{\mathfrak{W}}_+ = \{\left[\begin{smallmatrix} \varphi\hat{u} \\ \hat{u} \end{smallmatrix}\right] \mid \hat{u} \in H^2(\mathbb{D}^+;\mathcal{U})\}$ for some Schur function φ on $\mathbb{D}^+$ with values in $\mathcal{B}(\mathcal{U};\mathcal{Y})$. We denote the boundary function, defined on $\mathbb{T}$, by the same symbol φ, and we define the function $\tilde{\varphi}$ by $\tilde{\varphi}(z) = \varphi^*(1/\overline{z})$, $z \in \mathbb{D}^-$. The restriction of φ to $\mathbb{T}$ is an L^∞-function with values in $\mathcal{B}(\mathcal{U};\mathcal{Y})$, and $\tilde{\varphi}$ is a Schur function in $\mathbb{D}^-$ with values in $\mathcal{B}(\mathcal{Y};\mathcal{U})$. We define

$$\widehat{\mathfrak{W}} := \left\{ \begin{bmatrix} \varphi\hat{u} \\ \hat{u} \end{bmatrix} \middle| \hat{u} \in L^2(\mathbb{T};\mathcal{U}) \right\},$$
$$\widehat{\mathfrak{W}}_-^{[\perp]} := \left\{ \begin{bmatrix} \hat{y} \\ \tilde{\varphi}\hat{u} \end{bmatrix} \middle| \hat{y} \in H^2(\mathbb{D}^-;\mathcal{Y}) \right\}.$$

It turns out that neither $\widehat{\mathfrak{W}}$ nor $\widehat{\mathfrak{W}}_-^{[\perp]}$ depends on the particular fundamental decomposition $\mathcal{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$, that $\widehat{\mathfrak{W}}$ is a maximal nonnegative subspace of the Kreĭn space $L^2(\mathbb{T};\mathcal{W})$, and that $\widehat{\mathfrak{W}}_-^{[\perp]}$ is a maximal nonnegative subspace of the Kreĭn space $K^2(\mathbb{D}^-;-\mathcal{W})$.

6. THE PASSIVE FORWARD CONSERVATIVE CONTROLLABLE REALIZATION

The forward conservative controllable realization is in a certain sense dual to the backward conservative observable realization. The state space $\mathcal{X}_-$ of this realization is a subspace of the quotient space $K^2(\mathbb{D}^-;\mathcal{W})/\widehat{\mathfrak{W}}_-^{[\perp]}$. Let us denote the quotient map from $K^2(\mathbb{D}^-;\mathcal{W})$ to $K^2(\mathbb{D}^-;\mathcal{W})/\widehat{\mathfrak{W}}_-^{[\perp]}$ by P_-. Thus, $P_-w = w + \widehat{\mathfrak{W}}_-^{[\perp]}$, $w \in K^2(\mathbb{D}^-;\mathcal{W})$. The space $\mathcal{X}_-$ is the subspace of equivalence classes in $K^2(\mathbb{D}^-;\mathcal{W})/\widehat{\mathfrak{W}}_-^{[\perp]}$ which have finite $\mathcal{X}_-$-norm:

$$\mathcal{X}_- = \{P_-\hat{w} \in K^2(\mathbb{D}^-;\mathcal{W})/\widehat{\mathfrak{W}}_-^{[\perp]} \mid \|P_-\hat{w}\|_{\mathcal{X}_-} < \infty\}, \tag{11}$$

where

$$\|P_-\hat{w}\|^2_{\mathcal{X}_-} = \sup_{\hat{v} - \hat{w} \in \widehat{\mathfrak{W}}_-^{[\perp]}} [\hat{v}, \hat{v}]_{K^2(\mathbb{D}^-;\mathcal{W})}. \tag{12}$$

Let us denote the inverse image under P_- of $\mathcal{X}_-$ by $\mathcal{K}_-(\mathcal{W})$, i.e., $\mathcal{K}_- := \{w \in K^2(\mathbb{D}^-;\mathcal{W}) \mid P_-w \in \mathcal{X}_-\}$.

The definition of the generating subspace V_- of the passive forward conservative controllable s/s realization contains the *past/future* map $\widehat{\Gamma}$. This is a contraction $\mathcal{X}_- \to \mathcal{X}_+$ which is uniquely characterized by the fact that

$$P_+w = \widehat{\Gamma}P_-w \qquad w \in \widehat{\mathfrak{W}}.$$

The generating subspace $V_- \subset \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_- \\ \mathcal{W} \end{bmatrix}$ of the forward conservative controllable realization has the image representation

$$V_- = \left\{ \begin{bmatrix} P_-(\hat{w}_-(z) + w_0)/z \\ P_-\hat{w}_-(z) \\ w_0 \end{bmatrix} \middle| \begin{array}{l} \hat{w}_-(\cdot) \in \mathcal{K}_-(\mathcal{W}), \\ w_0 \in (\widehat{\Gamma}P_-\hat{w}_-)(0) \end{array} \right\}. \tag{13}$$

Explicitly, this means the following: Given any initial state $x_0 \in \mathcal{X}_-$ we first choose some representative $\hat{w}_-$ of x_0 (i.e., $P_-\hat{w}_- = x_0$). We then add a function $\hat{w}_+ \in \mathcal{K}_+(\mathcal{W})$ to w_- satisfying $P_+\hat{w}_+ = \widehat{\Gamma}P_-\hat{w}_-$. The signal at time zero is equal to $w_0 = \hat{w}_+(0)$, and the new state x_1 is obtained by first left-shifting $\hat{w}_- + \hat{w}_+$, then projecting the result onto $\mathcal{K}(\mathbb{D}^-;\mathcal{W})$, and finally applying the quotient map P_- to this function.

7. THE SIMPLE CONSERVATIVE REALIZATION

The simple conservative realization can be regarded as a "coupling" of the forward conservative controllable and the backward conservative observable realizations. Its state space is a subspace of the quotient space $L^2(\mathbb{T};\mathcal{W})/(\widehat{\mathfrak{W}}_+ + \widehat{\mathfrak{W}}_-^{[\perp]})$, where $L^2(\mathbb{T};\mathcal{W})$ is the Kreĭn space of $\mathcal{W}$-values L^2-functions on $\mathbb{T}$ with the indefinite inner product inherited from $\mathcal{W}$. Let us denote the orthogonal projections in $L^2(\mathbb{T};\mathcal{W})$ onto $K^2(\mathbb{D}_\pm;\mathcal{W})$ by $\pi_\pm$, and let P be the quotient map $L^2(\mathbb{T};\mathcal{W}) \to L^2(\mathbb{T};\mathcal{W})/(\widehat{\mathfrak{W}}_+ + \widehat{\mathfrak{W}}_-^{[\perp]})$. Then $P = P_+\pi_+ + P_-\pi_-$.

The state space $\mathcal{X}$ of the simple conservative s/s realization is the subspace of equivalence classes in $L^2(\mathbb{T};\mathcal{W})/(\widehat{\mathfrak{W}}_+ + \widehat{\mathfrak{W}}_-^{[\perp]})$ which have finite $\mathcal{X}$-norm:

$$\mathcal{X} = \left\{ P\hat{w} \in L^2(\mathbb{T};\mathcal{W})/(\widehat{\mathfrak{W}}_+ + \widehat{\mathfrak{W}}_-^{[\perp]}) \mid \|P\hat{w}\|_{\mathcal{X}} < \infty \right\}, \tag{14}$$

where

$$\begin{aligned} \left\|P\hat{w}\right\|_{\mathcal{X}}^2 &= -[\pi_+\hat{w}, \pi_+\hat{w}]_{K^2(\mathbb{D}^+;\mathcal{W})} \\ &\quad + \limsup_{n\to\infty} \|P_-\pi_-\hat{w}(z)z^{-n}\|_{\mathcal{X}_-}^2 \\ &= [\pi_-\hat{w}, \pi_-\hat{w}]_{K^2(\mathbb{D}^-;\mathcal{W})} \\ &\quad + \limsup_{n\to\infty} \|P_+\pi_+\hat{w}(z)z^{n}\|_{\mathcal{X}_+}^2. \end{aligned} \tag{15}$$

Equivalently, the space $\mathcal{X}$ is the range space of the non-negative operator $\begin{bmatrix} 1 & \widehat{\Gamma} \\ \widehat{\Gamma}^* & 1 \end{bmatrix}^{1/2}$ in $\mathcal{X}_+ \oplus \mathcal{X}_-$.

Let us denote the inverse image under P of $\mathcal{X}$ by $\mathcal{K}(\mathcal{W})$, i.e., $\mathcal{K} := \{w \in L^2(\mathbb{T};\mathcal{W}) \mid Pw \in \mathcal{X}\}$. The generating subspace $V \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ of the simple conservative realization has the image representation

$$V = \left\{ \begin{bmatrix} P\hat{w}(z)/z \\ P\hat{w}(z) \\ \hat{w}(0) \end{bmatrix} \middle| \hat{w}(\cdot) \in \mathcal{K}(\mathcal{W}) \right\}. \tag{16}$$

Explicitly, this means the following: Given any initial state $x_0 \in \mathcal{X}$ we choose some representative $\hat{w}$ of x_0 (i.e., $P\hat{w} = x_0$). Then $\hat{w}(0)$ represents an arbitrary admissible signal value at time $t = 0$, and the state at time one corresponding to the state x_0 and the signal $\hat{w}(0)$ at time zero is $P\hat{w}_+(z)/z$, where $\hat{w}(z)/z$ is the function that one gets by applying the left-shift to $\hat{w}$.

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