H-Passive Linear Discrete Time Invariant State/Signal Systems

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Summary

- Discrete time-invariant i/s/o systems
- H-passivity with different supply rates
- State/signal systems
- H-passive s/s systems
- The KYP inequality
- Signal behaviors
- Passive S/S Systems $\leftrightarrow$ Passive Behaviors
- Realization theory

Discrete time-invariant i/s/o systems

Discrete Time-Invariant I/S/O System

Linear discrete-time-invariant systems are typically modeled as i/s/o (input/state/output) systems of the type

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0, \\ y(n) &= Cx(n) + Du(n), \qquad n \in \mathbb{Z}^+. \end{aligned} \tag{1}$$

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By a trajectory of this system we mean a triple of sequences (u, x, y) satisfying (1).

We denote this system by $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$.

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The system (1) is forward H-passive if all trajectories satisfy the condition

$$\|\sqrt{H}x(n+1)\|_{\mathcal{X}}^2 - \|\sqrt{H}x(n)\|_{\mathcal{X}}^2 \leq \left\langle \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, J \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}, \; n \in \mathbb{Z}^+, \qquad (2)$$

where $H > 0$ and J is a given signature operator ($J = J^* = J^{-1}$).

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The positive quadratic form

$$E_H(x) = \|\sqrt{H}x\|_{\mathcal{X}}^2 = \langle x, Hx \rangle_{\mathcal{X}}$$

is called the storage function (Lyapunov function), and the indefinite bilinear form

$$j(u,y) = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}.$$

is called the supply rate.

The Three Most Common Supply Rates

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(i) The scattering supply rate $j_{\rm sca}(u,y) = \|u\|_{\mathcal{U}}^2 - \|y\|_{\mathcal{Y}}^2$ with signature operator $J_{\rm sca} = \begin{bmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}$.

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(ii) The impedance supply rate $j_{\mathrm{imp}}(u,y) = 2\Re\langle \Psi u, y\rangle_{\mathcal{U}}$ with signature operator $J_{\mathrm{imp}} = \begin{bmatrix} 0 & \Psi \\ \Psi^* & 0 \end{bmatrix}$, where Ψ is a unitary operator $\mathcal{U} \to \mathcal{Y}$.

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(iii) The transmission supply rate $j_{\text{tra}}(u,y) = \langle u, J_{\mathcal{U}} u\rangle_{\mathcal{U}} - \langle y, J_{\mathcal{Y}} y\rangle_{\mathcal{Y}}$ with signature operator $J_{\text{tra}} = \begin{bmatrix} -J_{\mathcal{Y}} & 0 \\ 0 & J_{\mathcal{U}} \end{bmatrix}$, where $J_{\mathcal{Y}}$ and $J_{\mathcal{U}}$ are signature operators in $\mathcal{Y}$ and $\mathcal{U}$, respectively.

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It is possible to combine all these cases into one single setting, called the s/s (state/signal) setting. The idea is to introduce a class of systems which does not distinguish between inputs and outputs.

State/Signal Systems

The Signal Space

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We start by combining the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ into one signal space $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. This signal space has a natural Kreĭn space inner product obtained from the signature operator J in the supply rate j, namely

$$\left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]_{\mathcal{W}} = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y' \\ u' \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}.$$

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The (forward) H-passivity-inequality (2) now becomes (with $w(k) = \left[\begin{smallmatrix} y(k) \\ u(k) \end{smallmatrix}\right]$)

$$\|\sqrt{H}x(k+1)\|_{\mathcal{X}}^2 - \|\sqrt{H}x(k)\|_{\mathcal{X}}^2 \leq [w(k), w(k)]_{\mathcal{W}}, \qquad k \in \mathbb{Z}^+.$$

State/Signal System: Definition

A linear discrete time-invariant s/s system Σ is modelled by a system of equations

$$x(n+1) = F \begin{bmatrix} x(n) \\ w(n) \end{bmatrix}, \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0, \tag{3}$$

Here F is a bounded linear operator with a closed domain $\mathcal{D}(F) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ ($\mathbb{Z}^+ = 0, 1, 2, \ldots$) and a certain additional property.

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In the case of an i/s/o system we take $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, $F\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right] = Ax + Bu$, and
$\mathcal{D}(F) = \left\{ \left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right] \;\middle|\; y = Cx + Du \right\}$.

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The Adjoint State/Signal System

Each state/signal system Σ has an adjoint state/signal system Σ_* with the same state space $\mathcal{X}$ and the Kreĭn signal space $\mathcal{W}_* = -\mathcal{W}$.

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This system is determined by the fact that $(x_*(\cdot), w_*(\cdot))$ is a trajectory of Σ_* if and only if

$$-\langle x(n+1), x_*(0)\rangle_{\mathcal{X}} + \langle x(0), x_*(n+1)\rangle_{\mathcal{X}} + \sum_{k=0}^{n}[w(k), w_*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$

for all trajectories $(x(\cdot), w(\cdot))$ of Σ.

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The adjoint of Σ_* is the original system Σ.

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Fact: Σ is observable if and only Σ_* is controllable.

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Let $H = H^* > 0$.[1] Here H and H^{-1} may be unbounded. A s/s system Σ is

(i) forward H-passive if every trajectory (x, w) of Σ with $x(0) \in \mathcal{D}(\sqrt{H})$ satisfies $x(n) \in \mathcal{D}(\sqrt{H})$ and

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(iv) passive if it is $1_{\mathcal{X}}$-passive ($1_{\mathcal{X}}$ is the identity operator in $\mathcal{X}$).

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The S/S KYP Inequality

It is not difficult to see that a s/s system Σ whose trajectories are defined by (3) is forward H-passive if and only if $H > 0$ is a solution of the generalized s/s KYP (Kalman–Yakubovich–Popov) inequality[2]

$$\|H^{1/2}F\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]\|_{\mathcal{X}}^2 - \|H^{1/2}x\|_{\mathcal{X}}^2 \le [w,w]_{\mathcal{W}}, \quad \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F), \quad x \in \mathcal{D}(H^{1/2}). \tag{4}$$

[2] In particular, in order for the first term in this inequality to be well-defined we require F to map $\{\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F) \mid x \in \mathcal{D}(H^{1/2})\}$ into $\mathcal{D}(H^{1/2})$.

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This inequality is named after Kalman [Kal63], Yakubovich [Yak62], and Popov [Pop61] (who at that time restricted themselves to the finite-dimensional input/state/output case).

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There is a rich literature on this version of the KYP inequality and the corresponding equality; see, e.g., [PAJ91], [IW93], and [LR95], and the references mentioned there.

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Infinite-Dimensional I/S/O KYP Inequality: History

In the seventies the classical results on the i/s/o KYP inequalities were extended to systems with $\dim \mathcal{X} = \infty$ by Yakubovich and his students and collaborators (see [Yak74, Yak75, LY76] and the references listed there).

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An continuous-time example is given in [AS06c] where both H and H^{-1} are unbounded for every generalized solution of the i/s/o KYP inequality. The same example can be converted to discrete time and to also to a s/s setting.

Signal Behaviors

(The time domain counterpart of the frequency domain subspace

$$\left\{ \begin{bmatrix} \hat{y}(z) \\ \hat{u}(z) \end{bmatrix} \;\middle|\; \hat{y}(z) = \mathfrak{D}(z)\hat{u}(z) \right\}.)$$

The Behavior Induced by a State/Signal System

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Easy: $\mathfrak{W}$ is a closed and right-shift invariant subspace of the Fréchet space $\mathcal{W}^{\mathbb{Z}^+}$.

Behavior: Definition

By a (general) behavior[3] on the signal space $\mathcal{W}$ we mean a closed right-shift invariant subspace of the Fréchet space $\mathcal{W}^{\mathbb{Z}^+}$.

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Thus, in particular, the set $\mathfrak{W}$ of all sequences w that are the signal part of some externally generated trajectory (x, w) of a given s/s system Σ is a behavior.

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Thus, in particular, the set $\mathfrak{W}$ of all sequences w that are the signal part of some externally generated trajectory (x, w) of a given s/s system Σ is a behavior.

We call this the behavior induced by Σ, and refer to Σ as a s/s realization of $\mathfrak{W}$, or, in the case where Σ is minimal, as a minimal s/s realization of $\mathfrak{W}$.

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A realizable behavior $\mathfrak{W}$ on the signal space $\mathcal{W}$ has a minimal s/s realization, which is determined by $\mathfrak{W}$ up to pseudo-similarity. (See [AS05, Section 7] for details.)

The Adjoint Behavior

Recall the "orthogonality" between a s/s system Σ and its adjoint Σ_*:

$$-\langle x(n+1), x_*(0)\rangle_{\mathcal{X}} + \langle x(0), x_*(n+1)\rangle_{\mathcal{X}} + \sum_{k=0}^{n} [w(k), w_*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$

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If $\mathfrak{W}$ is induced by Σ, then $\mathfrak{W}_*$ is (realizable and) induced by Σ_*, and the adjoint of $\mathfrak{W}_*$ is the original behavior $\mathfrak{W}$.[4]

[4] Is this statement true or false if $\mathfrak{W}$ is not realizable?

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In particular, every w in the behavior $\mathfrak{W}$ induced by Σ satisfies

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(iii) passive if it is realizable[5] and both forward and backward passive.

[5]We do not know if the realizability assumption is redundant or not.

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Thus, if Σ is backward H_2-passive for at least one H_2, then forward H-passivity implies backward H-passivity for all $H > 0$.

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(iii) says: It is possible to make the resulting system both passive and minimal.

Ordering of Solutions of KYP Inequality

We denote the set of all solutions $H = H^* > 0$ of the KYP inequality by M_Σ, and we let $M_\Sigma^{\min}$ be the set of $H \in M_\Sigma$ for which the system Σ_H in assertion (ii) of Theorem 2 is minimal by $\mathcal{L}_\Sigma^{\min}$.

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Theorem 3. Let Σ be a minimal s/s system with a passive behavior. Then $M_\Sigma^{\min} \neq \emptyset$ and $M_\Sigma^{\min}$ contains a minimal element $H_\circ$ and a maximal element $H_\bullet$, i.e., $H_\circ \preceq H \preceq H_\bullet$ for every $H \in M_\Sigma^{\min}$.

$H_1 \preceq H_2 \Leftrightarrow \mathcal{D}(\sqrt{H_2}) \subset \mathcal{D}(\sqrt{H_1})$ and $\|\sqrt{H_1}x\| \leq \|\sqrt{H_2}x\|$ $\forall x \in \mathcal{D}(\sqrt{H_2})$.

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$E_{H_\circ}(\cdot)$ is the available storage, and $E_{H_\bullet}(\cdot)$ is the required supply (Willems).

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$H_\circ$ is the optimal and $H_\bullet$ is the $*$-optimal solution of the KYP inequality (Arov).

Further Extensions

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Analogous results also hold for the quadratic cost minimization problem and its dual. The advantage with this approach is that we get rid of the finite cost condition. This is current joint work with Mark Opmeer.

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