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MAXWELL'S EQUATIONS AS A SCATTERING PASSIVE LINEAR SYSTEM*

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Abstract. We consider Maxwell's equations on a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary Γ, with boundary control and boundary observation. Relying on an abstract framework developed by us in an earlier paper, we define a scattering passive linear system that corresponds to Maxwell's equations and investigate its properties. The state of the system is $\left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right]$, where $\mathbf{B}$ and $\mathbf{D}$ are the magnetic and electric flux densities, and the state space of the system is $X = E \oplus E$, where $E = L^2(\Omega;\mathbb{R}^3)$. We assume that Γ_0 and Γ_1 are disjoint, relatively open subsets of Γ such that $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \Gamma$. We consider Γ_0 to be a superconductor, which means that on Γ_0 the tangential component of the electric field is forced to be zero. The input and output space U consists of tangential vector fields of class L^2 on Γ_1. The input and output at any moment are suitable linear combinations of the tangential components of the electric and magnetic fields. The semigroup generator has the structure $A = \left[\begin{smallmatrix}0 & -L\\ L^* & G-\gamma^* R\gamma\end{smallmatrix}\right] P$, where $L = \mathrm{rot}$ (with a suitable domain), γ is the tangential component trace operator restricted to Γ_1, R is a strictly positive pointwise multiplication operator on U (that can be chosen arbitrarily), and $P^{-1} = \left[\begin{smallmatrix}\mu & 0\\ 0 & \varepsilon\end{smallmatrix}\right]$ is another strictly positive pointwise multiplication operator (acting on X). The operator $-G$ is pointwise multiplication with the conductivity $g \geq 0$ of the material in Ω. The system is scattering conservative iff $g = 0$.

Key words. scattering passive system, scattering conservative system, Maxwell's equations, boundary control, boundary observation

AMS subject classifications. 93C20, 35Q61, 47A40

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1. Overview of the results.

The aim of this paper is to formulate the Maxwell equations on a bounded domain Ω as a linear scattering passive (and hence well-posed) distributed parameter system. We allow charges and currents in Ω, and the permittivity, permeability, and conductivity of the material in Ω are allowed to depend on the position. The input and output signals are defined such that interaction with a neighboring domain is easy and natural to formulate. We allow the presence of superconductors at the boundary of Ω (but not inside). Our main results are presented in section 5 in the form of Theorems 5.1, 5.4, and 5.6, where we apply the results from our paper [35] to construct two scattering passive systems induced by Maxwell's equations.

Below we give an overview of the results in this paper.

In section 2 we recall the concepts of system node, transfer function, classical solution, well-posed linear system, scattering passive system, scattering energy preserving system, and scattering conservative system. The last three concepts are introduced on Hilbert spaces with strictly positive weighting operators, as this is better suited for analyzing physically meaningful systems.

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For the purposes of this introductory section, we recall here the concept of a scattering passive linear system. Let U and X be Hilbert spaces and let $P \in \mathcal{L}(X)$, $R \in \mathcal{L}(U)$ be strictly positive (hence invertible) operators. A *scattering passive system node* with respect to the *storage operator* $P > 0$ and the *supply operator* $R > 0$ is a system node with state space X, input space U, and output space U described by the equations

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A \;\; B \\ \overline{C \;\; D} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{1.1}$$

such that, along classical solutions of the above equations,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Px(t), x(t) \rangle \leq \langle Ru(t), u(t) \rangle - \langle Ry(t), y(t) \rangle . \tag{1.2}$$

This implies, in particular, that our system node determines a well-posed linear system, called a *scattering passive linear system* (with respect to the same weighting operators). For the context, related concepts, and results see section 2.

At the end of section 2 we give slight generalizations of two propositions from Tucsnak and Weiss [36] concerning the relation between exponential stability, strong stability, and various controllability and observability properties of conservative linear systems. These results can be applied to the system corresponding to the Maxwell equations when the conductivity of the material in the domain Ω is zero.

In section 3 we review some results from [35], where we have introduced a special class of scattering passive linear systems and studied their properties. This section also contains a generalization of the main results from [35] to the case when weighting operators are used in the state and input/output spaces, as in (1.2). This generalization is needed for a rigorous treatent of Maxwell's equations. The proof of the new theorem uses the results of [35] and a new result from section 2.

Section 4 contains a collection of results about trace operators (mainly the tangential component and the normal trace operator) associated to a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. The *tangential component trace operator* π_τ associates to any continuous function $\mathbf{E} : \overline{\Omega} \to \mathbb{R}^3$ the tangential component (the projection onto the tangent plane) of its Dirichlet trace on the boundary Γ. Thus, $\pi_\tau \mathbf{E}$ is defined almost everywhere on Γ and $\pi_\tau \mathbf{E} \in L^\infty(\Gamma; \mathbb{R}^3)$. This operator is important for the study of Maxwell's equations. In section 4 we are mainly interested in properties of extensions of trace operators to large spaces of distributions, and density results for such spaces. Most of the results in this section are known, but some might be new.

In section 5, the main part of this paper, we consider Maxwell's equations on a bounded three-dimensional domain Ω with Lipschitz boundary Γ:

$$\frac{\partial \mathbf{D}}{\partial t} = \mathrm{rot}\, \mathbf{H} - J , \qquad \frac{\partial \mathbf{B}}{\partial t} = -\,\mathrm{rot}\, \mathbf{E},$$

$$\mathrm{div}\, \mathbf{D} = \rho , \qquad \mathrm{div}\, \mathbf{B} = 0 ,$$

$$J = g\mathbf{E}, \qquad \mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H},$$

where the positive scalar functions ε, μ, and g are such that

$$\varepsilon, \frac{1}{\varepsilon}, \mu, \frac{1}{\mu}, g \in L^\infty(\Omega) , \quad g \geq 0 .$$

For more explanations and background about these equations see section 5.

We allow an open part Γ_0 of the boundary to be a superconductor, which means that on Γ_0 the tangential component trace of the electric field $\pi_\tau \mathbf{E}$ is forced to be zero. We denote by Γ_1 another relatively open part of Γ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$, while $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \Gamma$. We assume that the surface measures of $\partial\Gamma_0$ and $\partial\Gamma_2$ are zero. The input and output space U consists of all the tangential vector fields of class L^2 on Γ_1. We assume that the system interacts with the external world via boundary control and boundary observation, described by

$$u = \frac{1}{\sqrt{2}}\Big(r(\nu \times \mathbf{H}) + \pi_\tau \mathbf{E}\Big), \qquad y = \frac{1}{\sqrt{2}}\Big(r(\nu \times \mathbf{H}) - \pi_\tau \mathbf{E}\Big), \tag{1.3}$$

both on Γ_1, where u is the input function and y is the output function, both with values in U. We have denoted by r a positive scalar function such that $r, \frac{1}{r} \in L^\infty(\Gamma_1)$, but otherwise r can be chosen arbitrarily. The physical dimension of r is the same as that of a resistance. We mention that using $\pi_\tau \mathbf{E}$ restricted to Γ_1 as the input and the tangential vector field $\nu \times \mathbf{H}$ restricted to Γ_1 as the output or the other way around, the system is not well-posed, but it is impedance passive in a certain generalized sense (as defined in [35, Theorem 5.2]).

Relying on the abstract framework developed in [35], we define in section 5 two scattering passive linear systems (with respect to certain weighting operators P and R and two different choices of E_0) that correspond to Maxwell's equations with the above input and output and we investigate their properties. Here we give a brief outline of this construction.

The state of the system corresponding to Maxwell's equations is $\left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right]$, and the state space of the system is $X = E \oplus E$, where $E = L^2(\Omega; \mathbb{R}^3)$. The semigroup generator has the structure

$$A = \begin{bmatrix} 0 & -L \\ L^* & G - \gamma^* R \gamma \end{bmatrix} P,$$

where L, γ, G, R, and P are defined in the following way.

The operator L is given by $L = \operatorname{rot}$ with domain $E_0 = \mathcal{D}(L)$ defined as follows: a function $\mathbf{E} \in E$ belongs to E_0 if $\operatorname{rot}\mathbf{E} \in L^2(\Omega; \mathbb{R}^3)$, its tangential component trace on Γ_0 is zero, and its tangential component trace on Γ_1 is in U. The space E_0 is a Hilbert space with the norm

$$\|\mathbf{E}\|_{E_0}^2 = \|\mathbf{E}\|_{L^2}^2 + \|\operatorname{rot}\mathbf{E}\|_{L^2}^2 + \|\pi_\tau \mathbf{E}\|_{L^2}^2 .$$

It is clear that we have $L \in \mathcal{L}(E_0, E)$, so that $L^* \in \mathcal{L}(E, E_0')$, where E_0' is the dual of E_0 with respect to the pivot space E. The operator L^* is not easy to understand intuitively, but if $\varphi \in \mathcal{D}(\Omega; \mathbb{R}^3)$ (a test function on Ω), then $L^*\varphi = \operatorname{rot}\varphi$.

The operator $\gamma \in \mathcal{L}(E_0, U)$ is defined as the tangential component trace operator π_τ followed by the operator of restriction to Γ_1: $\gamma\mathbf{E} = (\pi_\tau \mathbf{E})|_{\Gamma_1}$. The operators

$$P = \begin{bmatrix} \mu^{-1} & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}, \qquad R = \begin{bmatrix} r^{-1} \end{bmatrix}, \qquad G = [-g]$$

are pointwise multiplication operators on X, U, and E, respectively. We have

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \in P^{-1}\left[E \times E_0\right] \;\middle|\; L^*\mu^{-1}\mathbf{B} - \gamma^* R \gamma \varepsilon^{-1} \mathbf{D} \in E \right\}.$$

Note that $\left[\begin{smallmatrix}\mathbf{H}\\ \mathbf{E}\end{smallmatrix}\right] = P\left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right]$. The observation operator is

$$C = \sqrt{2}\,\begin{bmatrix}0 & -\gamma\end{bmatrix} P$$

with the domain $\mathcal{D}(A)$. This has a natural extension to $\overline{C} : P^{-1}[E \times E_0] \to Y$, given by the same formula as C, so that $\overline{C}\left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right] = -\sqrt{2}\,(\pi_\tau \mathbf{E})|_{\Gamma_1}$.

The control operator B of the system is $B = \sqrt{2}\,\left[\begin{smallmatrix}0\\ \gamma^* R\end{smallmatrix}\right]$. The state trajectories of the system are the solutions x of the state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \forall\, t \geq 0 \tag{1.4}$$

(the first part of (1.1)), where $x(t) = \left[\begin{smallmatrix}\mathbf{B}(t)\\ \mathbf{D}(t)\end{smallmatrix}\right]$ and u is the input function. The operator D from (1.1) is now the identity, so that the output equation is

$$y(t) = \overline{C}x(t) + u(t) = -\sqrt{2}\,\gamma\mathbf{E}(t) + u(t) \qquad \forall\, t \geq 0, \tag{1.5}$$

where y is the output function.

We show that classical solutions of the system equation $\dot{x} = Ax + Bu$ and the corresponding output function $y = \overline{C}x + u$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Px(t), x(t)\rangle = \langle Ru(t), u(t)\rangle - \langle Ry(t), y(t)\rangle + 2\langle G\mathbf{E}(t), \mathbf{E}(t)\rangle \tag{1.6}$$

and that the dual system satisfies a similar power balance equation (where R and P are replaced with their inverses). This power balance equation (1.6) corresponds exactly to what we would expect from physics. Indeed, the left-hand side is twice the derivative of the energy stored in the electromagnetic field, because

$$\langle Px(t), x(t)\rangle = \langle \mathbf{B}, \mathbf{H}\rangle_{L^2(\Omega)} + \langle \mathbf{D}, \mathbf{E}\rangle_{L^2(\Omega)}.$$

The first two terms on the right-hand side of (1.6) can be transformed into

$$\begin{aligned}\langle Ru(t), u(t)\rangle_U - \langle Ry(t), y(t)\rangle_U &= \langle R[u(t) + y(t)], u(t) - y(t)\rangle_U \\ &= 2\langle \nu \times \mathbf{H}, \pi_\tau \mathbf{E}\rangle_U = -2\langle \mathbf{E} \times \mathbf{H}, \nu\rangle_{L^2(\Gamma_1)},\end{aligned}$$

which can be recognized as being minus twice the (outward) flux of the Poynting vector field $\mathbf{E} \times \mathbf{H}$ through Γ_1. Since the Poynting vector field has zero flux through Γ_0, it follows that the above expression is (according to physics) twice the power flow coming into Ω through the boundary Γ. Finally, since g is the conductivity of the material in Ω, we recognize the term $2\langle G\mathbf{E}, \mathbf{E}\rangle = -2\langle J, E\rangle \leq 0$ appearing in (1.6) as being twice the power loss in Ω due to heating.

Notice that (1.6) implies that the system node corresponding to the Maxwell equations is scattering passive with respect to the storage operator P and the supply operator R. We will see that this system is scattering conservative (with respect to the same weighting operators) iff $g = 0$. (The concept of scattering conservative system node will be introduced in section 2.)

In section 6 we show that classical solutions of the state equation (1.4) and the output equation (1.5) are solutions of Maxwell's equations, except that $\operatorname{div}\mathbf{B}$ is not constrained to be zero, only to be constant in time. If the initial state of the system satisfies $\operatorname{div}\mathbf{B} = 0$ (as would be the case in any physically meaningful scenario), then this condition is preserved for all $t \geq 0$.

In section 6 we also indicate extensions of the results from section 5 and we make comments about controllability, observability, and stability. In Remark 6.5 we indicate how the results of this paper can be used to prove the well-posedness of various coupled systems, where the Maxwell system interacts with a mechanical system (a moving rigid body in the field that may be an electric conductor or made of ferromagnetic material), citing a recent paper on this topic. Finally, we give a detailed example representing the electromagnetic field in a coaxial cable, and we show how this model can be reduced, under additional symmetry conditions, to a one-dimensional wave equation.

We end this introduction with two remarks on the relevant literature.

Remark 1.1. Taking the input u to be zero results in the boundary condition $\pi_\tau \mathbf{E}(t) = -r(\nu \times \mathbf{H}(t))$ which is known as the *impedance* or *Leontovich boundary condition* after Leontovich [19]. It is often used in numerical computations to cut down the domain of computation from a larger (possibly infinite) domain to the numerically most important part of the domain. One typical example is the computation of the propagation of radio waves in the earth's athmosphere. In this application an impedance boundary condition is imposed on the lower boundary to model the influence of the surface conductivity of the earth, and another impedance boundary condition is imposed on the upper boundary to approximate a *perfectly absorbing* (or *transparent*) boundary condition. The perfectly absorbing boundary condition (which prevents all reflections from this part of the boundary) is more difficult to use since it is nonlocal, but one gets a first order approximation of such a condition by taking r to be equal to the characteristic impedance of free space. In this case the same boundary condition is also known as the *Silver–Müller boundary condition*. For more details see Eller, Lagnese, and Nicaise [9], Lafitte [16], or Levy [20]. The paper by Slodicka and Durand [29] considers well-posedness and numerical methods for nonlinear generalizations of the Silver–Müller boundary condition.

Remark 1.2. There are several papers that treat Maxwell's equations as an infinite-dimensional system, although usually inputs are defined differently from here (and outputs are often not defined). For instance, the paper by Phung [25] considers Maxwell's equations in a bounded smooth domain with constant μ, ε without charges or currents. One of the results is that taking $\nu \times \mathbf{H}$ on a suitably large part of the boundary as the input, the system is exactly controllable in some finite time. This is proved using microlocal analysis and the HUM method. The paper also contains results on exact controllability from a subdomain, boundary stabilization, and stabilization from a subdomain. Earlier studies in this direction were contained in Lagnese [17] and Komornik [15]. The paper by Eller, Lagnese, and Nicaise [10] considers the Maxwell system on a smooth domain with smoothly varying μ, ε with a possibly mildly nonlinear boundary feedback (on the whole boundary). In the linear case this feedback corresponds to taking $u = 0$ in our equations. The authors analyze the stability and decay rate of the solutions in the natural energy norm, in particular proving exponential stability under certain geometric conditions. These results were generalized in Nicaise and Pignotti [23] to functions μ, ε that depend also on time. Eller, Lagnese, and Nicaise in [9] considered Maxwell's equations on a Lipschitz domain with $\Gamma_0 = \emptyset$ (no superconductor), no currents or charges, $\mu, \varepsilon \in L^\infty(\Omega)$ (positive), and (in the linear part of the paper) the Silver–Müller boundary conditions (see Remark 1.1). They derived various sufficient conditions for exponential stability and showed that exponential stability is equivalent to exact observability of the corresponding impedance-passive system (mentioned after (1.3)); see [9, Theorem 3.3].

They also showed that exponential stability implies exact controllability of another related impedance-passive system; see [9, Theorem 4.1].

2. Scattering passive system nodes. In the first half of this section we recall (for easy reference) some basic facts about system nodes. This information is given in greater detail in our previous paper [35], so that we want to be really brief here. In the second half of this section we recall the concept of scattering passive system node with weighting operators (and some subclasses of these), and in this part the last two propositions are new.

Let $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on the Hilbert space X with generator A. We define on X a new norm by

$$\|x\|_{-1} = \|(\beta I - A)^{-1}x\|, \tag{2.1}$$

where $\beta \in \rho(A)$. Different choices of β lead to equivalent norms. The space X_1 is $\mathcal{D}(A)$ with the norm $\|x\|_1 = \|(\beta I - A)x\|$, while X_{-1} is the completion of X with respect to the norm $\|\cdot\|_{-1}$. We can extend A to an operator in $\mathcal{L}(X, X_{-1})$ and this generates a strongly continuous semigroup on X_{-1}, an extension of the original semigroup. We denote these extensions of A and of $\mathbb{T}_t$ by the same symbols. The space X_1^d is $\mathcal{D}(A^*)$ with the norm $\|z\|_1^d = \|(\overline{\beta}I - A^*)z\|$. Then X_{-1} is the dual of X_1^d with respect to the pivot space X (see [37, section 2.10] for details).

Definition 2.1. *Let U, X, and Y be Hilbert spaces. An operator*

$$S : \mathcal{D}(S) \to X \oplus Y \qquad \text{with } \mathcal{D}(S) \subset X \oplus U$$

is called a system node *on (U, X, Y) if it has the following properties:*

1. *S is closed (as an operator from $X \oplus U$ to $X \oplus Y$).*
2. *We partition $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. The operator $A : \mathcal{D}(A) \to X$ defined by*

$$Ax = A\&B \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \qquad \mathcal{D}(A) = \{x \in X \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{D}(S)\} \tag{2.2}$$

   is the generator of a strongly continuous semigroup on X.
3. *The operator $A\&B$ (with $\mathcal{D}(A\&B) = \mathcal{D}(S)$) can be extended to an operator $\begin{bmatrix} A & B \end{bmatrix} \in \mathcal{L}(X \oplus U, X_{-1})$, where A is the extension of the earlier A as discussed at the beginning of this section.*
4. $\mathcal{D}(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X \oplus U \;\middle|\; Ax + Bu \in X \right\}$.

It is easy to see that if S is a system node on (U, X, Y), then $\mathcal{D}(S)$ is dense in $X \oplus U$ and $A\&B$ is closed (with domain $\mathcal{D}(S)$). Hence, the graph norm on $\mathcal{D}(S)$ is equivalent to the graph norm of the operator $A\&B$ on the same domain, defined by

$$\|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\|_{\mathcal{D}(S)}^2 = \|x\|^2 + \|u\|^2 + \|Ax + Bu\|^2 . \tag{2.3}$$

The operator A is called the *semigroup generator* of S and B is called the *control operator* of S. The operator $C \in \mathcal{L}(X_1, Y)$ defined by

$$Cx = C\&D \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \qquad \forall\, x \in \mathcal{D}(A) \tag{2.4}$$

is called the *observation operator* of S. The *transfer function* of S is the $\mathcal{L}(U, Y)$-valued analytic function defined by

$$\mathbf{G}(s) = C\&D \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} \qquad \forall\, s \in \rho(A). \tag{2.5}$$

A system node S is usually associated with the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geq 0 \tag{2.6}$$

or, equivalently,

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = C\&D \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geq 0, \tag{2.7}$$

where A, B, and $C\&D$ are as in Definition 2.1.

DEFINITION 2.2. *Let U, X, and Y be Hilbert spaces. Let S be a closed linear operator from $X \oplus U$ to $X \oplus Y$ with domain $\mathcal{D}(S)$ (but S need not be a system node).*

A triple (x, u, y) is called a classical solution *of* (2.6) *on* $[0, \infty)$ *if*

(a) $x \in C^1([0, \infty); X)$,

(b) $u \in C([0, \infty); U)$, $y \in C([0, \infty); Y)$,

(c) $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{D}(S)$ *for all* $t \geq 0$,

(d) (2.6) *holds.*

The following proposition guarantees that for a system node, we have plenty of classical solutions of the system equation (2.6).

PROPOSITION 2.3. *Let S be a system node on (U, X, Y). If $u \in C^2([0, \infty); U)$ and $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$, then* (2.6) *has a unique classical solution (x, u, y) satisfying $x(0) = x_0$.*

For the proof we refer to Lemma 4.7.8 in [30] or Proposition 4.2.11 in [37]. (Various versions of (parts of) this proposition can be found in the literature.)

For any $\tau \geq 0$ and $u \in L^2([0, \infty); U)$, let us denote by $\mathbf{P}_\tau u$ the restriction of u to $[0, \tau]$. Let us denote by $\mathcal{D}$ the space of all the pairs $(x_0, u) \in X \oplus L^2([0, \infty); U)$ which satisfy the assumptions of Proposition 2.3. Notice that $\mathcal{D}$ is dense in $X \oplus L^2([0, \infty); U)$. Hence, the corresponding space $\mathcal{D}_\tau$ of pairs $(x_0, \mathbf{P}_\tau u)$ is dense in $X \oplus L^2([0, \tau]; U)$. The last proposition allows us to define the operators Σ_τ from $\mathcal{D}_\tau$ to $X \oplus L^2([0, \tau], Y)$ such that for any solution of (2.6) and for any $\tau \geq 0$,

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x_0 \\ \mathbf{P}_\tau u \end{bmatrix}. \tag{2.8}$$

DEFINITION 2.4. *The system node S is called* well-posed *if for some (hence, for every) $\tau > 0$, the operator Σ_τ from* (2.8) *has a continuous extension*

$$\Sigma_\tau \in \mathcal{L}(X \oplus L^2([0, \tau], U), X \oplus L^2([0, \tau], Y)).$$

In this case, the family $(\Sigma_\tau)_{\tau \geq 0}$ is called a well-posed linear system.

For such systems we refer to the monograph [30] and the references therein. Here we only mention a few facts about well-posed systems that will be needed later. We use the notation of Definition 2.1 and (2.5). Define the space

$$Z = D(A) + (\beta I - A)^{-1} BU, \tag{2.9}$$

which is a Hilbert space with the norm

$$\|z\|_Z^2 = \inf \left\{ \|x\|_1^2 + \|v\|^2 \mid x \in X_1,\, v \in U,\, z = x + (\beta I - A)^{-1} Bv \right\}. \tag{2.10}$$

If S is a well-posed system node, then C has (at least one) continuous extension $\overline{C} \in \mathcal{L}(Z, Y)$ (see [33, section 3]). Since X_1 need not be dense in Z, there may be many such extensions. For each such $\overline{C}$, we define $D \in \mathcal{L}(U,Y)$ by $D = \mathbf{G}(\beta) - \overline{C}(\beta I - A)^{-1}B$ and then D is independent of $\beta \in \rho(A)$. Then $C\&D$ and S can be split to take their form which is familiar from finite-dimensional systems theory:

$$C\&D \begin{bmatrix} x \\ v \end{bmatrix} = \overline{C}x + Dv\,, \qquad S = \begin{bmatrix} A & B \\ \overline{C} & D \end{bmatrix}, \tag{2.11}$$

and we have

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B + D \qquad \forall\, s \in \rho(A)\,.$$

For well-posed system nodes we have a stronger version of Proposition 2.3.

PROPOSITION 2.5. *Let S be a well-posed system node on (U,X,Y). Assume that $u \in \mathcal{H}^1_{\text{loc}}(0,\infty;U)$ and $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \mathcal{D}(S)$. Then (2.6) has a unique classical solution (x,u,y) satisfying $x(0) = x_0$. Moreover, we have*

$$\begin{bmatrix} x \\ u \end{bmatrix} \in C([0,\infty);\mathcal{D}(S))\,, \qquad y \in \mathcal{H}^1_{\text{loc}}(0,\infty;Y)\,.$$

Here, $\mathcal{H}^1_{\text{loc}}(0,\infty;U)$ denotes the space of functions $u : (0,\infty) \to U$ such that their restrictions to intervals of the form $(0,\tau)$, with $\tau > 0$, are in $\mathcal{H}^1((0,\tau);U)$. For the proof see Theorem 4.6.11 in [30] or Theorem 3.1 in [33].

DEFINITION 2.6. *Let U, X, and Y be Hilbert spaces and let R, P, and J be strictly positive (hence boundedly invertible) operators in $\mathcal{L}(U), \mathcal{L}(X)$, and $\mathcal{L}(Y)$, respectively. Let S be a system node on (U,X,Y). S is called* (R,P,J)-scattering passive *if for all $(x_0,u_0) \in \mathcal{D}(S)$, denoting $\dot{x}_0 = Ax_0 + Bu_0$ and $y_0 = C\&D \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$, we have*

$$2\text{Re}\,\langle Px_0, \dot{x}_0\rangle \le \langle Ru_0, u_0\rangle - \langle Jy_0, y_0\rangle\,. \tag{2.12}$$

Notice that if S is (R,P,J)-scattering passive and if (u,x,y) is a classical solution of (2.6) on $[0,\infty)$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Px(t), x(t)\rangle \le \langle Ru(t), u(t)\rangle - \langle Jy(t), y(t)\rangle \qquad \forall\, t \ge 0\,,$$

and in fact this is an equivalent formulation of the last definition.

Another equivalent condition to (2.12) is that the operators Σ_τ from (2.8) are contractions with respect to the appropriate weighted norms: if $(x_0,u) \in \mathcal{D}_\tau$ (this space was defined before (2.8)) and if $\begin{bmatrix} x(\tau) \\ y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x_0 \\ u \end{bmatrix}$, then

$$\langle Px(\tau), x(\tau)\rangle + \int_0^\tau \langle Jy(t), y(t)\rangle\,\mathrm{d}t \le \langle Px_0, x_0\rangle + \int_0^\tau \langle Ru(t), u(t)\rangle\,\mathrm{d}t\,.$$

In particular, it is clear that (R,P,J)*-scattering passive system nodes are well-posed*, and their semigroup is similar to a contraction semigroup (via the transformation $P^{\frac{1}{2}}$). If S is an (R,P,J)-scattering passive system node, then the corresponding family $(\Sigma_\tau)_{\tau\ge 0}$ is called an (R,P,J)*-scattering passive linear system*.

More sophisticated equivalent conditions for a system node to be (I,I,I)-scattering passive can be found in [33, Theorem 7.4] and [35, Theorem 4.6].

DEFINITION 2.7. *The system node S is called* (R,P,J)-scattering energy preserving *if we always have equality in* (2.12). *The corresponding family $(\Sigma_\tau)_{\tau\ge 0}$ is then called an* (R,P,J)-scattering energy preserving linear system.

The *dual* of a system node S on (U,X,Y) is simply its adjoint S^*. It can be verified that S^* is a system node on (Y,X,U). The semigroup generator A^d, the control operators B^d, the observation operator C^d, and transfer functions $\mathbf{G}^d$ of the dual system node S^* are related to the corresponding operators for S as follows:

$$A^d = A^*, \qquad B^d = C^*, \qquad C^d = B^*, \qquad \mathbf{G}^d(s) = \mathbf{G}(\overline{s})^*. \tag{2.13}$$

Definition 2.8. *With R, P, and J as in Definition* 2.6, *the system node S is called* (R, P, J)-scattering conservative *if S is (R, P, J)-scattering energy preserving and S^* is (J^{-1}, P^{-1}, R^{-1})-scattering energy preserving. The corresponding family $(\Sigma_\tau)_{\tau\geq 0}$ is then called an* (R, P, J)-scattering conservative linear system.

In the last three definitions, if it happens that $R = I$, $P = I$, and $J = I$, then we omit the prefix (R, P, J). For example, a *scattering passive linear system* means an (I, I, I)-scattering passive linear system. The terminology in the last two definitions follows Malinen, Staffans, and Weiss [22], which contains further comments and references.

We introduce another item of terminology. In section 1 we introduced scattering passive system nodes with respect to a storage operator P and a supply operator R. Such system nodes constitute a subclass of those defined in Definition 2.6, corresponding to $U = Y$ and $R = J$. (In fact, only this subclass is needed in this paper.) Similarly, a *scattering conservative system node* with repect to the storage operator P and the supply operator R (strictly positive operators on X and U, respectively) is a system node on (U, X, U) that is (R, P, R)-conservative.

Proposition 2.9. *Let S be a scattering passive (or energy preserving, or conservative) system node on (U, X, Y). Then S^* is a scattering passive (or energy preserving, or conservative) system node on (Y, X, U).*

Proof. It is easy to see that S is scattering passive (or energy preserving, or conservative) iff the operators Σ_τ that correspond to S^* (as in (2.8)) are contractive (or isometric, or unitary). For every $\tau > 0$, we denote by $\mathbf{Я}_\tau$ the time-reflection operator on the interval $[0, \tau]$: $(\mathbf{Я}_\tau v)(t) = v(\tau - t)$ for all $t \in [0, \tau]$. It follows from Theorems 3.4 and 3.5 in [34] that the operators Σ^d_τ that correspond to the dual system node S^* (as in the dual version of (2.8)) are given by

$$\Sigma^d_\tau = \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix} \Sigma^*_\tau \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix}.$$

Since $\mathbf{Я}_\tau$ is unitary, we see that Σ_τ being contractive (or isometric, or unitary) is equivalent to Σ^d_τ being contractive (or isometric, or unitary). ∎

Proposition 2.10. *Let S be a scattering passive (or energy preserving, or conservative) system node on (U, X, Y), and let $T \in \mathcal{L}(X)$ be invertible. Then the operator S_T defined by*

$$S_T = \begin{bmatrix} T^* & 0 \\ 0 & I \end{bmatrix} S \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}, \qquad \mathcal{D}(S_T) = \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \mathcal{D}(S)$$

is also a scattering passive (or energy preserving, or conservative) system node.

Proof. We decompose S_T using the decomposition (2.11) of S:

$$S_T = \begin{bmatrix} A_T & T^*B \\ \overline{C}T & D \end{bmatrix}, \qquad A_T = T^*AT, \qquad \mathcal{D}(A_T) = T^{-1}\mathcal{D}(A).$$

First we show that A_T is m-dissipative. It is clear that A_T is densely defined and dissipative. If A_T had a proper dissipative extension, then A would have one too, but this is not possible, since A is m-dissipative. Thus, A_T is m-dissipative.

The next step is to show that S_T is a system node. The first condition from Definition 2.1 (closedness) is an easy consequence of the formula defining S_T. The second condition (generation) follows from the fact that A_T is m-dissipative. For the third condition (the extension of $[A_T\ T^*B]$) we introduce $A_T^* = T^*A^*T$ with $\mathcal{D}(A_T^*) = T^{-1}\mathcal{D}(A^*)$. Recall the definition of X_1^d from the beginning of this section. Similarly we introduce the Hilbert space $X_{T,1}^d$ as being $\mathcal{D}(A_T^*)$ with the norm

$$\|z\|_{T,1}^d = \|(I - A_T^*)z\|$$

and then $X_{T,-1}$ is the dual of $X_{T,1}^d$ with respect to the pivot space X. Since $T \in \mathcal{L}(X_{T,1}^d, X_1^d)$, it follows that T^* can be extended such that $T^* \in \mathcal{L}(X_{-1}, X_{T,-1})$. Therefore we have an extension $[A_T\ T^*B] \in \mathcal{L}(X \oplus U, X_{T,-1})$, as required. Finally, the fourth condition in Definition 2.1 can be verified by inspection.

Now we show that the properties of being scattering passive (or energy preserving, of conservative) are inherited from S to S_T. Let $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S_T)$. Denote $\dot{x}_0 = A_T x_0 + T^*Bu_0$ and $y_0 = \overline{C}Tx_0 + Du_0$. Then

$$\begin{aligned} &2\mathrm{Re}\,\langle x_0, \dot{x}_0\rangle - \|u_0\|^2 + \|y_0\|^2 \\ &\quad = 2\mathrm{Re}\,\langle x_0, T^*(ATx_0 + Bu_0)\rangle - \|u_0\|^2 + \|\overline{C}Tx_0 + Du_0\|^2 \\ &\quad = 2\mathrm{Re}\,\langle z_0, Az_0 + Bu_0\rangle - \|u_0\|^2 + \|\overline{C}z_0 + Du_0\|^2, \end{aligned}$$

where $z_0 = Tx_0$, so that $\left[\begin{smallmatrix} z_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$. From the above equality we see that if S is scattering passive (or energy preserving), then S_T has the same property. Finally, if S is scattering conservative, then both S and S^* are scattering energy preserving, so that in particular S_T is scattering energy preserving. By the earlier argument applied to S_T^* (instead of S_T) we obtain that S_T^* is also scattering energy preserving, and hence S_T is scattering conservative. $\square$

The following proposition is essentially a very particular case of the main result of [27]. It tells us how we can transform a scattering passive (or scattering conservative) system node into an (R, P, J)-scattering passive (or scattering conservative) system node. We need this to prove the main result of section 3.

Proposition 2.11. *Let R, P, and J be strictly positive bounded operators on the Hilbert spaces U, X, and Y, respectively, and let S be a linear operator from $\mathcal{D}(S) \subset X \oplus U$ to $X \oplus Y$. Define the operator S_w by*

$$S_w = \begin{bmatrix} I & 0 \\ 0 & J^{-\frac{1}{2}} \end{bmatrix} S \begin{bmatrix} P & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix}, \qquad \mathcal{D}(S_w) = \begin{bmatrix} P^{-1} & 0 \\ 0 & R^{-\frac{1}{2}} \end{bmatrix} \mathcal{D}(S). \tag{2.14}$$

Then S is a scattering passive (or energy preserving, or conservative) system node on (U, X, Y) iff S_w is an (R, P, J)-scattering passive (or energy preserving, or conservative) system node.

Proof. Assume that S is a scattering passive system node. We partition S as in (2.11), and we partition S_w in a similar way, so that its components are

$$A_w = AP, \quad B_w = BR^{\frac{1}{2}}, \quad \overline{C}_w = J^{-\frac{1}{2}}\overline{C}P, \quad D_w = J^{-\frac{1}{2}}DR^{\frac{1}{2}},$$

with $\mathcal{D}(A_w) = P^{-1}\mathcal{D}(A)$. First we prove that S_w is a system node.

For this, the first step is to check that S_w is closed. This is an easy consequence of the closedness of S and the invertibility of R, P, and J.

The second step needed to show that S_w is a system node is to prove that A_w generates a strongly continuous semigroup on X. It is clear that A_w is closed. Introduce on X the inner product $\langle x, z\rangle_P = \langle Px, z\rangle$. We have

$$\langle A_w z, z\rangle_P = \langle APz, Pz\rangle \qquad \forall\, z \in P^{-1}\mathcal{D}(A).$$

Since A is dissipative, it follows that A_w is dissipative on X with respect to the new inner product. Hence (see [37, Lemma 3.1.4]), $I - A_w$ has closed range for the equivalent norm on X induced by this inner product. As a result, $I - A_w$ has closed range in X. It is easy to see that $A_w^* = PA^*$ with $\mathcal{D}(A_w^*) = \mathcal{D}(A^*)$. Since A^* is dissipative, it follows that A_w^* is dissipative with respect to the inner product $\langle x, z\rangle_{P^{-1}} = \langle P^{-1}x, z\rangle$. Therefore $I - A_w^*$ is injective, so that $I - A_w$ has dense range. Together with what we proved earlier, this implies that $\mathrm{Ran}\,(I - A_w) = X$. As a result, A_w is maximally dissipative with respect to $\langle\cdot,\cdot\rangle_P$, so that it generates a contraction semigroup on X with respect to this inner product.

Notice that the space X_{-1} for A_w is the same as for A. Indeed, as we mentioned at the beginning of this section, this is the dual of $\mathcal{D}(A_w^*) = \mathcal{D}(A^*)$ with respect to the pivot space X. The third step needed to show that S_w is a system node is to prove that the operator $[A_w\ B_w]$ (defined on $\mathcal{D}(S_w)$) can be extended to a bounded operator from $X \oplus U$ to X_{-1}. But this is clear from $[A_w\ B_w] = [AP\ \ BR^{\frac{1}{2}}]$, since $A \in \mathcal{L}(X, X_{-1})$ and $B \in \mathcal{L}(U, X_{-1})$.

The fourth and last step needed to show that S_w is a system node is to prove that $\mathcal{D}(S_w) = \left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X \oplus U \;\middle|\; A_w x + B_w u \in X\right\}$. According to item 4 in Definition 2.1, the condition $A_w x + B_w u \in X$ is equivalent to $\left[\begin{smallmatrix} P & 0 \\ 0 & R^{\frac{1}{2}} \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}(S)$, and from here we see that it is equivalent (by definition) to $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}(S_w)$.

Now we show that S_w is (R, P, J)-scattering passive. Let $\left[\begin{smallmatrix} x_w \\ u_w \end{smallmatrix}\right] \in \mathcal{D}(S_w)$ and denote $\dot{x}_w = A_w x_w + B_w u_w$, $y_w = \overline{C}_w x_w + D_w u_w$. According to Definition 2.6 we have to show that

$$2\mathrm{Re}\,\langle Px_w, \dot{x}_w\rangle \le \langle Ru_w, u_w\rangle - \langle Jy_w, y_w\rangle\,.$$

We define

$$x = Px_w\,, \quad u = R^{\frac{1}{2}}u_w\,, \quad y = J^{\frac{1}{2}}y_w\,;$$

then it is easy to see that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}(S)$ and

$$\dot{x}_w = Ax + Bu\,, \qquad y = \overline{C}x + Du\,.$$

We have, by using the scattering passivity of S,

$$\begin{aligned}
&2\mathrm{Re}\,\langle Px_w, \dot{x}_w\rangle - \langle Ru_w, u_w\rangle + \langle Jy_w, y_w\rangle \\
&\quad = 2\mathrm{Re}\,\langle Px_w, Ax + Bu\rangle - \langle R^{\frac{1}{2}}u_w, R^{\frac{1}{2}}u_w\rangle + \langle J^{\frac{1}{2}}y_w, J^{\frac{1}{2}}y_w\rangle \\
&\quad = 2\mathrm{Re}\,\langle x, Ax + Bu\rangle - \langle u, u\rangle + \langle y, y\rangle \le 0\,.
\end{aligned}$$

Thus we have shown that S_w is (R, P, J)-scattering passive. The converse direction (showing that S is scattering passive) is very similar, with the roles of the systems reversed, and we omit the details. To show the equivalence of S being scattering energy preserving with S_w being (R, P, J)-scattering energy preserving, we repeat the same computations, but with the inequality sign replaced by equality.

Now we show that if S is scattering conservative, then S_w is (R, P, J)-scattering conservative. According to what we have shown earlier, S_w is (R, P, J)-scattering energy preserving. The dual system of S_w is

$$S_w^* = \begin{bmatrix} P & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} S^* \begin{bmatrix} I & 0 \\ 0 & J^{-\frac{1}{2}} \end{bmatrix}, \qquad \mathcal{D}(S_w^*) = \begin{bmatrix} I & 0 \\ 0 & J^{\frac{1}{2}} \end{bmatrix} \mathcal{D}(S^*).$$

According to Proposition 2.9, the system node S^* is scattering conservative. From Proposition 2.10 we know that the operator

$$S_P^* = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} S^* \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}$$

is again a scattering conservative system node; in particular, it is scattering energy preserving. If we apply to S_P^* the transformation from (2.14) but with (R, P, J) replaced with (J^{-1}, P^{-1}, R^{-1}), then we obtain that S_w^* is (J^{-1}, P^{-1}, R^{-1})-scattering energy preserving. (Again we have used what we have shown earlier in this proof.) Therefore S_w is (R, P, J)-scattering conservative. □

PROPOSITION 2.12. *Let U, X, Y and R, P, J be as in the previous proposition. Let S be an (R, P, J)-scattering conservative system node on (U, X, Y) with semigroup $\mathbb{T}$, semigroup generator A, control operator B, and observation operator C. Then for each $\tau > 0$, the following statements are equivalent:*

(1) *The pair (A, B) is exactly controllable in time τ.*

(2) *The pair (A, C) is exactly observable in time τ.*

(3) $\|P^{\frac{1}{2}}\,\mathbb{T}_\tau P^{-\frac{1}{2}}\| < 1$ *(hence, $\mathbb{T}$ is exponentially stable).*

For $R = I$, $P = I$ and $J = I$, this is Proposition 3.2 in [36]. For more general R, P, and J it follows from the cited result by suitably renorming the spaces U, X, and Y. For the concepts of exact controllability and exact observability in time τ we refer to the literature, for instance, to [36] or [37].

PROPOSITION 2.13. *With the notation of Proposition 2.12, assume that the intersection $\sigma(A) \cap i\mathbb{R}$ is countable (this happens, for example, if $(\beta I - A)^{-1}$ is compact for some $\beta \in \rho(A)$). Then the following seven assertions are equivalent:*

(1) *$\mathbb{T}$ is strongly stable.*

(2) *The pair (A, C) is exactly observable in infinite time.*

(3) *The pair (A, C) is approximately observable in infinite time.*

(4) *$\mathbb{T}$ is weakly stable (equivalently, $\mathbb{T}^*$ is weakly stable).*

(5) *$\mathbb{T}^*$ is strongly stable.*

(6) *The pair (A, B) is exactly controllable in infinite time.*

(7) *The pair (A, B) is approximately controllable in infinite time.*

For $R = I$, $P = I$, and $J = I$ this is Proposition 3.4 in [36]. For more general R, P, and J the above proposition follows by renorming the spaces U, X, and Y. For the concepts appearing in the proposition we refer again to [36] or [37].

3. A special class of scattering passive systems.

In our recent article [35] we presented a class of passive linear system with a special structure observed in models of mathematical physics. If a certain operator is zero, then the systems in this class are conservative. It is often simpler to verify that a given system belongs to the above class and hence is passive (or even conservative), than to check the conditions for scattering passivity or conservativity known from the literature (such as Staffans and Weiss [33] or Malinen, Staffans, and Weiss [22]). The new class from [35] is an extension of another class of conservative systems ("from thin air") that was introduced earlier by Weiss and Tucsnak [38] and further studied by Tucsnak and Weiss [36] and Staffans [31]. We were led to introduce the class described in [35] by our failure to fit Maxwell's equations into the framework of [38]. The principal aim of this paper is to show that Maxwell's equations fit into a slightly extended version of the framework of [35].

In [35] we considered a linear system Σ whose state space X can be decomposed as $X = H \oplus E$, where H and E are Hilbert spaces. The Hilbert space U is both the input space and the output space of Σ. We identify H, E, and U with their duals H', E', and U'. The Hilbert space E_0 is a dense subspace of E and the embedding $E_0 \hookrightarrow E$ is continuous. We denote by E_0' the dual of E_0 with respect to the pivot space E so that

$$E_0 \subset E \subset E_0',$$

densely and with continuous embeddings. Such triples of Hilbert spaces are often encountered in the abstract treatment of partial differential equations. We denote $X_0 = H \oplus E_0$ so that $X_0' = H \oplus E_0'$. We decompose the state of Σ as follows:

$$x_0 = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, \qquad z_0 \in H, \quad w_0 \in E.$$

We assume that we have three bounded operators,

$$L \in \mathcal{L}(E_0, H), \quad K \in \mathcal{L}(E_0, U), \quad G \in \mathcal{L}(E_0, E_0'), \tag{3.1}$$

satisfying

$$\begin{bmatrix} L \\ K \end{bmatrix} : E \to H \oplus U \text{ (with domain } E_0\text{) is closed,} \tag{3.2}$$

$$\operatorname{Re} \langle G w_0, w_0 \rangle_{E_0', E_0} \le 0 \qquad \forall\, w_0 \in E_0, \tag{3.3}$$

and we define $\overline{A} \in \mathcal{L}(X_0, X_0')$, $B \in \mathcal{L}(U, X_0')$, and $\overline{C} \in \mathcal{L}(X_0, U)$ by

$$\overline{A} = \begin{bmatrix} 0 & -L \\ L^* & G - \frac{1}{2}K^*K \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ K^* \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 0 & -K \end{bmatrix}. \tag{3.4}$$

In [35] we studied the system given by the equations

$$\dot{x}(t) = \overline{A}x(t) + Bu(t), \qquad y(t) = \overline{C}x(t) + u(t), \tag{3.5}$$

where x is the state trajectory, u is the input function, and y is the output function. Note that the differential equation above is an equation in X_0'. We define the domain $\mathcal{D}(A)$ by

$$\mathcal{D}(A) = \{x_0 \in X_0 \mid \overline{A}x_0 \in X\} \tag{3.6}$$

and we denote by A and C the restrictions of $\overline{A}$ and $\overline{C}$ to $\mathcal{D}(A)$. More explicitly,

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in X_0 \;\middle|\; L^* z_0 + \big(G - \frac{1}{2}K^*K\big) w_0 \in E \right\}. \tag{3.7}$$

The following theorem is contained in Theorems 1.1, 1.3, and 1.4 of [35].

THEOREM 3.1. *The operators introduced above determine a scattering passive system node* $S = \left[\begin{smallmatrix} \overline{A} & B \\ C & I \end{smallmatrix}\right]$ *(with its natural domain). More explicitly,*

$$(3.8)\qquad \begin{aligned} S &= \begin{bmatrix} 0 & -L & 0 \\ L^* & (G - \frac{1}{2}K^*K) & K^* \\ 0 & -K & I \end{bmatrix}, \\ \mathcal{D}(S) &= \left\{ \begin{bmatrix} z_0 \\ w_0 \\ u_0 \end{bmatrix} \in H \times E_0 \times U \,\middle|\, L^* z_0 + \big(G - \frac{1}{2}K^*K\big) w_0 + K^* u_0 \in E \right\}. \end{aligned}$$

If $z_0 \in H$, $w_0 \in E_0$, *and* $u_0 \in U$ *satisfy* $\left[\begin{smallmatrix} z_0 \\ w_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$ *and if we denote*

$$\begin{bmatrix} \dot{z}_0 \\ \dot{w}_0 \\ y_0 \end{bmatrix} = S \begin{bmatrix} z_0 \\ w_0 \\ u_0 \end{bmatrix},$$

then

$$(3.9)\qquad 2\mathrm{Re}\,[\langle z_0, \dot{z}_0 \rangle + \langle w_0, \dot{w}_0 \rangle] = \|u_0\|^2 - \|y_0\|^2 + 2\mathrm{Re}\,\langle G w_0, w_0 \rangle.$$

The dual system node S^* *has the same structure* (3.8)*, but with* L, K*, and* G *replaced with* $-L, -K$*, and* G^**. Hence,* S^* *also satisfies* (3.9)*.*

Therefore, S *is scattering conservative iff*

$$(3.10)\qquad \mathrm{Re}\,\langle G w_0, w_0 \rangle_{E_0', E_0} = 0 \qquad \forall\, w_0 \in E_0.$$

The semigroup generator of S *is the operator* A *defined in* (3.4), (3.7)*. We denote by* X_1 *the space* $\mathcal{D}(A)$ *with the norm* $\|z\|_1 = \|(I - A)z\|$ *and we denote by* X_{-1} *the completion of* X *with respect to the norm* $\|z\|_{-1} = \|(I - A)^{-1} z\|$*. We have*

$$X_1 \subset X_0 \subset X \subset X_0' \subset X_{-1},$$

densely and with continuous embeddings. A *has a unique extension to an operator* $A \in \mathcal{L}(X, X_{-1})$*, whose restriction to* X_0 *is* $\overline{A}$ *from* (3.4)*.*

Above we have used that $I - A$ is boundedly invertible. This follows from the fact (explained in section 2) that (since S is scattering passive) A generates a contraction semigroup. The main result of this section, stated below, is a version of the previous theorem with weighting operators and extra details added.

THEOREM 3.2. *Let* H, E, U*, and* E_0 *be as at the beginning of this section, and let the operators* $L \in \mathcal{L}(E_0, H)$*,* $\gamma \in \mathcal{L}(E_0, U)$*, and* $G \in \mathcal{L}(E_0, E_0')$ *be such that*

$$(3.11)\qquad \begin{bmatrix} L \\ \gamma \end{bmatrix} : E \to H \oplus U \ \textit{(with domain } E_0\textit{) is closed,}$$

$$(3.12)\qquad \mathrm{Re}\,\langle G w_0, w_0 \rangle_{E_0', E_0} \le 0 \qquad \forall\, w_0 \in E_0.$$

Let P_H*,* P_E*, and* R *be strictly positive bounded operators on* H*,* E*, and* U*, respectively, and define* $P = \left[\begin{smallmatrix} P_H & 0 \\ 0 & P_E \end{smallmatrix}\right]$*. Define the operator* S_w *by*

$$(3.13)\qquad S_w = \begin{bmatrix} 0 & -LP_E & 0 \\ L^* P_H & (G - \gamma^* R \gamma) P_E & \sqrt{2}\,\gamma^* R \\ 0 & -\sqrt{2}\,\gamma P_E & I \end{bmatrix},$$

$$\mathcal{D}(S_w) = \left\{ \begin{bmatrix} z_0 \\ w_0 \\ u_0 \end{bmatrix} \in \begin{matrix} H \\ \times \\ P_E^{-1} E_0 \\ \times \\ U \end{matrix} \;\middle|\; \begin{matrix} L^* P_H z_0 + (G - \gamma^* R \gamma) P_E w_0 \\ + \sqrt{2}\, \gamma^* R u_0 \in E \end{matrix} \right\}. \tag{3.14}$$

Then S_w is a scattering passive system node with respect to the storage operator P and the supply operator R. S_w is scattering conservative with respect to the storage operator P and the supply operator R iff (3.10) *holds.*

Moreover, the following claims hold:

1. *If the input function u and the initial state $\left[\begin{smallmatrix} z(0) \\ w(0) \end{smallmatrix}\right]$ of S_w satisfy*

$$u \in \mathcal{H}^1_{\mathrm{loc}}(0, \infty; U), \qquad \begin{bmatrix} z(0) \\ w(0) \\ u(0) \end{bmatrix} \in \mathcal{D}(S_w), \tag{3.15}$$

then the corresponding state trajectory $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ and output function y of S_w satisfy

$$\begin{bmatrix} z \\ w \end{bmatrix} \in C^1([0,\infty); H \oplus E), \qquad \begin{bmatrix} z \\ w \\ u \end{bmatrix} \in C([0,\infty); \mathcal{D}(S_w)), \quad y \in \mathcal{H}^1_{\mathrm{loc}}(0, \infty; Y), \tag{3.16}$$

and

$$\begin{bmatrix} \dot z(t) \\ \dot w(t) \\ y(t) \end{bmatrix} = S_w \begin{bmatrix} z(t) \\ w(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geq 0. \tag{3.17}$$

2. *The semigroup generator A_w of S_w is the restriction of the operator*

$$\overline{A_w} = \begin{bmatrix} 0 & -L P_E \\ L^* P_H & (G - \gamma^* R \gamma) P_E \end{bmatrix} \tag{3.18}$$

(defined on $H \times P_E^{-1} E_0$ with values in $H \times E_0'$) to the domain

$$\mathcal{D}(A_w) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in H \times P_E^{-1} E_0 \;\middle|\; L^* P_H z_0 + \big(G - \gamma^* R \gamma\big) P_E w_0 \in E \right\}. \tag{3.19}$$

3. *We denote by X_{1w} the space $\mathcal{D}(A_w)$ with the norm $\|z\|_{1w} = \|(I - A_w) z\|$ and by X_{-1w} the completion of X with respect to the norm $\|z\|_{-1w} = \|(I - A_w)^{-1} z\|$. Then X_{-1w} is independent of P_H, P_E. We have*

$$X_{1w} \subset H \times P_E^{-1} E_0 \subset X \subset H \times E_0' \subset X_{-1w},$$

densely and with continuous embeddings. A_w has a unique extension to an operator $A_w \in \mathcal{L}(X, X_{-1w})$, whose restriction to $H \times P_E^{-1} E_0$ is $\overline{A_w}$ from (3.18).

4. *The control and observation operators of S_w are*

$$B_w = \begin{bmatrix} 0 \\ \sqrt{2}\, \gamma^* R \end{bmatrix}, \qquad C_w = \begin{bmatrix} 0 & -\sqrt{2}\, \gamma P_E \end{bmatrix} \qquad (\mathcal{D}(C_w) = \mathcal{D}(A_w)). \tag{3.20}$$

5. *If the functions $u, x = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ and y are as in* (3.15)*–*(3.17)*, then they satisfy the following power balance equation for every $t \geq 0$:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle P x(t), x(t) \rangle = \langle R u(t), u(t) \rangle - \langle R y(t), y(t) \rangle + 2 \mathrm{Re}\, \langle G P_E w(t), P_E w(t) \rangle. \tag{3.21}$$

Proof. Let S be the operator from (3.8), with $K = \sqrt{2}R^{\frac{1}{2}}\gamma$, so that and $K^* = \sqrt{2}\gamma^* R^{\frac{1}{2}}$. From (3.11) we see that (3.2) holds. According to Theorem 3.1 S is a scattering passive system node, and it is scattering conservative iff (3.10) holds.

Define S_w as in (2.14), with $J = R$, which in our case means that

$$S_w = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & R^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 0 & -L & 0 \\ L^* & (G - \gamma^* R\gamma) & \sqrt{2}\,\gamma^* R^{\frac{1}{2}} \\ 0 & -\sqrt{2}\,R^{\frac{1}{2}}\gamma & I \end{bmatrix} \begin{bmatrix} P_H & 0 & 0 \\ 0 & P_E & 0 \\ 0 & 0 & R^{\frac{1}{2}} \end{bmatrix} \tag{3.22}$$

with the natural domain, as in (2.14). A short computation shows that S_w is the operator from (3.13), (3.14). According to Proposition 2.11 the operator S_w is (R, P, R)-scattering passive, i.e., scattering passive with respect to the storage operator P and the supply operator R. Moreover, S_w is (R, P, R)-scattering conservative iff S is scattering conservative. We already know that the latter condition is equivalent to (3.10). Thus we have proved the main statement of the theorem.

Now we prove the first additional statement. The (R, P, R)-scattering passivity of S implies that it is well-posed, as explained in section 2. Statement 1 in the theorem is just a detailed restatement of Proposition 2.5 applied in our specific context.

The second additional statement follows from the definition of the semigroup generator of a system node (Definition 2.1 and the text after it).

We prove the third additional statement. Let A be the semigroup generator of S (S is the system node from (3.8) with $K = \sqrt{2}R^{\frac{1}{2}}\gamma$). We denote by X_1 the space $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(I - A)z\|$ and we denote by X_{-1} the completion of X with respect to the norm $\|z\|_{-1} = \|(I - A)^{-1}z\|$. Then $X_{-1w} = X_{-1}$ (with equivalent norms), as we have explained in the middle of the proof of Proposition 2.11. Since A does not depend on P_H, P_E, the same is true for X_{-1w}. We have $\mathcal{D}(A_w) = P^{-1}\mathcal{D}(A)$, as can be seen by comparing (3.19) with (3.7). We know from Theorem 3.1 that $X_1 \subset H \times E_0 \subset X$ holds densely and with continuous embeddings. Applying P^{-1} to these inclusions, we obtain that $X_{1w} \subset H \times P_E^{-1}E_0 \subset X$ holds densely and with continuous embeddings, as claimed. The remaining inclusions $X \subset H \times E_0' \subset X_{-1w}$ are the same as in Theorem 3.1, hence they are dense and with continuous embeddings.

Like any semigroup generator, A_w has a unique extension to an operator $A_w \in \mathcal{L}(X, X_{-1w})$. (This was recalled at the beginning of section 2.) Since $H \times P_E^{-1}E_0$ is continuously embedded in X, the restriction of A_w to $H \times P_E^{-1}E_0$ is continuous from $H \times P_E^{-1}E_0$ to X_{-1w}. Since $H \times E_0'$ is continuously embedded in X_{-1w}, the operator $\overline{A_w}$ may also be regarded as a bounded operator from $H \times P_E^{-1}E_0$ to X_{-1w}. Since the restrictions of these two operators to X_{1w} are equal and since X_{1w} is dense in $H \times P_E^{-1}E_0$, it follows that these two operators are in fact equal, as stated in the third additional statement of the theorem.

We prove the fourth additional statement. From the formula (3.13) we see that if we partition $S_w = \left[\begin{smallmatrix} (A\&B)_w \\ (C\&D)_w \end{smallmatrix}\right]$, then $(A\&B)_w$ can be extended to an operator $[\overline{A_w}\ B_w]$ that maps $H \times P_E^{-1}E_0$ to $H \times E_0'$. Here $\overline{A_w}$ is the operator from (3.18) and B_w is the operator from (3.20). Since $H \times E_0'$ is continuously embedded in X_{-1w} (see the third additional statement) and since $\overline{A_w}$ can be extended to an operator $A_w \in \mathcal{L}(X, X_{-1w})$ (see the fourth additional statement), it follows that $[A_w\ B_w]$ is an extension of $(A\&B)_w$ to a bounded operator from $X \oplus U$ to X_{-1w}. According to the definition of the control operator of a system node (see the first half of section 2), B_w is the control operator of S_w. The fact that C_w from (3.20) is the observation operator of S_w follows easily from (3.13) and the definition of the observation operator of a system node (see (2.4)).

To prove the fifth additional statement, take $z_w \in H$, $w_w \in P_E^{-1}E_0$, and $u_w \in U$ such that $\left[\begin{smallmatrix} z_w \\ w_w \\ u_w \end{smallmatrix}\right] \in \mathcal{D}(S_w)$ and denote

$$\begin{bmatrix} \dot{z}_w \\ \dot{w}_w \\ y_w \end{bmatrix} = S_w \begin{bmatrix} z_w \\ w_w \\ u_w \end{bmatrix}.$$

We reason as in the proof of Proposition 2.11. We define

$$z = P_H z_w, \quad w = P_E w_w, \quad u = R^{\frac{1}{2}} u_w, \quad y = R^{\frac{1}{2}} y_w;$$

then it is easy to see from (3.22) that $\left[\begin{smallmatrix} z \\ w \\ u \end{smallmatrix}\right] \in \mathcal{D}(S)$ and

$$\begin{bmatrix} \dot{z}_w \\ \dot{w}_w \\ y \end{bmatrix} = S \begin{bmatrix} z \\ w \\ u \end{bmatrix}.$$

We have, by using (3.9),

$$\begin{aligned} 0 &= 2\mathrm{Re}\,[\langle z, \dot{z}_w\rangle + \langle w, \dot{w}_w\rangle] - \|u\|^2 + \|y\|^2 - 2\mathrm{Re}\,\langle Gw, w\rangle \\ &= 2\mathrm{Re}\,[\langle P_H z_w, \dot{z}_w\rangle + \langle P_E w_w, \dot{w}_w\rangle] - \|R^{\frac{1}{2}} u_w\|^2 + \|R^{\frac{1}{2}} y_w\|^2 \\ &\quad - 2\mathrm{Re}\,\langle G P_E w_w, P_E w_w\rangle, \end{aligned}$$

which means that

$$2\mathrm{Re}\,[\langle P_H z_w, \dot{z}_w\rangle + \langle P_E w_w, \dot{w}_w\rangle] = \|R^{\frac{1}{2}} u_w\|^2 - \|R^{\frac{1}{2}} y_w\|^2 + 2\mathrm{Re}\,\langle G P_E w_w, P_E w_w\rangle.$$

Clearly this implies (3.21). ∎

Remark 3.3. We use the assumptions and the notation of Theorem 3.2. Define $\overline{C_w} \in \mathcal{L}(H \times P_E^{-1}E_0, U)$ by $\overline{C_w} = [0 \;\; -\sqrt{2}\,\gamma P_E]$ so that $\overline{C_w}$ is an extension of C_w. The space Z from (2.9) (with A_w and B_w in place of A and B) is a subspace of $H \times P_E^{-1}E_0$ and $\overline{C_w}$, when restricted to Z, is in $\mathcal{L}(Z, U)$. These facts follow from Remark 6.4 in [35], using that $\overline{C_w} = \sqrt{2}\,R^{-\frac{1}{2}}\overline{C}P$, where $C = [0 - \sqrt{2}\,R^{\frac{1}{2}}\gamma]$ is the observation operator of S from (3.8) with $K = \sqrt{2}R^{\frac{1}{2}}\gamma$.

4. Boundary traces. The Dirichlet trace operator. We need to look more carefully at the boundary trace of vector-valued functions before we can discuss Maxwell's equations. In this discussion all the functions have real (scalar or vector) values. Throughout this section, Ω is a bounded open subset of $\mathbb{R}^3$ with Lipschitz boundary Γ.

It is well known that the *Dirichlet trace operator* γ_0 can be defined such that

$$\gamma_0 \in \mathcal{L}(\mathcal{H}^1(\Omega), \mathcal{H}^{\frac{1}{2}}(\Gamma)),$$

and γ_0 is onto; see, for instance, Adams [1] or Grisvard [13].

PROPOSITION 4.1. *Let ℓ be a bounded linear functional on $\mathcal{H}^{\frac{1}{2}}(\Gamma)$. If $\ell\gamma_0$ (which is defined on $\mathcal{H}^1(\Omega)$) has a bounded extension to $L^2(\Omega)$, then $\ell = 0$.*

Proof. For every $x \in \mathbb{R}^3$ and $\varepsilon > 0$, let $B(x,\varepsilon)$ denote the open ball with center x and radius ε. Denote

$$\Gamma_\varepsilon = \bigcup_{x\in\Gamma} B(x,\varepsilon)$$

so that Γ_ε is open and contains Γ. Since Γ is compact, according to Proposition 13.1.5 in [37], for every $\varepsilon > 0$ there exists a test function $\eta_\varepsilon \in \mathcal{D}(\Gamma_{2\varepsilon})$ such that $\eta_\varepsilon(x) \in [0,1]$ for all $x \in \Gamma_{2\varepsilon}$ and $\eta_\varepsilon(x) = 1$ for all $x \in \Gamma_\varepsilon$. We extend η_ε to all $\mathbb{R}^3$ by setting it to be zero everywhere outside $\Gamma_{2\varepsilon}$ and clearly $\eta_\varepsilon \in C^\infty(\mathbb{R}^3)$.

Since γ_0 is onto, it is enough to prove that $\ell\gamma_0\varphi = 0$ for every $\varphi \in \mathcal{H}^1(\Omega)$. It can be verified that $\gamma_0\eta_\varepsilon\varphi = \gamma_0\varphi$. Indeed, this is true for all $\varphi \in C^\infty(\overline{\Omega})$ and $C^\infty(\overline{\Omega})$ is dense in $\mathcal{H}^1(\Omega)$; see Theorem 1.4.2.1 in [13] (reproduced as Theorem 13.5.4 in [37]). Thus

$$\ell\gamma_0\varphi = \ell\gamma_0\eta_\varepsilon\varphi. \tag{4.1}$$

It is easy to verify that $\lim_{\varepsilon\to 0}\|\eta_\varepsilon\varphi\|_{L^2(\Omega)} = 0$ (this is because the Lebesgue measure of $\Gamma_{2\varepsilon}$ shrinks to zero). Since, by assumption, $\ell\gamma_0$ has a continuous extension to $L^2(\Omega)$, it follows from (4.1) that

$$\ell\gamma_0\varphi = \lim_{\varepsilon\to 0}\ell\gamma_0\eta_\varepsilon\varphi = 0. \qquad \square$$

The tangential component trace operator. We continue to use the notation γ_0 for the Dirichlet trace. We denote by ν the unit normal outward vector field on Γ so that $\nu \in L^\infty(\Gamma,\mathbb{R}^3)$. We introduce a space of tangential vector fields on Γ:

$$L^2_\tau(\Gamma) = \left\{\psi \in L^2(\Gamma;\mathbb{R}^3) \mid \psi\cdot\nu = 0\right\}$$

with the norm inherited from $L^2(\Gamma;\mathbb{R}^3)$. If $\mathbf{E} \in \mathcal{H}^1(\Omega;\mathbb{R}^3)$, then we denote the *tangential component of the Dirichlet trace of* $\mathbf{E}$ by

$$\pi_\tau\mathbf{E} = (\nu\times\gamma_0\mathbf{E})\times\nu = \gamma_0\mathbf{E} - (\gamma_0\mathbf{E}\cdot\nu)\nu \tag{4.2}$$

so that

$$\pi_\tau \in \mathcal{L}(\mathcal{H}^1(\Omega;\mathbb{R}^3), L^2_\tau(\Gamma)).$$

Note that $\nu\times\gamma_0\mathbf{E} \in L^2_\tau(\Gamma;\mathbb{R}^3)$ is the same as $\pi_\tau\mathbf{E}$ rotated $90°$ around the direction of ν. In several references (such as Buffa, Costabel, and Sheen [5] and Dautray and Lions [8]) the operator $-\nu\times\gamma_0$ is denoted by γ_τ and it is called the tangential trace operator. To avoid confusion, and following [5], we adopt the more clumsy name *tangential component trace operator* for π_τ.

Following [5, section 2] we introduce a subspace of $L^2_\tau(\Gamma)$ as follows:

$$V_\pi = \pi_\tau\mathcal{H}^1(\Omega;\mathbb{R}^3)$$

with the norm

$$\|\psi\|_{V_\pi} = \inf\left\{\|\gamma_0\mathbf{H}\|_{\mathcal{H}^{\frac12}(\Gamma;\mathbb{R}^3)} \;\middle|\; \mathbf{H}\in\mathcal{H}^1(\Omega;\mathbb{R}^3),\quad \psi = \pi_\tau\mathbf{H}\right\}.$$

Since obviously $\operatorname{Ker}\gamma_0 \subset \operatorname{Ker}\pi_\tau$, it follows that there exists an operator P in $\mathcal{L}(\mathcal{H}^{\frac12}(\Gamma;\mathbb{R}^3); L^2_\tau(\Gamma))$ such that $\pi_\tau = P\gamma_0$. It is easy to see that P is the operator of eliminating the normal component of a vector field. (In [5], the operator P is denoted by the same symbol as π_τ, which we find a bit confusing.) Since γ_0 is onto $\mathcal{H}^{\frac12}(\Gamma;\mathbb{R}^3)$, we see that V_π with the above norm is isomorphic to the orthogonal complement of $\operatorname{Ker}P$ in $\mathcal{H}^{\frac12}(\Gamma;\mathbb{R}^3)$, so that it is a Hilbert space. Clearly $\|\pi_\tau\varphi\|_{V_\pi} \le \|\gamma_0\varphi\|_{\mathcal{H}^{\frac12}(\Gamma;\mathbb{R}^3)}$ for all $\varphi\in\mathcal{H}^1(\Omega;\mathbb{R}^3)$, so that $\pi_\tau \in \mathcal{L}(\mathcal{H}^1(\Omega;\mathbb{R}^3), V_\tau)$.

If Γ is smooth, then V_π is a subspace of $\mathcal{H}^{\frac{1}{2}}(\Gamma;\mathbb{R}^3)$ but, in general, this is not the case. Indeed, based on a two-dimensional domain constructed by Filonov [11], in [5, section 6] there is an example of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ with $\operatorname{Ker} P = \{0\}$. If it were true for this Ω that $V_\pi \subset \mathcal{H}^{\frac{1}{2}}(\Gamma;\mathbb{R}^3)$, then we would have $P^2 = P$ and hence $P(I-P) = 0$, whence $P = I$, which is absurd.

The density of $\mathcal{H}^{\frac{1}{2}}(\Gamma;\mathbb{R}^3)$ in $L^2(\Gamma;\mathbb{R}^3)$ easily implies that V_π is dense in $L^2_\tau(\Gamma)$. The embedding is continuous, since clearly for every $\psi \in V_\pi$, $\|\psi\|_{L^2} \leq \|P\| \cdot \|\psi\|_{V_\pi}$. Thus, we can define V'_π as the dual of V_π with respect to the pivot space $L^2_\tau(\Omega;\mathbb{R}^3)$.

We shall need a more sophisticated version of Proposition 4.1, where we work with tangential traces on a part of the boundary. We assume that Γ_0 is a relatively open subset of the Lipschitz boundary Γ. We define

$$\mathcal{H}^1_{\Gamma_0}(\Omega) = \left\{ \varphi \in \mathcal{H}^1(\Omega) \mid \gamma_0 \varphi|_{\Gamma_0} = 0 \right\} .$$

For more details on this space we refer to section 13.6 in [37]. The vector-valued version $\mathcal{H}^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ is defined similarly. We also define a slightly larger space as follows:

$$\mathcal{H}^1_{\tau,\Gamma_0}(\Omega) = \left\{ \varphi \in \mathcal{H}^1(\Omega;\mathbb{R}^3) \;\middle|\; (\pi_\tau \varphi)|_{\Gamma_0} = 0 \right\} .$$

PROPOSITION 4.2. *Let $\ell \in V'_\pi$ (i.e., it is a bounded linear functional on V_π). If $\ell\pi_\tau$, when restricted to be a linear functional on $\mathcal{H}^1_{\tau,\Gamma_0}(\Omega)$, has a bounded extension to $L^2(\Omega;\mathbb{R}^3)$, then $\ell\pi_\tau\varphi = 0$ for every $\varphi \in \mathcal{H}^1_{\tau,\Gamma_0}(\Omega)$.*

Proof. For all $\varepsilon > 0$ let the smooth functions η_ε be defined as in the proof of Proposition 4.1. It can be verified that $\pi_\tau \eta_\varepsilon \varphi = \pi_\tau \varphi$ for all $\varphi \in \mathcal{H}^1(\Omega;\mathbb{R}^3)$, for similar reasons as we had for the equality $\gamma_0 \eta_\varepsilon \varphi = \gamma_0 \varphi$ in the proof of Proposition 4.1. Hence in particular we have

$$\ell\pi_\tau\varphi = \ell\pi_\tau\eta_\varepsilon\varphi \qquad \forall\, \varphi \in \mathcal{H}^1_{\tau,\Gamma_0}(\Omega).$$

It is easy to verify that

$$\lim_{\varepsilon \to 0} \|\eta_\varepsilon \varphi\|_{L^2(\Omega;\mathbb{R}^3)} = 0$$

(because the Lebesgue measure of $\Gamma_{2\varepsilon}$ tends to zero). Since $\eta_\varepsilon\varphi \in \mathcal{H}^1_{\tau,\Gamma_0}(\Omega)$ and, by assumption, the relevant restriction of $\ell\pi_\tau$ has a continuous extension to $L^2(\Omega;\mathbb{R}^3)$, it follows that

$$\ell\pi_\tau\varphi = \lim_{\varepsilon \to 0} \ell\pi_\tau\eta_\varepsilon\varphi = 0. \qquad \square$$

Extending the operator $\boldsymbol{\nu \times \gamma_0}$. The operator $\nu \times \gamma_0$ is bounded from $\mathcal{H}^1(\Omega;\mathbb{R}^3)$ to $L^2_\tau(\Gamma;\mathbb{R}^3)$, but we need an extension of this operator.

PROPOSITION 4.3. *There exists $c > 0$ (which depends on Ω) such that for every $\mathbf{E} \in \mathcal{H}^1(\Omega;\mathbb{R}^3)$,*

$$\|\nu \times \gamma_0 \mathbf{E}\|_{V'_\pi} \leq c \left(\|\mathbf{E}\|^2_{L^2} + \|\operatorname{rot}\mathbf{E}\|^2_{L^2} \right)^{\frac{1}{2}} . \tag{4.3}$$

Proof. From the basic formula of integration by parts (see, for instance, Theorem 13.7.1 in [37]) it follows by simple manipulations (adding six different instances of the basic formula) that for all $\mathbf{E}, \varphi \in \mathcal{H}^1(\Omega;\mathbb{R}^3)$,

$$\langle \operatorname{rot}\mathbf{E}, \varphi \rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle \mathbf{E}, \operatorname{rot}\varphi \rangle_{L^2(\Omega;\mathbb{R}^3)} = \langle \nu \times \gamma_0 \mathbf{E}, \pi_\tau \varphi \rangle_{L^2_\tau(\Gamma;\mathbb{R}^3)} . \tag{4.4}$$

For a moment, we identify $\mathcal{H}^1(\Omega;\mathbb{R}^3)$ and V_π with their duals, then

$$\pi_\tau^* \in \mathcal{L}(V_\pi, \mathcal{H}^1(\Omega;\mathbb{R}^3)).$$

Since π_τ is onto, it follows that $\pi_\tau\pi_\tau^*$ is a strictly positive (hence invertible) operator on V_π (see, for instance, Proposition 12.1.3 in [37]).

For an arbitrary $\psi \in V_\pi$, we define $\tilde{\psi} \in \mathcal{H}^1(\Omega;\mathbb{R}^3)$ by $\tilde{\psi} = \pi_\tau^*(\pi_\tau\pi_\tau^*)^{-1}\psi$. Denoting $c_1 = \|\pi_\tau^*(\pi_\tau\pi_\tau^*)^{-1}\|$, we have

$$\pi_\tau\tilde{\psi} = \psi, \qquad \|\tilde{\psi}\|_{\mathcal{H}^1(\Omega;\mathbb{R}^3)} \le c_1\|\psi\|_{V_\pi}. \tag{4.5}$$

Substituting $\tilde{\psi}$ in place of φ in (4.4) we obtain that for all $\mathbf{E} \in \mathcal{H}^1(\Omega;\mathbb{R}^3)$,

$$\langle \nu \times \gamma_0\mathbf{E}, \psi\rangle_{L^2_\tau(\Gamma;\mathbb{R}^3)} = \langle \mathrm{rot}\,\mathbf{E}, \tilde{\psi}\rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle \mathbf{E}, \mathrm{rot}\,\tilde{\psi}\rangle_{L^2(\Omega;\mathbb{R}^3)},$$

whence (using (4.5) and the Cauchy–Schwarz inequality)

$$\begin{aligned} |\langle \nu \times \gamma_0\mathbf{E}, \psi\rangle_{L^2}| &\le \left(\|\mathbf{E}\|_{L^2}^2 + \|\mathrm{rot}\,\mathbf{E}\|_{L^2}^2\right)^{\frac{1}{2}} \cdot \left(\|\tilde{\psi}\|_{L^2} + \|\mathrm{rot}\,\tilde{\psi}\|_{L^2}^2\right)^{\frac{1}{2}} \\ &\le \left(\|\mathbf{E}\|_{L^2}^2 + \|\mathrm{rot}\,\mathbf{E}\|_{L^2}^2\right)^{\frac{1}{2}} \cdot \sqrt{2}c_1\|\psi\|_{V_\pi}. \end{aligned} \tag{4.6}$$

In the last inequality we have used that for any $\alpha \in \mathcal{H}^1(\Omega;\mathbb{R}^3)$,

$$\|\mathrm{rot}\,\alpha\|_{L^2}^2 \le 2\left(\left\|\frac{\partial\alpha}{\partial x_1}\right\|_{L^2}^2 + \left\|\frac{\partial\alpha}{\partial x_2}\right\|_{L^2}^2 + \left\|\frac{\partial\alpha}{\partial x_3}\right\|_{L^2}^2\right).$$

Since (4.6) holds for all $\psi \in V_\pi$, it follows that (4.3) holds with $c = \sqrt{2}c_1$. ◻

The spaces $\mathcal{H}(\mathrm{rot},\Omega)$ and $\mathcal{H}(\mathrm{rot},\Omega,\tau)$. We define the space $\mathcal{H}(\mathrm{rot},\Omega)$ by

$$\mathcal{H}(\mathrm{rot},\Omega) = \left\{\mathbf{E} \in L^2(\Omega;\mathbb{R}^3) \mid \mathrm{rot}\,\mathbf{E} \in L^2(\Omega,\mathbb{R}^3)\right\}$$

with the norm

$$\|\mathbf{E}\|_{\mathcal{H}(\mathrm{rot},\Omega)} = \left(\|\mathbf{E}\|_{L^2}^2 + \|\mathrm{rot}\,\mathbf{E}\|_{L^2}^2\right)^{\frac{1}{2}}.$$

It is easy to check that $\mathcal{H}(\mathrm{rot},\Omega)$ is complete. Much material about this space can be found in Dautray and Lions [8, Chapter IX], Girault and Raviart [12], Amrouche et al. [2], and Buffa, Costabel, and Sheen [5]. From Proposition 4.3 we see that the operator $\nu \times \gamma_0$ (originally defined on $\mathcal{H}^1(\Omega;\mathbb{R}^3)$) can be extended so that

$$\nu \times \gamma_0 \in \mathcal{L}(\mathcal{H}(\mathrm{rot},\Omega), V_\pi'). \tag{4.7}$$

This extension is unique, because $\mathcal{H}^1(\Omega;\mathbb{R}^3)$ contains $C^\infty(\overline{\Omega};\mathbb{R}^3)$, which in turn is dense in $\mathcal{H}(\mathrm{rot},\Omega)$, according to Theorem 2 on p. 204 of [8]. (See also [2, section 2] for this and related results.) By continuous extension, we see from (4.4) that for any $\mathbf{E} \in \mathcal{H}(\mathrm{rot},\Omega)$ and any $\varphi \in \mathcal{H}^1(\Omega;\mathbb{R}^3)$,

$$\langle \mathrm{rot}\,\mathbf{E}, \varphi\rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle \mathbf{E}, \mathrm{rot}\,\varphi\rangle_{L^2(\Omega;\mathbb{R}^3)} = \langle \nu \times \gamma_0\mathbf{E}, \pi_\tau\varphi\rangle_{V_\pi',V_\pi}. \tag{4.8}$$

For a sophisticated generalization of (4.8) to $\varphi \in \mathcal{H}(\mathrm{rot},\Omega)$ see [5, section 5].

Remark 4.4. Since $\mathcal{H}^{\frac{1}{2}}(\Gamma;\mathbb{R}^3)$ is dense in $L^2(\Gamma;\mathbb{R}^3)$ (see, for instance, Corollary 13.6.11 in [37]), it has a dual with respect to the pivot space $L^2(\Gamma;\mathbb{R}^3)$, denoted by $\mathcal{H}^{-\frac{1}{2}}(\Gamma;\mathbb{R}^3)$. Proposition 4.3 has a version in which we use the space $\mathcal{H}^{-\frac{1}{2}}(\Gamma;\mathbb{R}^3)$ in place of V_π'. It is in fact this version that is more commonly found in the literature; see, for instance, [8]. The proof is very similar to the proof of Proposition 4.3, but the details are often omitted from texts dealing with the subject. Similarly, the integration by parts formula (4.8) has a version in which $\pi_\tau\varphi$ is replaced with $\gamma_0\varphi$, V_π is replaced with $\mathcal{H}^{\frac{1}{2}}(\Gamma;\mathbb{R}^3)$, and V_π' is replaced with $\mathcal{H}^{-\frac{1}{2}}(\Gamma;\mathbb{R}^3)$. This is a well-known formula; see, for instance, Chapter I of [12] or p. 207 of [8].

We define a subspace of $L^2(\Omega;\mathbb{R}^3)$ as follows:

$$\mathcal{H}(\mathrm{rot},\Omega,\tau) = \left\{\mathbf{E}\in\mathcal{H}(\mathrm{rot},\Omega) \mid \nu\times\gamma_0\mathbf{E}\in L^2(\Gamma;\mathbb{R}^3)\right\} \tag{4.9}$$

with the norm defined by

$$\|\mathbf{E}\|^2_{\mathcal{H}(\mathrm{rot},\Omega,\tau)} = \|\mathbf{E}\|^2_{L^2} + \|\mathrm{rot}\,\mathbf{E}\|^2_{L^2} + \|\nu\times\gamma_0\mathbf{E}\|^2_{L^2}.$$

Clearly this space contains $\mathcal{H}^1(\Omega;\mathbb{R}^3)$. The space $C^\infty(\overline{\Omega},\mathbb{R}^3)$ *is dense in* $\mathcal{H}(\mathrm{rot},\Omega,\tau)$, as was proved in Ben Belgacem et al. [4]. (Interesting related density results can be found in Costabel and Dauge [7].)

PROPOSITION 4.5. *The space* $\mathcal{H}(\mathrm{rot},\Omega,\tau)$ *is complete.*

Proof. Let $(\mathbf{E}_n)$ be a Cauchy sequence in $\mathcal{H}(\mathrm{rot},\Omega,\tau)$. From the completeness of $\mathcal{H}(\mathrm{rot},\Omega)$ we conclude that there exists $\mathbf{E}_0\in\mathcal{H}(\mathrm{rot},\Omega)$ such that $\mathbf{E}_n\to\mathbf{E}_0$ in $\mathcal{H}(\mathrm{rot},\Omega)$. Moreover, it is clear that there exists $\psi\in L^2(\Gamma;\mathbb{R}^3)$ such that $\nu\times\gamma_0\mathbf{E}_n\to\psi$ in $L^2(\Gamma;\mathbb{R}^3)$. Substituting $\mathbf{E}_n$ in place of $\mathbf{E}$ in (4.8) and then taking limits, we obtain that for any $\varphi\in\mathcal{H}^1(\Omega;\mathbb{R}^3)$,

$$\langle\mathrm{rot}\,\mathbf{E}_0,\varphi\rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle\mathbf{E}_0,\mathrm{rot}\,\varphi\rangle_{L^2(\Omega;\mathbb{R}^3)} = \langle\psi,\pi_\tau\varphi\rangle_{L^2(\Gamma;\mathbb{R}^3)}.$$

Since $\pi_\tau\varphi$ may be any element of V_π and V_π is dense in $L^2_\tau(\Gamma;\mathbb{R}^3)$, we conclude from the above formula and from (4.8) that $\psi=\nu\times\gamma_0\mathbf{E}_0$. This means that $\mathbf{E}_0\in\mathcal{H}(\mathrm{rot},\Omega,\tau)$ and $\mathbf{E}_n\to\mathbf{E}_0$ in $\mathcal{H}(\mathrm{rot},\Omega,\tau)$. ☐

Since the pointwise vector product with ν is a bounded operator on $L^2(\Gamma;\mathbb{R}^3)$, we can extend π_τ (defined in (4.2)) such that

$$\pi_\tau \in \mathcal{L}(\mathcal{H}(\mathrm{rot},\Omega,\tau), L^2_\tau(\Gamma;\mathbb{R}^3)).$$

The normal trace operator. It is well known that for all $\mathbf{D}\in\mathcal{H}^1(\Omega;\mathbb{R}^3)$ and $\varphi\in\mathcal{H}^1(\Omega)$ we have (integration by parts)

$$\langle\mathrm{div}\,\mathbf{D},\varphi\rangle_{L^2(\Omega)} + \langle\mathbf{D},\mathrm{grad}\,\varphi\rangle_{L^2(\Omega;\mathbb{R}^3)} = \langle\nu\cdot\gamma_0\mathbf{D},\gamma_0\varphi\rangle_{L^2(\Gamma)}; \tag{4.10}$$

see, for instance, p. 206 in [8] or (2.6) in [2] or (13.7.2) in [37]. We introduce the *normal trace operator* $\gamma_\nu=\nu\cdot\gamma_0$, which is in $\mathcal{L}(\mathcal{H}^1(\Omega;\mathbb{R}^3),L^2(\Gamma))$. We mention that according to (4.2), for every $\mathbf{D}\in\mathcal{H}^1(\Omega;\mathbb{R}^3)$,

$$\gamma_0\mathbf{D} = \pi_\tau\mathbf{D} + \nu\gamma_\nu\mathbf{D}.$$

For any $\psi\in\mathcal{H}^{\frac{1}{2}}(\Gamma)$ we can find $\tilde\psi\in\mathcal{H}^1(\Omega)$ such that

$$\gamma_0\tilde\psi = \psi, \qquad \|\tilde\psi\|_{\mathcal{H}^1(\Omega)} \le c_2\|\psi\|_{\mathcal{H}^{\frac{1}{2}}(\Gamma)}, \tag{4.11}$$

where $c_2 > 0$ is independent of ψ. The proof of this fact is very similar to the proof of (4.5) and we omit it. Substituting $\tilde{\psi}$ in place of φ in (4.10) and then using the Cauchy–Schwarz inequality, we obtain that for all $\mathbf{D} \in \mathcal{H}^1(\Omega;\mathbb{R}^3)$ and $\psi \in \mathcal{H}^{\frac{1}{2}}(\Gamma)$,

$$\left|\langle \gamma_\nu \mathbf{D}, \psi\rangle_{L^2(\Gamma)}\right| \;\leq\; \left(\|\mathbf{D}\|_{L^2}^2 + \|\mathrm{div}\,\mathbf{D}\|_{L^2}^2\right)^{\frac{1}{2}} \cdot \left(\|\tilde{\psi}\|_{L^2} + \|\mathrm{grad}\,\tilde{\psi}\|_{L^2}^2\right)^{\frac{1}{2}} .$$

Using (4.11) and an argument similar to the one in the proof of Proposition 4.3, we obtain that

$$\|\gamma_\nu \mathbf{D}\|_{\mathcal{H}^{-\frac{1}{2}}(\Gamma)} \;\leq c_2 \left(\|\mathbf{D}\|_{L^2}^2 + \|\mathrm{div}\,\mathbf{D}\|_{L^2}^2\right)^{\frac{1}{2}} . \tag{4.12}$$

We introduce the space $\mathcal{H}(\mathrm{div}\,,\Omega)$ by

$$\mathcal{H}(\mathrm{div}\,,\Omega) \;=\; \left\{\mathbf{D} \in L^2(\Omega;\mathbb{R}^3) \mid \mathrm{div}\,\mathbf{D} \in L^2(\Omega)\right\}$$

with the norm

$$\|\mathbf{D}\|_{\mathcal{H}(\mathrm{div}\,,\Omega)} \;=\; \left(\|\mathbf{D}\|_{L^2}^2 + \|\mathrm{div}\,\mathbf{D}\|_{L^2}^2\right)^{\frac{1}{2}} .$$

It is easy to check that this space is complete. For more details about this space we refer to [8, Chapter IX], [12], [2]. According to (4.12), γ_ν can be extended so that

$$\gamma_\nu \,\in\, \mathcal{L}(\mathcal{H}(\mathrm{div}\,,\Omega), \mathcal{H}^{-\frac{1}{2}}(\Gamma)). \tag{4.13}$$

This extension is unique, because $\mathcal{H}^1(\Omega;\mathbb{R}^3)$ contains $C^\infty(\overline{\Omega};\mathbb{R}^3)$, which in turn is dense in $\mathcal{H}(\mathrm{div}\,,\Omega)$, according to Theorem 1 on p. 204 of [8] (see also [2]).

Taking limits in (4.10), we obtain that for every $\mathbf{D} \in \mathcal{H}(\mathrm{div}\,,\Omega)$ and $\varphi \in \mathcal{H}^1(\Omega)$,

$$\langle \mathrm{div}\,\mathbf{D}, \varphi\rangle_{L^2(\Omega)} + \langle \mathbf{D}, \mathrm{grad}\,\varphi\rangle_{L^2(\Omega;\mathbb{R}^3)} \;=\; \langle \gamma_\nu \mathbf{D}, \gamma_0\varphi\rangle_{\mathcal{H}^{-\frac{1}{2}}(\Gamma),\mathcal{H}^{\frac{1}{2}}(\Gamma)} . \tag{4.14}$$

Let Γ_0 be a relatively open subset of the boundary Γ. We introduce a space of "test functions" on Γ_0 as follows:

$$\mathcal{D}(\Gamma_0) \;=\; \left\{\gamma_0\varphi \mid \varphi \in C^\infty(\overline{\Omega}),\; \mathrm{supp}\,\varphi \cap \Gamma \subset \Gamma_0\right\} .$$

We also introduce a space of "tangential test functions" on Γ_0 as follows:

$$\mathcal{D}_\tau(\Gamma_0) \;=\; \left\{\pi_\tau\varphi \mid \varphi \in C^\infty(\overline{\Omega};\mathbb{R}^3),\; \mathrm{supp}\,\varphi \cap \Gamma \subset \Gamma_0\right\} .$$

PROPOSITION 4.6. *If* $\mathbf{E} \in \mathcal{H}(\mathrm{rot}\,,\Omega)$*, then* $\mathrm{rot}\,\mathbf{E} \in \mathcal{H}(\mathrm{div}\,,\Omega)$*, so that* $\gamma_\nu(\mathrm{rot}\,\mathbf{E}) \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$*. Now suppose that* Γ_0 *is a relatively open subset of* Γ*. Let* $\mathbf{E} \in \mathcal{H}(\mathrm{rot}\,,\Omega)$ *such that, in the sense of distributions,*

$$\nu \times \gamma_0 \mathbf{E} \,=\, 0 \quad on \quad \Gamma_0\,, \tag{4.15}$$

meaning that $\langle \nu \times \gamma_0\mathbf{E}, \xi\rangle_{V_\pi', V_\pi} = 0$ *for all* $\xi \in \mathcal{D}_\tau(\Gamma_0)$*. Then we have, again in the sense of distributions,*

$$\gamma_\nu(\mathrm{rot}\,\mathbf{E}) = 0 \quad on \quad \Gamma_0\,,$$

which means that $\langle \gamma_\nu(\mathrm{rot}\,\mathbf{E}), \psi\rangle_{\mathcal{H}^{-\frac{1}{2}}(\Gamma),\mathcal{H}^{\frac{1}{2}}(\Gamma)} = 0$ *for all* $\psi \in \mathcal{D}(\Gamma_0)$*.*

This proposition can be regarded as a consequence of the fact that $\gamma_\nu(\mathrm{rot}\,\mathbf{E})$ can be expressed as a "tangential div operator" div_Γ applied to $-\nu \times \gamma_0 \mathbf{E}$; see the beginning of section 4 in [5]. Below we give a direct proof that does not use the concept of "tangential div operator."

Proof. The first sentence follows from $\mathrm{div}\,\mathrm{rot} = 0$ and from (4.13).

If we take $\mathbf{D} = \mathrm{rot}\,\mathbf{E}$ in (4.14), then we get that for all $\varphi \in \mathcal{H}^1(\Omega)$

$$\langle \mathrm{rot}\,\mathbf{E}, \mathrm{grad}\,\varphi\rangle_{L^2(\Omega;\mathbb{R}^3)} = \langle \gamma_\nu(\mathrm{rot}\,\mathbf{E}), \gamma_0\varphi\rangle_{\mathcal{H}^{-\frac{1}{2}}(\Gamma),\mathcal{H}^{\frac{1}{2}}(\Gamma)}.$$

In what follows we assume that $\varphi \in C^\infty(\overline{\Omega})$. We rewrite the left side of the above formula using (4.8), using that $\mathrm{rot}\,\mathrm{grad} = 0$:

$$\langle \nu \times \gamma_0\mathbf{E}, \pi_\tau(\mathrm{grad}\,\varphi)\rangle_{V'_\pi,V_\pi} = \langle \gamma_\nu(\mathrm{rot}\,\mathbf{E}), \gamma_0\varphi\rangle_{\mathcal{H}^{-\frac{1}{2}}(\Gamma),\mathcal{H}^{\frac{1}{2}}(\Gamma)}. \tag{4.16}$$

If $\mathrm{supp}\,\varphi \cap \Gamma \subset \Gamma_0$, then $\pi_\tau(\mathrm{grad}\,\varphi) \in \mathcal{D}_\tau(\Gamma_0)$. We see from (4.15) and (4.16) that

$$\langle \gamma_\nu(\mathrm{rot}\,\mathbf{E}), \gamma_0\varphi\rangle_{\mathcal{H}^{-\frac{1}{2}}(\Gamma),\mathcal{H}^{\frac{1}{2}}(\Gamma)} = 0,$$

which is exactly the desired identity. □

5. Systems corresponding to Maxwell's equations. In this section we consider Maxwell's equations on a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary Γ. We assume that Γ_0 and Γ_1 are relatively open subsets of Γ such that

$$\Gamma_0 \cap \Gamma_1 = \emptyset, \qquad \overline{\Gamma_0} \cup \overline{\Gamma_1} = \Gamma \tag{5.1}$$

and the surface measures of the boundaries $\partial\Gamma_0$ and $\partial\Gamma_1$ are zero. This implies that $L^2(\Gamma) = L^2(\Gamma_0) \oplus L^2(\Gamma_1)$. Intuitively, the domain Ω is filled with a material with limited (possibly zero) conductivity, while the surface Γ_0 is superconductive.

We denote the *electric and magnetic field intensities* by $\mathbf{E}$ and $\mathbf{H}$, respectively. ρ is the *charge density* and J is the *current density*. We denote the *electric and magnetic flux densities* by $\mathbf{D}$ and $\mathbf{B}$, respectively. Usually, $\mathbf{E}$ and $\mathbf{H}$ are integrated along curves, while $\mathbf{D}$ and $\mathbf{B}$ are integrated on surfaces. Maxwell's equations are

$$\frac{\partial \mathbf{D}}{\partial t} = \mathrm{rot}\,\mathbf{H} - J, \qquad \frac{\partial \mathbf{B}}{\partial t} = -\,\mathrm{rot}\,\mathbf{E}, \tag{5.2}$$

$$\mathrm{div}\,\mathbf{D} = \rho, \qquad \mathrm{div}\,\mathbf{B} = 0. \tag{5.3}$$

Assuming that the material in Ω has linear and isotropic behavior and there are no external sources of electric field, we have in Ω the following equations:

$$J = g\mathbf{E}, \tag{5.4}$$

$$\mathbf{D} = \varepsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}, \tag{5.5}$$

where the positive scalar functions ε, μ, and g are such that

$$\varepsilon, \frac{1}{\varepsilon}, \mu, \frac{1}{\mu}, g \in L^\infty(\Omega).$$

g is called the *conductivity* of the material. The function ε is called the *electric permittivity*, while μ is called the *magnetic permeability* of the material in Ω.

Note that the first halves of (5.2) and (5.3) imply that

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} J\,. \tag{5.6}$$

Equation (5.4) is known as *Ohm's law*. For more details on the electromagnetic field we refer to Dautray and Lions [8], Jackson [14], Orfanidis [24], or Simonyi [28]. There exist materials with negative ε or μ [39], as well as materials where $\mathbf{D}$ and $\mathbf{B}$ are nonlinear functions of $\mathbf{E}$ and $\mathbf{H}$ (possibly with memory effects), but we do not consider them. We also do not consider media where (5.4) is replaced by a different formula, for instance, $J = g(\mathbf{E} - \mathbf{E}_0)$ (with $\mathbf{E}_0$ fixed) inside batteries.

We denote by $\pi_\tau \mathbf{E}$ the tangential component trace of $\mathbf{E}$ on Γ, as in (4.2). Since Γ_0 represents a superconductor, there will be no electric field along this surface:

$$\pi_\tau \mathbf{E} = 0 \quad \text{on } \Gamma_0\,. \tag{5.7}$$

Under suitable assumptions (see Proposition 4.6) it follows that $\gamma_\nu(\operatorname{rot}\mathbf{E}) = 0$ on Γ_0. Taking normal traces of both sides in the second formula in (5.2), we get that $\gamma_\nu\left(\frac{\partial \mathbf{B}}{\partial t}\right) = 0$ on Γ_0. This implies that $\gamma_\nu \mathbf{B}$ is constant in time on Γ_0.

The active boundary is Γ_1, i.e., this is where the input acts and where the output is measured. We choose a positive scalar function r such that

$$r, \frac{1}{r} \in L^\infty(\Gamma_1)\,.$$

The input function imposes a boundary condition on $\mathbf{H}$ and $\mathbf{E}$ via

$$u = \frac{1}{\sqrt{2}}\Big(r(\nu \times \gamma_0 \mathbf{H}) + \pi_\tau \mathbf{E}\Big) \quad \text{on } \Gamma_1\,. \tag{5.8}$$

The output function is defined by

$$y = \frac{1}{\sqrt{2}}\Big(r(\nu \times \gamma_0 \mathbf{H}) - \pi_\tau \mathbf{E}\Big) \quad \text{on } \Gamma_1\,. \tag{5.9}$$

All the spaces that we use to analyze Maxwell's equations are real Hilbert spaces, consisting of real-valued functions. The input and output spaces consist of tangential vector fields on Γ_1:

$$U, Y = \{u \in L^2(\Gamma_1; \mathbb{R}^3) \mid u \cdot \nu = 0\}\,. \tag{5.10}$$

The state space X is defined (as in section 1) by

$$X = E \oplus E\,, \qquad \text{where} \qquad E = L^2(\Omega; \mathbb{R}^3)\,.$$

For a state $x = \left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right]$, $\langle \mathbf{H}, \mathbf{B}\rangle + \langle \mathbf{E}, \mathbf{D}\rangle$ is twice the physical energy.

Systems of this type or related (usually with $\Gamma_0 = \emptyset$) were considered by several authors; see, for instance, [4, 8, 9, 18, 26]. Our approach to constructing a system node that corresponds to these equations is to use Theorem 3.2 with $H = E$.

We define a subspace of E using the space $\mathcal{H}(\operatorname{rot}, \Omega, \tau)$ from (4.9),

$$E_0 = \{\mathbf{E} \in \mathcal{H}(\operatorname{rot}, \Omega, \tau) \mid (\pi_\tau \mathbf{E})|_{\Gamma_0} = 0\}$$

with the norm inherited from $\mathcal{H}(\operatorname{rot}, \Omega, \tau)$:

$$\|\mathbf{E}\|_{E_0}^2 = \|\mathbf{E}\|_E^2 + \|\operatorname{rot}\mathbf{E}\|_E^2 + \|(\pi_\tau \mathbf{E})|_{\Gamma_1}\|_U^2\,.$$

Recall from section 4 that for $\mathbf{E} \in \mathcal{H}(\mathrm{rot}, \Omega, \tau)$ the tangential component trace $\pi_\tau \mathbf{E}$ exists and it is in $L^2(\Gamma; \mathbb{R}^3)$, so that the above definition of E_0 and its norm makes sense. Clearly E_0 is dense in E, since it contains the test functions on Ω. Moreover, as a closed subspace of $\mathcal{H}(\mathrm{rot}, \Omega, \tau)$, E_0 is complete.

The following theorem introduces the system node that corresponds to Maxwell's equations with boundary control and boundary observation as in (5.8) and (5.9).

THEOREM 5.1. *With the spaces* U, E, E_0, X *and the functions* ε, μ, g, r *defined as above, we introduce the operators* $L \in \mathcal{L}(E_0, E)$ *and* $\gamma \in \mathcal{L}(E_0, U)$ *by*

$$L\mathbf{E} = \mathrm{rot}\,\mathbf{E}, \qquad \gamma\mathbf{E} = (\pi_\tau \mathbf{E})|_{\Gamma_1}. \tag{5.11}$$

Define $G \in \mathcal{L}(E)$ *by*

$$(G\mathbf{E})(\xi) = -g(\xi)\mathbf{E}(\xi) \quad \text{for almost every } \xi \in \Omega. \tag{5.12}$$

The operators $P_H, P_E \in \mathcal{L}(E)$, *and* $R \in \mathcal{L}(U)$ *are pointwise multiplication operators with the functions* $1/\mu$, $1/\varepsilon$, *and* $1/r$, *respectively.*

Then the operator S_w *from* (3.13), (3.14) *is a scattering passive system node on* (U, X, U) *with respect to the storage operator*

$$P = \begin{bmatrix} P_H & 0 \\ 0 & P_E \end{bmatrix} = \begin{bmatrix} \mu^{-1} & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}$$

and the supply operator R. S_w *is scattering conservative (with respect to the same weighting operators) iff* $g = 0$.

The five additional statements in Theorem 3.2 *also hold.*

Proof. From the way the space E_0 and the operators L and γ are defined, it is easy to see that $\left[\begin{smallmatrix} L \\ \gamma \end{smallmatrix}\right]$ is closed, as required in (3.11). Indeed, the norm on E_0 is equivalent to the graph norm of $\left[\begin{smallmatrix} L \\ \gamma \end{smallmatrix}\right]$. (It can be shown that L alone is not closed.) Clearly G satisfies (3.12). Clearly P_H, P_E, and R are strictly positive and hence $P \in \mathcal{L}(X)$ is also strictly positive. According to Theorem 3.2 the operator S_w is a scattering passive system node with respect to the storage operator P and the supply operator R. This system node is scattering conservative (with respect to the same weighting operators) iff $g = 0$ (i.e., there is no conductivity in Ω and hence there are no currents). Clearly the five additional statements also hold. ∎

Remark 5.2. The range of the operator γ introduced in the last theorem is not U but rather a dense subspace of U. This range can be characterized using the "tangential rot operator" rot_Γ defined in section 3 of [5]. A function $u \in U$ belongs to $\mathrm{Ran}\,\gamma$ iff $\mathrm{rot}_\Gamma u \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$. This follows from Theorem 4.1 in [5], which is based on a surjectivity result of Luc Tartar from 1997.

It is quite a challenge to show that the system node S_w introduced above does indeed correspond to Maxwell's equations with the boundary input and boundary output. More precisely, we have to show that the classical solutions of

$$\begin{bmatrix} \dot{\mathbf{B}}(t) \\ \dot{\mathbf{D}}(t) \\ y(t) \end{bmatrix} = S_w \begin{bmatrix} \mathbf{B}(t) \\ \mathbf{D}(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geq 0 \tag{5.13}$$

together with the functions $\mathbf{H}(t) = \frac{1}{\mu}\mathbf{B}(t)$ and $\mathbf{E}(t) = \frac{1}{\varepsilon}\mathbf{D}(t)$ and with $J(t) = g\mathbf{E}(t)$ (as in (5.4)) indeed satisfy Maxwell's equations and the input/output formulas (5.8)–(5.9). This will be a consequence of Theorem 5.4 below.

DEFINITION 5.3. *Recall the tangential trace spaces V_π and V_π' introduced in section* 4. *While for elements of V_π the meaning of a truncation to a subset of Γ is obvious, this is not the case for elements of V_π'. We have to define what we mean by the* truncation of an element $f \in V_\pi'$ to Γ_1, *denoted by $f|_{\Gamma_1}$: this is the restriction of f (regarded as a functional on V_π) to $\pi_\tau \mathcal{H}^1_{\tau,\Gamma_0}(\Omega)$, where $\mathcal{H}^1_{\tau,\Gamma_0}(\Omega)$ is the space introduced before Proposition* 4.2. *We regard U from* (5.10) *as a subspace of $L^2_\tau(\Gamma;\mathbb{R}^3)$ by extending functions in U to be zero on Γ_0, and then*

$$\pi_\tau \mathcal{H}^1_{\tau,\Gamma_0}(\Omega) = V_\pi \cap U$$

(as can be seen from the definitions). For $f \in V_\pi'$, the truncation $f|_{\Gamma_1}$ may have a continuous extension to U, in which case the extension is unique. Indeed, $\pi_\tau \mathcal{H}^1_{\tau,\Gamma_0}(\Omega)$ is dense in U. This density follows from the fact that $\gamma_0 \mathcal{H}^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ is dense in $L^2(\Gamma_1;\mathbb{R}^3)$ according to Remark 13.6.14 *in* [37]. *If $f|_{\Gamma_1}$ has a continuous extension to U, then (by identifying U with its dual) we say that $f|_{\Gamma_1} \in U$. The definition of the truncation $f|_{\Gamma_0}$ (for $f \in V_\pi'$) is of course similar.*

THEOREM 5.4. *We use the assumptions and the notation of Theorem* 5.1. *The scattering passive system node S_w from* (3.13), (3.14) *has the following alternative description. If $\begin{bmatrix} \mu\mathbf{H}_0 \\ \varepsilon\mathbf{E}_0 \\ u_0 \end{bmatrix} \in \mathcal{D}(S_w)$, then*

$$S_w \begin{bmatrix} \mu\mathbf{H}_0 \\ \varepsilon\mathbf{E}_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} -\operatorname{rot}\mathbf{E}_0 \\ \operatorname{rot}\mathbf{H}_0 - g\mathbf{E}_0 \\ \frac{1}{\sqrt{2}}\Big(r(\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_1} - (\pi_\tau\mathbf{E}_0)\big|_{\Gamma_1}\Big) \end{bmatrix}, \tag{5.14}$$

and $\mathcal{D}(S_w) \subset \mathcal{D}_w$, where

$$\mathcal{D}_w = \left\{ \begin{bmatrix} \mu\mathbf{H}_0 \\ \varepsilon\mathbf{E}_0 \\ u_0 \end{bmatrix} \in E\times E\times U \;\middle|\; \begin{array}{c} \operatorname{rot}\mathbf{H}_0 \in E,\ \mathbf{E}_0 \in E_0, \\ (\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_1} + \frac{1}{r}(\pi_\tau\mathbf{E}_0)\big|_{\Gamma_1} = \frac{\sqrt{2}}{r}u_0 \end{array} \right\}. \tag{5.15}$$

Remark 5.5. For $\mathbf{H}_0$ as on the right-hand side of (5.15) we have $\mathbf{H}_0 \in \mathcal{H}(\operatorname{rot},\Omega)$ so that $\nu\times\gamma_0\mathbf{H}_0 \in V_\pi'$ (see (4.7)). From the last condition in (5.15) we see that $(\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_1} \in U$, i.e., this truncation is an L^2 function. However, the truncation $(\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_0}$ need not be an L^2 function. The physical interpretation of $(\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_0}$ is the surface current density flowing in Γ_0. (There is no current inside a superconductor, only on its surface, because $\mathbf{E}=0$ and $\mathbf{H}=0$ inside.)

Proof of Theorem 5.4. Take $\begin{bmatrix} \mathbf{B}_0 \\ \mathbf{D}_0 \\ u_0 \end{bmatrix} \in \mathcal{D}(S_w)$ and define $\mathbf{H}_0 = \frac{1}{\mu}\mathbf{B}_0$, $\mathbf{E}_0 = \frac{1}{\varepsilon}\mathbf{D}_0$; then clearly $\mathbf{H}_0 \in E$, $\mathbf{E}_0 \in E_0$, and $u_0 \in U$. To show that $\mathcal{D}(S_w) \subset \mathcal{D}_w$ we only have to prove that $\operatorname{rot}\mathbf{H}_0 \in E$ and that the last condition in (5.15) holds. Let us denote

$$\begin{bmatrix} \dot{\mathbf{B}}_0 \\ \dot{\mathbf{D}}_0 \\ y_0 \end{bmatrix} = S_w \begin{bmatrix} \mathbf{B}_0 \\ \mathbf{D}_0 \\ u_0 \end{bmatrix}$$

so that $\dot{\mathbf{B}}_0, \dot{\mathbf{D}}_0 \in E$, $y_0 \in U$. By (3.13) and (5.11),

$$\dot{\mathbf{B}}_0 = -\operatorname{rot}\mathbf{E}_0,$$

$$\dot{\mathbf{D}}_0 = L^*\mathbf{H}_0 + (G - \gamma^* R\gamma)\,\mathbf{E}_0 + \sqrt{2}\,\gamma^* R u_0, \tag{5.16}$$

$$y_0 = -\sqrt{2}\,\gamma \mathbf{E}_0 + u_0 . \tag{5.17}$$

Notice that we have already obtained the first line of (5.14). Equation (5.16) implies that for all $\varphi \in E_0$, using the inner products on E and U,

$$\langle \dot{\mathbf{D}}_0, \varphi\rangle = \langle \mathbf{H}_0, L\varphi\rangle + \langle G\mathbf{E}_0, \varphi\rangle - \langle R\gamma\mathbf{E}_0, \gamma\varphi\rangle + \sqrt{2}\,\langle Ru_0, \gamma\varphi\rangle . \tag{5.18}$$

In particular, if we take a test function $\varphi \in \mathcal{D}(\Omega;\mathbb{R}^3)$ and recall (5.12), then this becomes $\langle \dot{\mathbf{D}}_0, \varphi\rangle = \langle \mathbf{H}_0, \operatorname{rot}\varphi\rangle - \langle g\mathbf{E}_0, \varphi\rangle$. Thus, in the sense of distributions,

$$\dot{\mathbf{D}}_0 = \operatorname{rot}\mathbf{H}_0 - g\mathbf{E}_0 ,$$

which is the second line of (5.14). Since $\dot{\mathbf{D}}_0, \mathbf{E}_0 \in E$, it follows that $\operatorname{rot}\mathbf{H}_0 \in E$, as required in (5.15), and hence $\mathbf{H}_0 \in \mathcal{H}(\operatorname{rot}, \Omega)$. According to (4.7) we have $\nu \times \gamma_0 \mathbf{H}_0 \in V_\pi'$. We may apply (4.8) (with $\mathbf{H}_0$ in place of $\mathbf{E}$) to the first term on the right-hand side of (5.18), obtaining that for all $\varphi \in \mathcal{H}^1_{\tau,\Gamma_0}(\Omega) \subset E_0$,

$$\begin{aligned}\langle \dot{\mathbf{D}}_0, \varphi\rangle &= \langle \operatorname{rot}\mathbf{H}_0, \varphi\rangle - \langle \nu\times\gamma_0\mathbf{H}_0, \pi_\tau\varphi\rangle_{V_\pi', V_\pi} \\ &\quad - \langle g\mathbf{E}_0, \varphi\rangle - \langle R\gamma\mathbf{E}_0, \gamma\varphi\rangle + \sqrt{2}\,\langle Ru_0, \gamma\varphi\rangle .\end{aligned}$$

(All the above pairings are in the L^2 sense, except the one indicated differently.) Using (5.17), we rewrite this:

$$\begin{aligned}&\langle \dot{\mathbf{D}}_0, \varphi\rangle + \langle g\mathbf{E}_0, \varphi\rangle - \langle \operatorname{rot}\mathbf{H}_0, \varphi\rangle \\ &\quad = \frac{1}{\sqrt{2}}\langle R(u_0+y_0), \gamma\varphi\rangle - \langle \nu\times\gamma_0\mathbf{H}_0, \pi_\tau\varphi\rangle_{V_\pi', V_\pi} .\end{aligned} \tag{5.19}$$

Remember that u_0 and y_0 are L^2 functions defined on Γ_1. We extend them to be zero on all other points of Γ. Now we rewrite the right-hand side of (5.19) using the fact that we may replace $\gamma\varphi$ in (5.19) with $\pi_\tau\varphi$. Define $\ell \in V_\pi'$ by

$$\ell = \frac{1}{\sqrt{2}} R(u_0+y_0) - \nu\times\gamma_0\mathbf{H}_0 .$$

Then (5.19) can be rewritten in the form

$$\langle \dot{\mathbf{D}}_0, \varphi\rangle + \langle g\mathbf{E}_0, \varphi\rangle - \langle \operatorname{rot}\mathbf{H}_0, \varphi\rangle = \langle \ell, \pi_\tau\varphi\rangle_{V_\pi', V_\pi} \qquad \forall\, \varphi \in \mathcal{H}^1_{\tau,\Gamma_0}(\Omega).$$

The left-hand side of this equality has a continuous extension to all $\varphi \in L^2(\Omega;\mathbb{R}^3)$. Therefore, according to Proposition 4.2 we have

$$\langle \ell, \pi_\tau\varphi\rangle_{V_\pi', V_\pi} = 0 \qquad \forall\, \varphi \in \mathcal{H}^1_{\tau,\Gamma_0}(\Omega).$$

According to Definition 5.3, this means that $\ell|_{\Gamma_1} = 0$. Thus,

$$\frac{1}{\sqrt{2}} R(u_0+y_0) = (\nu\times\gamma_0\mathbf{H}_0)|_{\Gamma_1} .$$

In particular, $(\nu\times\gamma_0\mathbf{H}_0)|_{\Gamma_1} \in U$. From this and (5.11) and (5.17), we can easily obtain the formulas

$$u_0 = \frac{1}{\sqrt{2}}\Big(r(\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_1} + (\pi_\tau\mathbf{E}_0)\big|_{\Gamma_1}\Big), \quad y_0 = \frac{1}{\sqrt{2}}\Big(r(\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_1} - (\pi_\tau\mathbf{E}_0)\big|_{\Gamma_1}\Big).$$

The first of these two formulas is just what we needed to complete the proof of $\mathcal{D}(A_w) \subset \mathcal{D}_w$. The second formula is the third (and last) line of (5.14). ∎

Theorem 5.4 is incomplete as it does not give an explicit formula for $\mathcal{D}(S_w)$. We shall discuss this further at the end of this section, where we show that if

$$\mathcal{H}^1_{\tau,\Gamma_0}(\Omega) \text{ is dense in } E_0\,, \tag{5.20}$$

then $\mathcal{D}(S_w) = \mathcal{D}_w$. If condition (5.20) is *not* satisfied, then it turns out that it is possible to also construct another scattering passive system node that corresponds to the same set of Maxwell's equations, with the same boundary conditions and the same state space—a rather shocking situation, for which we have no physical interpretation. Here is the construction of this alternative system node.

THEOREM 5.6. *Theorem* 5.1 *remains true if we throughout replace the space* E_0 *with the space*

$$F_0 = \text{ the closure of } \mathcal{H}^1_{\tau,\Gamma_0}(\Omega) \text{ in } E_0\,. \tag{5.21}$$

If we denote the corresponding system node by $S_w^{\rm alt}$*, then* $\mathcal{D}(S_w^{\rm alt}) = \mathcal{D}_w^{\rm alt}$*, where* $\mathcal{D}_w^{\rm alt}$ *is defined by the right-hand side of* (5.15) *with* E_0 *replaced with* F_0*, i.e.,*

$$\mathcal{D}_w^{\rm alt} = \left\{ \begin{bmatrix} \mu\mathbf{H}_0 \\ \varepsilon\mathbf{E}_0 \\ u_0 \end{bmatrix} \in E \times E \times U \;\middle|\; \begin{array}{c} \operatorname{rot}\mathbf{H}_0 \in E,\ \mathbf{E}_0 \in F_0\,, \\ (\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_1} + \frac{1}{r}(\pi_\tau\mathbf{E}_0)\big|_{\Gamma_1} = \frac{\sqrt{2}}{r}u_0 \end{array} \right\}, \tag{5.22}$$

and (5.14) *holds for all* $\left[\begin{smallmatrix} \mu\mathbf{H}_0 \\ \varepsilon\mathbf{E}_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S_w^{\rm alt}) = \mathcal{D}_w^{\rm alt}$.

Proof. The space F_0 is dense in E because it contains the test functions on Ω, which are dense in E. Since F_0 is a closed subspace of E_0, and since the norm in E_0 is equal to the graph norm of $\left[\begin{smallmatrix} L \\ \gamma \end{smallmatrix}\right]$, the restriction of $\left[\begin{smallmatrix} L \\ \gamma \end{smallmatrix}\right]$ to F_0 is still closed. and this is the only condition in Theorem 5.1 which contains an explicit reference to E_0. This modified version of Theorem 5.1 gives us another scattering passive system node $S_w^{\rm alt}$ (which is conservative if $g = 0$). By repeating the proof of Theorem 5.4 with E_0 replaced by F_0 we find that (5.14) holds with S_w replaced with $S_w^{\rm alt}$ and that $\mathcal{D}(S_w^{\rm alt}) \subset \mathcal{D}_w^{\rm alt}$. It is clear that $\mathcal{D}_w^{\rm alt} \subset \mathcal{D}_w$.

Suppose that $\left[\begin{smallmatrix} \mathbf{B}_0 \\ \mathbf{D}_0 \\ u_u \end{smallmatrix}\right] \in \mathcal{D}_w^{\rm alt}$, and denote $\mathbf{H}_0 = \frac{1}{\mu}\mathbf{B}_0 = P_H\mathbf{B}_0$, $\mathbf{E}_0 = \frac{1}{\varepsilon}\mathbf{D}_0 = P_E\mathbf{D}_0$. Then it is clear that $\mathbf{H}_0 \in E$, $\mathbf{E}_0 \in F_0$ and $u_0 \in U$. Thus, in order to show that $\left[\begin{smallmatrix} \mathbf{B}_0 \\ \mathbf{D}_0 \\ u_u \end{smallmatrix}\right] \in \mathcal{D}(S_w^{\rm alt})$ we only have to show that the last condition (the second line) in (3.14) holds. Since G is a bounded operator on E, we may ignore it and the condition that we must verify becomes

$$L^*\mathbf{H}_0 + \gamma^* R\left(-\gamma\mathbf{E}_0 + \sqrt{2}\,u_0\right) \in E\,.$$

We know from the last condition in (5.22) that the bracket appearing above (which is obviously in U) is equal to $r(\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_1}$, whence $(\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_1} \in U$. Thus, the condition that must be verified simplifies to

$$L^*\mathbf{H}_0 + \gamma^*(\nu\times\gamma_0\mathbf{H}_0)\big|_{\Gamma_1} \in E\,.$$

Equivalently, we have to show that the expression

$$\begin{aligned}\mathcal{Q} &= \left\langle L^*\mathbf{H}_0 + \gamma^*(\nu \times \gamma_0 \mathbf{H}_0)\big|_{\Gamma_1}, \varphi \right\rangle_{F_0', F_0} \\ &= \langle \mathbf{H}_0, \mathrm{rot}\,\varphi\rangle_E + \left\langle (\nu \times \gamma_0 \mathbf{H}_0)\big|_{\Gamma_1}, (\pi_\tau \varphi)\big|_{\Gamma_1} \right\rangle_U ,\end{aligned}$$

which is defined for $\varphi \in F_0$, has a continuous extension to $\varphi \in E$, i.e., there exists $k > 0$ such that

$$|\mathcal{Q}| \leq k\|\varphi\|_E \qquad \forall\, \varphi \in F_0\,. \tag{5.23}$$

Now assume that $\varphi \in \mathcal{H}^1_{\tau,\Gamma_0}(\Omega) \subset F_0$. Using Definition 5.3 we rewrite $\mathcal{Q}$:

$$\mathcal{Q} \,=\, \langle \mathbf{H}_0, \mathrm{rot}\,\varphi\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle \nu \times \gamma_0 \mathbf{H}_0, \pi_\tau \varphi\rangle_{V_\pi', V_\pi}\,.$$

Recall from Remark 5.5 that $\mathbf{H}_0 \in \mathcal{H}(\mathrm{rot}\,,\Omega)$. According to (4.8) (integration by parts) we get that for $\varphi \in \mathcal{H}^1_{\tau,\Gamma_0}(\Omega)$,

$$\mathcal{Q} \,=\, \langle \mathrm{rot}\,\mathbf{H}_0, \varphi\rangle_{L^2}\,,$$

so that (5.23) holds with $k = \|\mathrm{rot}\,\mathbf{H}_0\|_{L^2}$ for all $\varphi \in \mathcal{H}^1_{\tau,\Gamma_0}(\Omega)$. By continuous extension we obtain that (5.23) holds for all $\varphi \in F_0$, as required. ☐

The situation encountered in this section, with two different applications of Theorem 3.1 corresponding to two different choices of the subspace E_0 leading to two different system nodes, was also encountered in [35, Examples 2.4 and 2.5]. Both these examples deal with an Euler–Bernoulli beam, but the space E_0 in Example 2.5 is larger than that in Example 2.4. Even if we assume for simplicity that $K = 0$ in both examples (so that the two systems have no relevant input or output) we get a clamped beam in Example 2.4 and a free beam in Example 2.5, which are rather different systems. Unlike in this section, the difference between the two cited examples in [35] is easy to interpret as being due to different boundary conditions.

Open problem. We do not know if there are domains that satisfy our standing assumptions stated around (5.1) but do not satisfy (5.20). This might have something to do with the regularity of $\overline{\Gamma_0} \cap \overline{\Gamma_1}$, the common boundary between Γ_0 and Γ_1. We also do not know if it is possible that $\mathcal{D}(S_w) = \mathcal{D}_w$ in Theorem 5.4 without having (5.20). If we could show that $\mathcal{D}(S_w^{\mathrm{alt}}) \subset \mathcal{D}(S_w)$, then by a simple argument (see [32]) we could show that in fact $S_w^{\mathrm{alt}} = S_w$, and hence $\mathcal{D}(S_w) = \mathcal{D}_w$.

Remark 5.7. If condition (5.20) holds, then the system node in Theorem 5.6 coincides with the one in Theorems 5.1 and 5.4, and $\mathcal{D}(S_w) = \mathcal{D}_w$ in Theorem 5.4. Indeed, if (5.20) holds, then clearly $E_0 = F_0$, and hence the mentioned system nodes coincide. Thus, it follows from Theorem 5.6 that $\mathcal{D}(S_w) = \mathcal{D}_w$.

Remark 5.8. We explain here the little that we know about the above open problem. The main result of [4] is equivalent to the fact that (5.20) holds if $\Gamma_0 = \emptyset$. We are able to prove that (5.20) holds if $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. This can be done by using a function $\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for all x in a neighborhood of Γ_1 and $\varphi(x) = 0$ for all x in a neighborhood of Γ_0 (see, for instance, Proposition 13.1.5 in [37]). Take $\mathbf{E} \in E_0$ and let $\delta > 0$; then according to the result of [4] we can find $\mathbf{H} \in C^\infty(\overline{\Omega};\mathbb{R}^3)$ such that $\|\mathbf{E} - \mathbf{H}\|_{\mathcal{H}(\mathrm{rot}\,,\Omega,\tau)} < \delta$. Since $\pi_\tau(\mathbf{E} - \varphi\mathbf{E}) = 0$, according to Lemma 1 on p. 206 of [8] we can find $\psi \in \mathcal{D}(\Omega;\mathbb{R}^3)$ such that $\|\mathbf{E} - \varphi\mathbf{E} - \psi\|_{\mathcal{H}(\mathrm{rot}\,,\Omega)} < \delta$. We claim that if δ is small, then $\varphi\mathbf{H} + \psi \in C^\infty(\overline{\Omega};\mathbb{R}^3) \cap \mathcal{H}^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ is a good approximation of $\mathbf{E}$ in the space E_0. For this we use the decomposition

$$\mathbf{E} - (\varphi\mathbf{H} + \psi) \,=\, (\mathbf{E} - \varphi\mathbf{E} - \psi) + \varphi(\mathbf{E} - \mathbf{H})\,.$$

The norm of the first term on the right-hand side in E_0 is the same as its norm in $\mathcal{H}(\mathrm{rot},\Omega)$, which (as we already know) is $< \delta$. The norm of the second term is $< m\delta$, where m is the norm of the pointwise multiplication operator with φ.

A proof of (5.20) for a more general class of Ω and Γ_0, or an example of Ω and Γ_0 for which (5.20) does not hold, would be of interest, in our opinion.

6. Classical solutions, extensions, and an example. In the previous section we introduced the scattering passive system nodes S_w and S_w^{alt}, but it is still not clear how these are related to Maxwell's equations. In the following proposition we show that classical solutions of these system nodes do satisfy Maxwell's equations and the boundary conditions (5.8) and (5.9).

PROPOSITION 6.1. *If* $(\left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right], u, y)$ *is a classical solution of* (5.13) *(as introduced in Definition* 2.2*), either in the setting of Theorem* 5.1 *or in the setting of Theorem* 5.6*, and if we define* $\mathbf{H}(t) = \frac{1}{\mu}\mathbf{B}(t)$*,* $\mathbf{E}(t) = \frac{1}{\varepsilon}\mathbf{D}(t)$*,* $J(t) = g\mathbf{E}(t)$*, then for every* $t \geq 0$*, the functions* $\mathbf{B}, \mathbf{D}, \mathbf{H}, \mathbf{E}$*, and* J *satisfy Maxwell's equations except that* $\mathrm{div}\,\mathbf{B}$ *is an arbitrary function of space (constant in time), and not necessarily zero, as required by* (5.3)*. Moreover,*

$$u(t) = \frac{1}{\sqrt{2}}\Big(r(\nu \times \gamma_0 \mathbf{H}(t))\big|_{\Gamma_1} + (\pi_\tau \mathbf{E}(t))\big|_{\Gamma_1}\Big), \tag{6.1}$$

$$y(t) = \frac{1}{\sqrt{2}}\Big(r(\nu \times \gamma_0 \mathbf{H}(t))\big|_{\Gamma_1} - (\pi_\tau \mathbf{E}(t))\big|_{\Gamma_1}\Big). \tag{6.2}$$

Proof. This follows easily from Theorems 5.1, 5.4, and 5.6, and the arguments are essentially the same for the system node S_w and for S_w^{alt}. Below we go through some of the details in the case of S_w.

We start by proving that $\mathbf{B}, \mathbf{D}, \mathbf{H}$, and $\mathbf{E}$ satisfy (5.2)–(5.3) if J is defined by (5.4). Let the input function u and the initial state $\left[\begin{smallmatrix}\mathbf{B}(0)\\ \mathbf{D}(0)\end{smallmatrix}\right]$ of S_w satisfy (3.15). Then, according to the first additional statement in Theorem 3.2, the corresponding state trajectory $\left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right]$ and output function y of S_w satisfy the smoothness assumptions (3.16) and they are a classical solution of (5.13), which is in fact (3.17). In particular, we have $[\mathbf{B}\ \mathbf{D}\ u]^T \in C([0,\infty); \mathcal{D}(S_w))$ and, according to (5.14),

$$\dot{\mathbf{B}}(t) = -\,\mathrm{rot}\,\mathbf{E}(t), \qquad \dot{\mathbf{D}}(t) = \mathrm{rot}\,\mathbf{H}(t) - g\mathbf{E}(t),$$

which is (5.2). The equation $\mathrm{div}\,\mathbf{D} = \rho$ holds simply because ρ is defined by this formula. The equation $\mathrm{div}\,\mathbf{B} = 0$ does *not* follow from our assumptions: our system node S_w allows classical solutions with "magnetic charges" (which would have the density $\mathrm{div}\,\mathbf{B}$). However, it follows from the second half of (5.2) (by applying div to both sides) that $\mathrm{div}\,\mathbf{B}$ is constant in time. (The physical interpretation is that our system does not allow "magnetic currents" and hence the magnetic charge density stays constant.) Thus, if we assume that $\mathrm{div}\,\mathbf{B} = 0$ holds at some initial time, then this remains valid at any later time. We could restrict the state space to impose $\mathrm{div}\,\mathbf{B} = 0$, and this would indeed produce a new scattering passive system, but this new system would no longer have the neat abstract structure from Theorem 3.2.

Equations (5.4) and (5.5) hold by the definition of J, $\mathbf{H}$, and $\mathbf{E}$. Finally, (6.1) and (6.2) follow from the last lines of (5.15) and (5.14). ∎

Remark 6.2. Using Theorem 5.4 we can reformulate the first two equations from (3.17) (which describe the state trajectories for classical solutions of the Maxwell system) as a boundary control system in the sense of [37, Chapter 10]. Relevant material about well-posed (or conservative) boundary control systems is in [21, 30].

Remark 6.3. In section 5 we took μ, ε, g, and r to be positive scalar L^∞ functions which (with the exception of g) are bounded away from zero. This is the so called "isotropic case." However, all the results that we present remain true in the anisotropic case where these four scalar functions are replaced by 3×3 real matrix-valued positive L^∞ functions which (with the exception of g) must have uniformly bounded inverses. It would, in fact, be possible to extend the results even further by relaxing the requirement that the operators P_H, P_E, G, and R are pointwise multiplication operators, as long as P_H, P_E, and R remain positive and boundedly invertible, while G remains dissipative. In this way it would in principle be possible to replace the term $r(\nu \times \gamma_0 \mathbf{H}(t))\big|_{\Gamma_1}$ in the formulas for $u(t)$ and $y(t)$ by a "nonlocal boundary feedback" $R^{-1}(\nu \times \gamma_0 \mathbf{H}(t))\big|_{\Gamma_1}$.

Remark 6.4. Proposition 6.1 together with Theorem 5.1 allow us to apply Propositions 2.12 and 2.13 in the case when $g = 0$. For example, consider the system S_w from (5.13) but with constant μ, ε and $g = 0$. ($g = 0$ implies that the charge density ρ is constant; see (5.6).) We assume that all the variables that are known to be constant in time, namely,

$$\operatorname{div} \mathbf{B}, \ \operatorname{div} \mathbf{D}, \ \gamma_\nu \mathbf{B} \ \text{ on } \ \Gamma_0,$$

are zero from the start. (About $\gamma_\nu \mathbf{B}$ see the explanation after (5.7).) We assume that Γ_1 satisfies a geometric optics condition introduced in the microlocal analysis of the wave equation (roughly speaking, every ray in Ω encounters Γ_1, possibly after some reflections on Γ_0); see Phung [25, Definition 1.2]. Then the Maxwell system is exponentially stable according to [25, Theorem 4.1]. According to Proposition 2.12 the system is also exactly controllable and exactly observable in some finite time.

Remark 6.5. An interesting research problem is to investigate the well-posedness of the coupled system that arises when a conducting rigid body moves in vacuo in a bounded domain subject to an electromagnetic field (both in the moving body and in the vacuum) described by Maxwell's equations. This field creates a force and a torque acting on the rigid body, whose movement is described by a system of Lagrangian equations that incorporate external forces and possibly constraints. In this case the operator G from (5.12) depends on the position of the rigid body, which may be regarded as an output of the (finite-dimensional) mechanical system. The force and torque induced by the electromagnetic field are now a new output of the Maxwell system that acts as an input to the mechanical system. To investigate this complex nonlinear time-invariant system, a first step is to model the time-varying Maxwell system that arises when the rigid body is moving along a given path. This is done by Chen and Weiss [6] by regarding this time-varying Maxwell system as a perturbation of the system studied here, using the main results of this paper.

Example 6.6. We consider the special case where Ω is the region occupied by the insulator in a straight coaxial cable of length L, and we put additional symmetry conditions on the solution of Maxwell's equations. The domain Ω is a cylindrical region situated between a central conductor with radius a and an external conductor with radius b (so that any cross section taken perpendicularly to the axis of the cable is an annulus). Both the central and the external conductors are assumed to be superconducting, which is a reasonable approximation for a copper wire and shielding. The region Ω between the conductors is filled with a homogeneous nonconductive material. Thus in Ω, the permittivity ε and the permeability μ are supposed to be constant and $g = 0$ (hence $J = 0$). We also assume that there is no trapped charge in Ω, so that $\rho = 0$. For a discussion of Maxwell's equations in a coaxial cable we refer to Chapter 9 of Orfanidis [24].

We use a cylindrical coordinate system in Ω, so that every point there is represented by a triple (v, ϕ, z) with $a < v < b$, $0 \le \phi < 2\pi$, and $0 < z < L$. Here v is the *radial* variable, ϕ the *azimuthal* variabe, and z the *axial* variable. At each point, we denote the orthonormal basis vectors corresponding to the cylindrical coordinate system by $\mathbf{e}_v$, $\mathbf{e}_\phi$, and $\mathbf{e}_z$, respectively. The inactive part Γ_0 of the boundary consists of the two superconducting surfaces $v = a$ and $v = b$ with $0 < z < L$ (the surface of the central conductor and the outer surface), and the active part Γ_1 of the boundary consists of the two end sections where $z = 0$ and $z = L$.

We shall not treat the full Maxwell's equations, but only one important special case where we have some additional symmetry. Both the electric field intensity $\mathbf{E}$ and the magnetic field intensity $\mathbf{H}$ are supposed to be *transversal*, i.e., they are orthogonal to the z-axis. In the literature this is called the TEM mode (this stands for "transversal electric magnetic"); see, for example, [24]. In addition we assume *rotational symmetry*, i.e., all the variables are independent of the azimuthal variable ϕ. Thus, we may write the fields in the form

$$\begin{aligned} \mathbf{E}(v,z,t) &= E_v(v,z,t)\mathbf{e}_v + E_\phi(v,z,t)\mathbf{e}_\phi, \\ \mathbf{H}(v,z,t) &= H_v(v,z,t)\mathbf{e}_v + H_\phi(v,z,t)\mathbf{e}_\phi, \end{aligned} \qquad a \le v \le b, \quad 0 \le z \le L.$$

Using the standard formulas for the gradient, divergence, and rotor in cylindrical coordinates (see, for example, Arfken and Weber [3]), we get

$$\begin{aligned} \operatorname{div}\mathbf{E} &= \frac{1}{v}\frac{\partial}{\partial v}(vE_v), \qquad \operatorname{div}\mathbf{H} = \frac{1}{v}\frac{\partial}{\partial v}(vH_v), \\ \operatorname{rot}\mathbf{E} &= -\mathbf{e}_v\frac{\partial}{\partial z}E_\phi + \mathbf{e}_\phi\frac{\partial}{\partial z}E_v + \mathbf{e}_z\frac{1}{v}\frac{\partial}{\partial v}(vE_\phi), \\ \operatorname{rot}\mathbf{H} &= -\mathbf{e}_v\frac{\partial}{\partial z}H_\phi + \mathbf{e}_\phi\frac{\partial}{\partial z}H_v + \mathbf{e}_z\frac{1}{v}\frac{\partial}{\partial v}(vH_\phi). \end{aligned}$$

The two divergence conditions $\operatorname{div}(\varepsilon\mathbf{E}) = 0$ and $\operatorname{div}(\mu\mathbf{H}) = 0$ with constant ε and μ imply that vE_v and vH_v do not depend on v, and hence

$$E_v(v,z,t) = \frac{a}{v}E_v(a,z,t), \qquad H_v(v,z,t) = \frac{a}{v}H_v(a,z,t).$$

The two equations $\frac{\partial}{\partial t}(\varepsilon\mathbf{E}) = \operatorname{rot}\mathbf{H}$ and $\frac{\partial}{\partial t}(\mu\mathbf{H}) = -\operatorname{rot}\mathbf{E}$ with constant ε and μ imply that both $\operatorname{rot}\mathbf{E}$ and $\operatorname{rot}\mathbf{H}$ are perpendicular to the z-axis, and hence neither vE_ϕ nor vH_ϕ depends on v. Therefore

$$E_\phi(v,z,t) = \frac{a}{v}E_\phi(a,z,t), \qquad H_\phi(v,z,t) = \frac{a}{v}H_\phi(a,z,t).$$

According to (5.7) we must have $E_\phi(a,z,t) = 0$, and so $E_\phi = 0$. The remaining two components of the equations $\frac{\partial}{\partial t}(\varepsilon\mathbf{E}) = \operatorname{rot}\mathbf{H}$ and $\frac{\partial}{\partial t}(\mu\mathbf{H}) = -\operatorname{rot}\mathbf{E}$ in the radial and azimuthal directions give

$$\varepsilon\frac{\partial}{\partial t}E_v(a,z,t) = -\frac{\partial}{\partial z}H_\phi(a,z,t), \qquad \mu\frac{\partial}{\partial t}H_\phi(a,z,t) = -\frac{\partial}{\partial z}E_v(a,z,t), \tag{6.3}$$

$$\frac{\partial}{\partial t}H_v(a,z,t) = \frac{\partial}{\partial z}H_v(a,z,t) = 0. \tag{6.4}$$

Thus, $H_v(v,z,t) = \frac{a}{v}H_v(a,z,t)$ does not depend on z or t. (This agrees with our earlier observation that $\nu \cdot \gamma_0\mathbf{B}$ is constant in time on Γ_0.) Since this term does not contribute to the dynamical behavior of the equation we normalize it to be zero. This leaves us with a purely radial electric field intensity and a purely tangential magnetic field intensity:

$$\mathbf{E}(v,z,t) = \frac{a}{v}E_v(a,z,t)\mathbf{e}_v, \qquad \mathbf{H}(v,z,t) = \frac{a}{v}H_\phi(a,z,t)\mathbf{e}_\phi.$$

As is well known, the solution of (6.3) can be parametrized by the d'Alembert formula

$$\begin{aligned} H_\phi(a,z,t) &= h_r(z-ct) + h_l(z+ct), \\ E_v(a,z,t) &= \eta h_r(z-ct) - \eta h_l(z+ct), \end{aligned} \tag{6.5}$$

where

$$c := \frac{1}{\sqrt{\varepsilon\mu}}, \qquad \eta := \sqrt{\frac{\mu}{\varepsilon}}$$

are the speed of light in the filling material and the characteristic impedance, respectively, and h_r and h_l can be chosen arbitrarily (subject to the smoothness conditions that we want to impose on the solution). Here h_r and h_l are the magnetic components of waves traveling to the right and left, respectively.

We take the input and the output to be given by (5.8) and (5.9), respectively, but allow the coefficient r to be different at the two end surfaces. However, because of the extra symmetry conditions that we have imposed on the system, the input and output also must have a special structure, and they cannot be arbitrary functions in $L^2(\Gamma_1)$. Instead both the input u and the output y must be scalar multiples of the radial field $\frac{a}{r}\mathbf{e}_r$ at each end of the cable. Thus, if we denote $\Gamma_1 = \Gamma^0 \cup \Gamma^L$, where Γ^0 is the left and Γ^L is the right end section of the cable, then

$$\begin{aligned} u(v,t)|_{\Gamma^0} &= u_0(t)\frac{a}{v}\mathbf{e}_v, \qquad & u(v,t)|_{\Gamma^L} &= u_L(t)\frac{a}{v}\mathbf{e}_v, \\ y(v,t)|_{\Gamma^0} &= y_0(t)\frac{a}{v}\mathbf{e}_v, \qquad & y(v,t)|_{\Gamma^L} &= y_L(t)\frac{a}{v}\mathbf{e}_v, \end{aligned}$$

where u_0, u_L, y_0, and y_L are scalar functions. At Γ^0 the outward normal is $\nu_0 = -\mathbf{e}_z$, and at Γ^L it is $\nu_L = \mathbf{e}_z$. Combining these facts with (5.8) and (5.9) we get

$$\begin{aligned} \sqrt{2}u_0(t) &= r_0 H_\phi(a,0,t) + E_v(a,0,t), \quad & \sqrt{2}u_L(t) &= -r_L H_\phi(a,L,t) + E_v(a,L,t), \\ \sqrt{2}y_0(t) &= r_0 H_\phi(a,0,t) - E_v(a,0,t), \quad & \sqrt{2}y_L(t) &= -r_L H_\phi(a,0,t) - E_v(a,L,t), \end{aligned}$$

which combined with (6.5) gives

$$\begin{aligned} u_0(t) &= \frac{1}{\sqrt{2}}(r_0+\eta)\, h_r(-ct) + \frac{1}{\sqrt{2}}(r_0-\eta)\, h_l(ct), \\ u_L(t) &= -\frac{1}{\sqrt{2}}(r_L-\eta)\, h_r(L-ct) - \frac{1}{\sqrt{2}}(r_L+\eta)\, h_l(L+ct), \\ y_0(t) &= \frac{1}{\sqrt{2}}(r_0-\eta)\, h_r(-ct) + \frac{1}{\sqrt{2}}(r_0+\eta)\, h_l(ct), \\ y_L(t) &= -\frac{1}{\sqrt{2}}(r_L+\eta)\, h_r(L-ct) - \frac{1}{\sqrt{2}}(r_L-\eta)\, h_l(L+ct). \end{aligned}$$

Thus, if we choose $r_0 = r_L = \eta$, then the amplitudes of the incoming waves through Γ^0 and Γ^L are proportional to u_0 and u_L, respectively, and the amplitudes of the outgoing waves through Γ^0 and Γ^L are proportional to y_0 and y_L, respectively. For all other values of r_0 or r_L there will be reflections. See [24, section 10.15] for a detailed discussion of the nature of these reflections for different values of r_0 and r_L.

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