

# Affine Input/State/Output Representations of State/Signal Systems

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**Abstract**—A linear state/signal system in discrete time has a state space  $\mathcal{X}$  and a signal space  $\mathcal{W}$ , where the state space is used to represent internal properties of the system, and the signal space describes interactions with the surrounding world. It resembles an input/state/output system apart from the fact that inputs and outputs are not separated from each other. By decomposing the signal space  $\mathcal{W}$  into a direct sum of an input space  $\mathcal{U}$  and an output space  $\mathcal{Y}$  one gets a standard input/state/output system, provided the decomposition is *admissible*. Here we discuss the nonadmissible case. Instead of ordinary input/state/output representations of the system we then get right and left affine representations, both of the system itself, and of the corresponding transfer function. In particular, in the case of a *passive* system we get right and left coprime representations of the generalized transfer functions corresponding to nonadmissible decompositions of the signal space, and we end up with transfer functions which are, e.g., generalized Potapov or Nevanlinna class functions.

The evolution of a *linear discrete time-invariant s/s* (= state/signal) system  $\Sigma$  with a Hilbert state space  $\mathcal{X}$  and a Kreĭn signal space  $\mathcal{W}$  is described by the system of equations

$$x(n+1) = F \begin{bmatrix} x(n) \\ w(n) \end{bmatrix}, \quad n \in \mathbb{Z}^+, \quad x(0) = x_0, \quad (1)$$

where the initial state  $x_0 \in \mathcal{X}$  may be arbitrary and  $F$  is a bounded linear operator with a closed domain  $\mathcal{D}(F) \subset [\mathcal{X}_W]$  ( $\mathbb{Z}^+ = 0, 1, 2, \dots$ ). By a *trajectory*  $(x(\cdot), w(\cdot))$  of this system we mean pairs of sequences  $x(n) \in \mathcal{X}$  and  $w(n) \in \mathcal{W}$  satisfying (1). If  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  is any i/o (= input/output) decomposition of  $\mathcal{W}$  as the direct sum of an input space  $\mathcal{U}$  and an output space  $\mathcal{Y}$  then it is natural to consider i/s/o (= input/state/output) trajectories  $(x(\cdot), u(\cdot), y(\cdot))$  of  $\Sigma$ , where the sequence  $x(\cdot)$  is the same as in (1) and

$$u(n) = P_{\mathcal{U}}^{\mathcal{Y}} w(n), \quad y(n) = P_{\mathcal{Y}}^{\mathcal{U}} w(n), \quad n \in \mathbb{Z}^+;$$

here  $P_{\mathcal{U}}^{\mathcal{Y}}$  is the projection onto  $\mathcal{U}$  along  $\mathcal{Y}$ , and  $P_{\mathcal{Y}}^{\mathcal{U}}$  is the complementary projection. A decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  is called *admissible* if the set of i/s/o trajectories  $(x(\cdot), u(\cdot), y(\cdot))$  has an alternative i/s/o description of the standard form

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \quad n \in \mathbb{Z}^+, \\ x(0) &= x_0, \end{aligned} \quad (2)$$

i.e., if this set coincides with the set of trajectories of the i/s/o system  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ , where  $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in$

$\mathcal{B}([\mathcal{X}_U]; [\mathcal{Y}])$ . In this case we call  $\Sigma_{i/s/o}$  an i/s/o representation of  $\Sigma$ , corresponding to the i/o decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ . The four block function

$$\begin{aligned} &\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} \\ &= \begin{bmatrix} (1_{\mathcal{X}} - zA)^{-1} & z(1_{\mathcal{X}} - zA)^{-1}B \\ C(1_{\mathcal{X}} - zA)^{-1} & zC(1_{\mathcal{X}} - zA)^{-1}B + D \end{bmatrix} \end{aligned} \quad (3)$$

defined on the set  $\Lambda_A$  consisting of those  $z \in \mathbb{C}$  for which  $1_{\mathcal{X}} - zA$  has a bounded inverse (including  $z = \infty$  if  $A$  has a bounded inverse), is called the four block transfer function of  $\Sigma$  corresponding to the i/o decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ . The bottom right block  $\mathfrak{D}(z) = zC(1_{\mathcal{X}} - zA)^{-1}B + D$  is the i/o transfer function of  $\Sigma$  corresponding to this i/o decomposition.

Not every i/o decomposition of  $\mathcal{W}$  is admissible. To be able to treat also the nonadmissible case we introduce right and left affine generalizations of the notions of i/s/o representations and their transfer functions. These are defined for arbitrary i/o decompositions  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ . By a *right affine i/s/o representation* of  $\Sigma$  we mean an i/s/o system

$$\Sigma_{i/s/o}^r = \left( \begin{bmatrix} A' & B' \\ C' & D' \\ C_u & D_u \end{bmatrix}; \mathcal{X}, \mathcal{L}, [\mathcal{Y}_U] \right)$$

(where the new input space  $\mathcal{L}$  is an auxiliary Hilbert space) with the following two properties: 1)  $D' = \begin{bmatrix} D'_y \\ D'_u \end{bmatrix}$  has a bounded left-inverse, and 2)  $\left( x(\cdot), \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix} \right)$  is a trajectory of  $\Sigma$  if and only if  $\left( x(\cdot), \ell(\cdot), \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix} \right)$  is a trajectory of  $\Sigma_{i/s/o}^r$  for some sequence  $\ell(\cdot)$  with values in  $\mathcal{L}$ . By a *left affine i/s/o representation* of  $\Sigma$  we mean an i/s/o system

$$\Sigma_{i/s/o}^l = \left( \begin{bmatrix} A'' & B''_y & B''_u \\ C'' & D''_y & D''_u \end{bmatrix}; \mathcal{X}, [\mathcal{Y}_U], \mathcal{K} \right)$$

(where the new output space  $\mathcal{K}$  is another auxiliary Hilbert space) with the following two properties: 1)  $D'' = \begin{bmatrix} D''_y & D''_u \end{bmatrix}$  has a bounded right-inverse, and 2)  $\left( x(\cdot), \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix} \right)$  is a trajectory of  $\Sigma$  if and only if  $\left( x(\cdot), \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix}, 0 \right)$  is a trajectory of  $\Sigma_{i/s/o}^l$  (i.e., the output is identically zero in  $\mathcal{K}$ ). The (four block or i/o) transfer functions of  $\Sigma_{i/s/o}^r$  and  $\Sigma_{i/s/o}^l$  are called the right, respectively left, (four block or i/o) affine transfer functions

of  $\Sigma$  corresponding to the (possibly non-admissible) i/o decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ . Note, in particular, that the right and left affine i/o transfer functions are now decomposed into  $\mathfrak{D}' = \begin{bmatrix} \mathfrak{D}'_{\mathcal{Y}} \\ \mathfrak{D}'_{\mathcal{U}} \end{bmatrix}$  and  $\mathfrak{D}'' = \begin{bmatrix} \mathfrak{D}''_{\mathcal{Y}} & \mathfrak{D}''_{\mathcal{U}} \end{bmatrix}$ , respectively.

Let

$$\Omega(\Sigma^r_{i/s/o}) = \{z \in \Lambda_{A'} \mid \mathfrak{D}'_{\mathcal{U}}(z) \text{ has a bounded inverse}\},$$

$$\Omega(\Sigma^l_{i/s/o}) = \{z \in \Lambda_{A''} \mid \mathfrak{D}''_{\mathcal{Y}}(z) \text{ has a bounded inverse}\},$$

and let

$$\Omega^r(\Sigma; \mathcal{U}, \mathcal{Y}) \text{ be the union of the above sets } \Omega(\Sigma^r_{i/s/o}),$$

$$\Omega^l(\Sigma; \mathcal{U}, \mathcal{Y}) \text{ be the union of the above sets } \Omega(\Sigma^l_{i/s/o}).$$

Our main results are related to the notions of right and left generalized four block transfer functions of  $\Sigma$  with input space  $\mathcal{U}$  and output space  $\mathcal{Y}$ , defined on the sets  $\Omega^r(\Sigma; \mathcal{U}, \mathcal{Y})$  and  $\Omega^l(\Sigma; \mathcal{U}, \mathcal{Y})$ , respectively, by the formulas

$$\begin{aligned} & \begin{bmatrix} \mathfrak{A}_r(z) & \mathfrak{B}_r(z) \\ \mathfrak{C}_r(z) & \mathfrak{D}_r(z) \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'_{\mathcal{Y}}(z) & \mathfrak{D}'_{\mathcal{Y}}(z) \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \mathfrak{C}'_{\mathcal{U}}(z) & \mathfrak{D}'_{\mathcal{U}}(z) \end{bmatrix}^{-1}, \end{aligned} \quad (4)$$

$$\begin{aligned} & \begin{bmatrix} \mathfrak{A}_l(z) & \mathfrak{B}_l(z) \\ \mathfrak{C}_l(z) & \mathfrak{D}_l(z) \end{bmatrix} \\ &= \begin{bmatrix} 1_{\mathcal{X}} & -\mathfrak{B}''_{\mathcal{Y}}(z) \\ 0 & -\mathfrak{D}''_{\mathcal{Y}}(z) \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{A}''(z) & \mathfrak{B}''_{\mathcal{U}}(z) \\ \mathfrak{C}''(z) & \mathfrak{D}''_{\mathcal{U}}(z) \end{bmatrix}. \end{aligned} \quad (5)$$

Here the entries on the right-hand sides of (4) and (5) are obtained from the four block transfer functions of some right and left affine i/s/o representations  $\Sigma^r_{i/s/o}$  and  $\Sigma^l_{i/s/o}$  of  $\Sigma$  with the property that  $z \in \Omega(\Sigma^r_{i/s/o})$  or  $z \in \Omega(\Sigma^l_{i/s/o})$ , respectively. In particular, the generalized right and left i/o transfer functions are given by

$$\begin{aligned} \mathfrak{D}_r(z) &= \mathfrak{D}'_{\mathcal{Y}}(z) \mathfrak{D}'_{\mathcal{U}}(z)^{-1}, \\ \mathfrak{D}_l(z) &= -\mathfrak{D}''_{\mathcal{Y}}(z)^{-1} \mathfrak{D}''_{\mathcal{U}}(z), \end{aligned} \quad (6)$$

respectively.

**Theorem 1.** *The right-hand side of (4) does not depend on the choice of  $\Sigma^r_{i/s/o}$  as long as  $\Omega(\Sigma^r_{i/s/o}) \ni z$ , and the right-hand side of (5) does not depend on the choice of  $\Sigma^l_{i/s/o}$  as long as  $\Omega(\Sigma^l_{i/s/o}) \ni z$ .*

**Theorem 2.** *The right and left generalized four block transfer functions defined by (4) and (5), respectively, coincide on*

$$\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) = \Omega^r(\Sigma; \mathcal{U}, \mathcal{Y}) \cap \Omega^l(\Sigma; \mathcal{U}, \mathcal{Y})$$

(whenever this set is nonempty). If the decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  is admissible, and if  $A$  is the main operator of the corresponding i/s/o representation of  $\Sigma$ , then

$$\Omega^r(\Sigma; \mathcal{U}, \mathcal{Y}) = \Omega^l(\Sigma; \mathcal{U}, \mathcal{Y}) = \Lambda_A,$$

and the right and left generalized four block transfer functions coincide with the ordinary four block transfer function corresponding to the decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ .

The case where the s/s system  $\Sigma$  is stabilizable, or detectable, or LFT-stabilizable in the sense of [AS05] is of special interest (LFT stands for Linear Fractional Transformation). An i/s/o system  $\Sigma_{i/s/o} = ([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$  is *stable* if the trajectories  $(x(\cdot), u(\cdot), y(\cdot))$  of this system has the property that  $x(\cdot) \in \ell^\infty(\mathcal{X})$  and  $y(\cdot) \in \ell^2(\mathcal{Y})$  whenever  $u(\cdot) \in \ell^2(\mathcal{U})$ . A right or left affine i/s/o representation is stable if it is stable when regarded as an i/s/o system. It is easy to see that the main operator  $A$  of a stable system has the property that  $\mathbb{D} \subset \Lambda_A$  and that its i/o transfer function belongs to  $H^\infty$  over the unit disk  $\mathbb{D}$ . This also applies to right and left affine i/s/o representations.

A s/s system  $\Sigma$  is *stabilizable* if it has a stable right affine i/s/o representation, it is *detectable* if it has a stable left affine i/s/o representation, and it is *LFT-stabilizable* if it has a stable i/s/o representation. Every LFT-stabilizable system is both stabilizable and detectable, since an i/s/o representation of a s/s system can be interpreted both as a left affine and as a right affine i/s/o representation of this system. In particular, every s/s system which is *passive* in the sense of [AS06a] is LFT-stabilizable. The four block (right or left or standard) transfer functions of these stable representations are defined in the full unit disk  $\mathbb{D}$ , and the corresponding right and left i/o transfer functions belong to  $H^\infty$  over  $\mathbb{D}$ . In the LFT-stabilizable case these right and left affine i/o transfer functions are even *right or left coprime in  $H^\infty$* , respectively.

By applying our theory to passive s/s systems we obtain right and left coprime transmission representations of these systems, and in the case where the positive and negative dimensions of the signal space  $\mathcal{W}$  are the same we also obtain right and left coprime impedance representations. The corresponding right and left coprime affine i/o transfer functions will be generalized Potapov and Nevanlinna class functions, respectively.

It is also possible to give an unbounded i/s/o *impedance representation* of a passive s/s system in the case where the *impedance function* is single-valued, but the *values are unbounded maximal accretive operators*. In this representation the bounded block operator  $[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$  is replaced by an unbounded operator, and the theory resembles the continuous-time system node theory presented in [Sta05].

Further details will be given in [AS06b], [AS06c], and [Sta06].

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