Passive and Conservative State/Signal Systems

Olof Staffans,
Åbo Akademi University, Finland

Leiden, Dec 16, 2009

Based on work by
Damir Z. Arov,
Mikael Kurula,
Olof Staffans

olof.staffans@abo.fi

http://web.abo.fi/~staffans

The Dynamics Induced by a Boundary Relation

Olof Staffans,
Åbo Akademi University, Finland

Leiden, Dec 16, 2009

Based on work by
Damir Z. Arov,
Mikael Kurula,
Olof Staffans

olof.staffans@abo.fi

http://web.abo.fi/~staffans

State/Signal System

A state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ in the forward time direction has a

state space $\mathcal{X}$ (a Hilbert space),

signal space $\mathcal{W}$ (a Krein space),

and the dynamics of the system is described by the equation

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \right. \tag{1}$$

where the generating subspace V is a closed subspace of the node space $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

$x(t) \in \mathcal{X}$ is the state at time $t \in \mathbb{R}^+$,

$x_0 \in \mathcal{X}$ is the initial state at time zero,

$w(t) \in \mathcal{W}$ is the signal at time $t \in \mathbb{R}^+$.

Example: Boundary Control S/S System

On Monday we discussed the boundary control system

$$\Sigma : \begin{cases} \dot{x}(t) = Lx(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \geq 0; \qquad x(0) = x_0. \tag{2}$$

L is the main operator (always unbounded),
Γ is the boundary operator (also unbounded),
L and Γ have the *same domain*
$\mathrm{Dom}\,(L) = \mathrm{Dom}\,(\Gamma) = \mathrm{Dom}\left(\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]\right) \subset \mathcal{X}$.
We can rewrite this as a state/signal system by defining

$$V := \left\{ \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \in \mathfrak{K} \;\middle|\; x \in \mathrm{Dom}\left(\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]\right) \right\}. \tag{3}$$

Example: Classical I/S/O System

Consider the classical input/state/output system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{4}$$

Here A, B, C, and D are bounded linear operators.
We can rewrite this as a state/signal system by taking $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ $(= \mathcal{Y} \times \mathcal{U})$ and defining

$$V := \left\{ \begin{bmatrix} z \\ x \\ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{aligned} z &= Ax + Bu \\ y &= Cx + Du \end{aligned} \right\}. \tag{5}$$

Example: A System Node

A system node is a construction used in the theory of well-posed (and non-wellposed) linear systems. It has a
state space $\mathcal{X}$ (a Hilbert space),
input space $\mathcal{U}$ (a Hilbert space),
ouput space $\mathcal{Y}$ (a Hilbert space).
It is a closed operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$. The dynamics of a system node is described by

$$\Sigma : \begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = S\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \qquad (6)$$

We can rewrite this as a state/signal system by taking $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ and defining

$$V := \left\{ \left[\begin{smallmatrix} z\\ x\\ \left[\begin{smallmatrix} y\\ u\end{smallmatrix}\right]\end{smallmatrix}\right] \subset \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} z\\ y\end{smallmatrix}\right] = S\left[\begin{smallmatrix} x\\ u\end{smallmatrix}\right] \right\}. \qquad (7)$$

Classical and Generalized Trajectories

Back to the general case:

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \right. \tag{1}$$

- $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory of Σ if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \begin{bmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{X}) \end{bmatrix}$ and (2) holds for all $t \in \mathbb{R}^+$.
- $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a generalized trajectory of Σ if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$ and there exists a sequence of classical trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ such that $x_n \to x$ uniformly on bounded intervals and $w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W})$.

(For the moment L^1_{loc} would also be OK, but later we need L^2_{loc} in the integrated power inequality.)

General Assumptions on the Generating Subspace V

In this talk I focus on state/signal systems which are passive or conservative, as studied in Kurula (2009).
They are well-posed in the sense of Kurula and Staffans (2009).

In the equation describing the dynamics

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \right. \tag{1}$$

I througout require that the present state $x(t)$ and the present signal $w(t)$ determine the value of $\dot{x}(t)$ uniquely. This leads to the condition

$$\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0. \tag{8}$$

This condition says that V always must have a graph representation over its last two components:

$$V = \left\{ \begin{bmatrix} G\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \\ x \\ w \end{bmatrix} \in \mathfrak{K} \;\middle|\; \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathrm{Dom}\,(G) \right\} \tag{9}$$

for some closed operator G. In general G is not densely defined. (G is closed since we assume that V is closed.)

Graph Representation of V over State

If we replace the condition

$$\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0. \qquad (8)$$

by the stronger condition that V has a graph representation over its middle component, then it becomes a boundary control s/s node:

$$V := \left\{ \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \in \mathfrak{K} \;\middle|\; x \in \mathrm{Dom}\left(\begin{bmatrix} L \\ \Gamma \end{bmatrix}\right) \right\}. \qquad (3)$$

In the general passive case there will always exist a direct sum decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ such that V has a graph representation over $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$:

$$V := \left\{ \begin{bmatrix} z \\ x \\ y+u \end{bmatrix} \subset \mathfrak{K} \;\middle|\; u \in \mathcal{U},\; y \in \mathcal{Y},\; \begin{array}{l} \left[\begin{smallmatrix} z \\ u \end{smallmatrix}\right] \in \mathrm{Dom}\,(S)\,, \\ \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right], \end{array} \right\} \tag{10}$$

where S is closed and densely defined. This is a system node representation.

However, it will not have a system node representation with respect to every decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$, only with respect to some decompositions (such as scattering decompositions).

The Kreĭn Signal Space $\mathcal{W}$

- Recall: I take the state space $\mathcal{X}$ to be a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{X}}$ and norm $\|\cdot\|_{\mathcal{X}} = \sqrt{(\cdot,\cdot)_{\mathcal{X}}}$.
- However, I take the signal space $\mathcal{W}$ to be a Kreĭn space, and not a Hilbert space.

Roughly speaking, a Kreĭn space $\mathcal{W}$ is a topological vector space which a (unique) indefinite inner product $[\cdot,\cdot]_{\mathcal{W}}$. It also has a Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}}$ such that

$$[w_1, w_2]_{\mathcal{W}} = (w_1, J_{\mathcal{W}} w_2)_{\mathcal{W}}, \qquad w_1,\ w_2 \in \mathcal{W}. \tag{11}$$

where $J_{\mathcal{W}}$ is a boundedly invertible self-adjoint operator in $\mathcal{W}$ (often taken to be a signature operator, i.e., $J_{\mathcal{W}} = J_{\mathcal{W}}^* = J_{\mathcal{W}}^{-1}$). However, the Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}}$ and the signature operator $J_{\mathcal{W}}$ are not unique! (One can always replace the given inner product $(\cdot,\cdot)_{\mathcal{W}}$ by another equivalent inner product, if one at the same time changes the operator $J_{\mathcal{W}}$ accordingly.)

The Power Inequality

We shall throughout suppose the the s/s system

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \right. \tag{1}$$

has no internal energy sources, or more precisely, it satisfies the power inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|_{\mathcal{X}}^2 \le [w(t), w(t)]_{\mathcal{W}}. \tag{12}$$

Here $\|x(t)\|_{\mathcal{X}}^2$ is the internal energy stored state at time t (= the Hamiltonian), and $[w(t), w(t)]_{\mathcal{W}}$ represents the energy flowing into the system from the outside world. Thus, if we want to allow the energy to flow in both directions, then we must allow the right-hand side to take both postive and negative values, and we cannot replace the indefinite inner product $[\cdot,\cdot]_{\mathcal{W}}$ in $\mathcal{W}$ by a positive definite Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}}$ in $\mathcal{W}$.

The Node Space $\mathfrak{K}$

By carrying out the differentiation in

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \le [w(t), w(t)]_{\mathcal{W}} \tag{12}$$

we get the inequality

$$-(\dot{x}(t), x(t))_{\mathcal{X}} - (x(t), \dot{x}(t))_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} \ge 0. \tag{13}$$

At $t = 0$ the vector $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix}$ can be an arbitrary vector in V, and hence (13) with $t = 0$ implies

$$-(z, x)_{\mathcal{X}} - (x, z)_{\mathcal{X}} + [w, w]_{\mathcal{W}} \ge 0, \qquad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V. \tag{14}$$

This inequality says that V is a nonnegative subspace of the node space $\mathfrak{K}$ with respect to a suitable indefinite inner product!

Define

$$\left[\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix}\right]\right]_{\mathfrak{K}} = \left(\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, J_{\mathfrak{K}} \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix}\right)_{\mathfrak{K}}, \quad J_{\mathfrak{K}} := \begin{bmatrix} 0 & -1_{\mathcal{X}} & 0 \\ -1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & J_{\mathcal{W}} \end{bmatrix}. \tag{15}$$

Then

$$-(z, x)_{\mathcal{X}} - (x, z)_{\mathcal{X}} + [w, w]_{\mathcal{W}} \geq 0, \qquad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \tag{14}$$

says that

$$\left[\left[\begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix}\right]\right]_{\mathfrak{K}} \geq 0, \qquad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V. \tag{16}$$

In other words, V is a nonnegative subspace of the node space $\mathfrak{K}$ with respect to the inner product (15).

Weak Forward Well-Posedness

By integrating the power inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 \le [w(t), w(t)]_{\mathcal{W}} \tag{12}$$

we get a weak forward well-posedness condition

$$\begin{aligned} \|x(t)\|_{\mathcal{X}}^2 &\le \|x(0)\|_{\mathcal{X}}^2 + \int_0^t [w(s), w(s)]_{\mathcal{W}}\, ds \\ &\le \|x(0)\|_{\mathcal{X}}^2 + \|J_{\mathcal{W}}\| \int_0^t \|w(s)\|_{\mathcal{W}}^2\, ds, \qquad t \in \mathbb{R}^+. \end{aligned} \tag{17}$$

Thus, if $w \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{W})$, then x is bounded on each finite interval.
Recall that (12) is equivalent to the nonnegativity of V. Thus, nonnegativity of V implies (17).

However, the inequality

$$\|x(t)\|_{\mathcal{X}}^2 \le \|x(0)\|_{\mathcal{X}}^2 + \|J_{\mathcal{W}}\| \int_0^t \|w(s)\|_{\mathcal{W}}^2 \, ds, \qquad t \in \mathbb{R}^+. \tag{17}$$

does not yet imply that the system

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \right. \tag{1}$$

has any nontrival solution.

Counter example: $V = \{0\}$ is nonnegative, but with this V the only solution is $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \equiv 0$.

Dimension of the Generating Subspace V

We need another condition, besides the nonnegativity of V, which says that "V is large enough to generate interesting trajectories". For example, in the case of the finite-dimensional i/s/o system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{4}$$

the generating subspace V given by

$$V := \left\{ \begin{bmatrix} z \\ x \\ \begin{bmatrix} y \\ u \end{bmatrix} \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \,\middle|\, \begin{matrix} z = Ax + Bu \\ y = Cx + Du \end{matrix} \right\}. \tag{5}$$

has dimension $\dim \mathcal{X} + \dim \mathcal{U}$ and co-dimension $\dim \mathcal{X} + \dim \mathcal{Y}$. (Note that the dimension of the node space $\mathfrak{K} = \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ is $\dim \mathfrak{K} = 2 \times (\dim \mathcal{X}) + \dim \mathcal{W}$.)

To also cover the general (possibly infinite-dimensional) passive case we reformulate this as follows:

Definition

The state/signal system

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \right. \tag{1}$$

is passive (in the forward time direction) if its generating subspace V satisfies the following two conditions:

- $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0$.
- V is a maximal nonnegative subspace of the node space $\mathfrak{K}$.

As we shall see in a moment, passive systems are well-posed.

Well-Posed Input/State/Output Decompositions

Definition

The direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is input/state/output well-posed (in the forward time direction) for the state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ if the following two conditions hold:

- For every $x_0 \in \mathcal{X}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{U})$ there exists a generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$ with $x(0) = x_0$ and $P^{\mathcal{Y}}_{\mathcal{U}} w = u$;
- There exists a positive function K such that every generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$ satisfies

$$\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|P^{\mathcal{U}}_{\mathcal{Y}} w(s)\|^2_{\mathcal{W}}\, ds \le \|x(0)\|^2_{\mathcal{X}} + K(t) \int_0^t \|P^{\mathcal{Y}}_{\mathcal{U}} w(s)\|^2_{\mathcal{W}}\, ds, \quad t \in \mathbb{R}^+. \tag{18}$$

Passive Systems are Well-Posed

Definition

The state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is well-posed (in the forward time direction) if there exists at least one input/state/output well-posed decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of the signal space $\mathcal{W}$.

The following result is proved in Kurula (2009):

Theorem

Every passive s/s system is well-posed.

The Backward Time Direction

Instead of working in the forward time direction we may equally well be working in the backward time direction. In order to get backward well-posedness it is natural to assume that Σ has no internal energy sinks, or more precisely, that the direction of the power inequality is reversed:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 \geq [w(t), w(t)]_{\mathcal{W}}. \tag{19}$$

This condition combined with the appropriate maximality condition will imply well-posedness in the backward time direction.

Definition

The state/signal system

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^{-}, \qquad x(0) = x_0, \right. \tag{20}$$

is passive in the backward time direction if its generating subspace V satisfies the following two conditions:

- $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0$.
- V is a maximal nonpositive subspace of the node space $\mathfrak{K}$.

Definition

The direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is input/state/output well-posed in the backward time direction for the state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ if the following two conditions hold:

- For every $x_0 \in \mathcal{X}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^-; \mathcal{U})$ there exists a generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^-;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^-$ with $x(0) = x_0$ and $P^{\mathcal{Y}}_{\mathcal{U}} w = u$;
- There exists a positive nonincreasing function K such that every generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^-;\mathcal{X}) \\ L^2_{\mathrm{loc}}(\mathbb{R}^-;\mathcal{W}) \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^-$ satisfies

$$\|x(t)\|^2_{\mathcal{X}} + \int_t^0 \|P^{\mathcal{U}}_{\mathcal{Y}} w(s)\|^2_{\mathcal{W}}\, ds \le \|x(0)\|^2_{\mathcal{X}} + K(t) \int_t^0 \|P^{\mathcal{Y}}_{\mathcal{U}} w(s)\|^2_{\mathcal{W}}\, ds, \quad t \in \mathbb{R}^-.$$

Backward Passive Systems are Backward Well-Posed

Definition

The state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is well-posed in the backward time direction if there exists at least one backward input/state/output well-posed decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of the signal space $\mathcal{W}$.

Theorem

Every backward passive s/s system is backward well-posed.

Finite Dimension: Forward $\Leftrightarrow$ Backward Well-Posed

If the system $\Sigma = (V; \mathcal{X}; \mathcal{W})$ is finite-dimensional, i.e., if both $\mathcal{X}$ and $\mathcal{W}$ are finite-dimensional, then
forward well-posed $\Leftrightarrow$ backward well-posed.
However, forward passive $\not\Leftrightarrow$ backward passive.

Infinite Dimension: Forward $\not\Leftrightarrow$ Backward Well-Posed

Warning: A decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ may very well be forward i/s/o-well-posed but not backward i/s/o-well-posed, and conversely.

Fact 1: If $\mathcal{W} = \mathcal{Y}_1 \dotplus \mathcal{U}_1$ and $\mathcal{W} = \mathcal{Y}_2 \dotplus \mathcal{U}_2$ are two forward i/s/o-well-posed decompositions, then $\dim \mathcal{U}_1 = \dim \mathcal{U}_2$ and $\dim \mathcal{Y}_1 = \dim \mathcal{Y}_2$. Thus, every passive s/s system has a well-defined input dimension $\dim_{\mathrm{in}} \mathcal{W}$ and a well-defined output dimension $\dim_{\mathrm{out}} \mathcal{W}$ in the forward time direction, with $\dim_{\mathrm{in}} \mathcal{W} + \dim_{\mathrm{out}} \mathcal{W} = \dim \mathcal{W}$.

Fact 2: If Σ is both forward and backward passive, then the forward input dimension = backward output dimension and the other way around.

Conclusion: A necessary (but not sufficient) condition for the existence of a decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ which is both forward and backward i/s/o-well-posed is that the input and output dimensions of Σ are the same (finite or infinite).

Systems that are Both Forward and Backward Passive

Above we already mentioned the possibility that Σ is both forward passive and backward passive. This means that V is both maximal nonnegative and maximal nonpositive.

Lemma

A subspace V of a Kreĭn space $\mathfrak{K}$ is both maximal nonnegative and maximal nonpositive if and only if $V = V^{[\perp]}$, where

$$V^{[\perp]} = \left\{ \kappa^\dagger \in \mathfrak{K} \;\middle|\; [\kappa, \kappa^\dagger]_{\mathfrak{K}} = 0 \text{ for all } \kappa \in V \right\}. \tag{22}$$

We call $V^{[\perp]}$ the orthogonal companion to V.
A subspace V which satisfies $V = V^{[\perp]}$ is called Lagrangian (or hypermaximal neutral) (or hypermaximal W-symmetric) (or self-adjoint relation) (or unitary relation) (or Dirac structure).

Conservative State/Signal System

Definition

The state/signal system

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \end{cases} \tag{1}$$

is conservative (both in the forward and the backward time directions) if its generating subspace V satisfies the following two conditions:

- $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0$.
- V is a Lagrangian subspace of the node space $\mathfrak{K}$, i.e., $V = V^{[\perp]}$.

Thus, conservative s/s systems are well-posed both in the forward and in the backward time directions.

Different Types of Decompositions of the Signal Space $\mathcal{W}$

The proof of forward well-posedness is based on the use of a scattering representation of Σ. This is an input/state/output representation corresponding to a fundamental decomposition of $\mathcal{W}$.

- A fundamental decomposition of $\mathcal{W}$ is of the type $\mathcal{W} = \mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$, where $\mathcal{W}_-$ is an anti-Hilbert space and $\mathcal{W}_+$ is a Hilbert space with respect to the inner products inherited from $\mathcal{W}$. (Anti-Hilbert means that it becomes a Hilbert space after we change the sign of the inner product, and "$[\dot{+}]$" means that $\mathcal{W}_-$ and $\mathcal{W}_+$ are orthogonal.)
- A Lagrangian decomposition of $\mathcal{W}$ is of the type $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$, where both $\mathcal{Y}$ and $\mathcal{U}$ are Lagrangian subspaces of $\mathcal{W}$ (but they are not orthogonal to each other).
- A general orthogonal decomposition of $\mathcal{W}$ is of the type $\mathcal{W} = \mathcal{Y} \,[\dot{+}]\, \mathcal{U}$, where both $\mathcal{Y}$ and $\mathcal{U}$ are Kreĭn spaces with respect to the the inner products inherited from $\mathcal{W}$.

Well-Posedness of Fundamental Decompositions

Let $\mathcal{W} = \mathcal{W}_- \, [\dot{+}] \, \mathcal{W}_+$ be a fundamental decomposition of $\mathcal{W}$. Let $\mathcal{Y} := |\mathcal{W}_-|$ be the Hilbert space that we get by changing the sign of the inner product in $\mathcal{W}_-$, and let $\mathcal{U} := \mathcal{W}_+$. Then each $w \in \mathcal{W}$ has a unique decomposition $w = y + u$ with $y \in \mathcal{Y}$ and $u \in \mathcal{U}$, and

$$\begin{aligned}[w,w]_{\mathcal{W}} &= [y+u, y+u]_{\mathcal{W}} = [y,y]_{\mathcal{W}} + [u,u]_{\mathcal{W}} \\ &= -(y,y)_{\mathcal{Y}} + (u,u)_{\mathcal{U}} = -\|y\|^2_{\mathcal{Y}} + \|u\|^2_{\mathcal{U}}.\end{aligned}$$

Thus, with respect to this decomposition of $\mathcal{W}$ the integrated forward power inequality

$$\|x(t)\|^2_{\mathcal{X}} \le \|x(0)\|^2_{\mathcal{X}} + \int_0^t [w(s), w(s)]_{\mathcal{W}} \, ds, \qquad t \in \mathbb{R}^+, \quad (17)$$

becomes

$$\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|y(s)\|^2_{\mathcal{W}} \, ds \le \|x(0)\|^2_{\mathcal{X}} + \int_0^t \|u(s)\|^2_{\mathcal{W}} \, ds, \qquad t \in \mathbb{R}^+, \quad (23)$$

The inequality

$$\|x(t)\|_{\mathcal{X}}^2+\int_0^t \|y(s)\|_{\mathcal{W}}^2\,ds \le \|x(0)\|_{\mathcal{X}}^2+\int_0^t \|u(s)\|_{\mathcal{W}}^2\,ds, \qquad t \in \mathbb{R}^+, \tag{23}$$

immediately implies the inequality used in the definition of i/s/o-well-posedness in forward time

$$\begin{aligned} &\|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|P_{\mathcal{Y}}^{\mathcal{U}} w(s)\|_{\mathcal{W}}^2\,ds \\ &\qquad \le \|x(0)\|_{\mathcal{X}}^2 + K(t)\int_0^t \|P_{\mathcal{U}}^{\mathcal{Y}} w(s)\|_{\mathcal{W}}^2\,ds,\ \ t \in \mathbb{R}^+ \end{aligned} \tag{18}$$

with $K(t) \equiv 1$.

Well-Posedness of Fundamental Decompositions

This leads to the following result.

Lemma

Let Σ be a (forward) passive s/s system. Then every fundamental decomposition $\mathcal{W} = \mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ is i/s/o-well-posed for Σ with input space $\mathcal{W}_+$ and output space $\mathcal{W}_-$.

An analogous argument shows that

Lemma

Let Σ be a backward passive s/s system. Then every fundamental decomposition $\mathcal{W} = \mathcal{W}_- \,[\dot{+}]\, \mathcal{W}_+$ is i/s/o-well-posed for Σ with input space $\mathcal{W}_-$ and output space $\mathcal{W}_+$.

In the conservative case both of these lemmas apply. Note that the roles of $\mathcal{W}_-$ and $\mathcal{W}_+$ change when we reverse the direction of time!

Scattering Representations

The i/s/o-well-posedness of the fundamental decomposition $\mathcal{W} = \mathcal{W}_- \, [\dot{+}] \, \mathcal{W}_+$ implies that the inclusion

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \right. \tag{1}$$

can be rewritten in the following i/s/o form, with input space $\mathcal{U} = \mathcal{W}_+$ and output space $\mathcal{Y} = -\mathcal{W}_-$:

$$\Sigma_{i/s/o} : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \tag{24}$$

where the (forward) system node S is closed and densely defined. This operator has a number of additional properties. See Staffans (2005) or Kurula (2009) for details.

We call $\Sigma_{i/s/o}$ a (forward) scattering representation of Σ.

In general a Lagrangian decomposition need not be (forward or backward) i/s/o-well-posed. If it is (forward or backward) i/s/o-well-posed, then Σ again has a (forward or backward) system node representation

$$\Sigma_{i/s/o} : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \tag{24}$$

where the (forward or backward) system node S is closed and densely defined.

In this case we call $\Sigma_{i/s/o}$ a (forward or backward) impedance representation of Σ.

Boundary Triplet is an Impedance Representation

Note that the equations arising from a boundary controlled Schrödinger equation (or boundary triplet)

$$\Sigma_{i/s/o}: \begin{cases} \dot{x}(t) = iA^*x(t), \\ u(t) = \Gamma_1 x(t), & t \geq 0 \\ y(t) = \Gamma_2 x(t), \\ \quad x(0) = x_0. \end{cases} \qquad (25)$$

can be interpreted as an impedance representation of the corresponding s/s boundary control system

$$\Sigma: \begin{cases} \dot{x}(t) = iA^*x(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \geq 0; \qquad x(0) = x_0. \qquad (2)$$

with respect to the Lagrangian decomposition $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ 0 \end{smallmatrix}\right] \dotplus \left[\begin{smallmatrix} 0 \\ \mathcal{U} \end{smallmatrix}\right]$ of $\mathcal{W}$ (if this decomposition is i/s/o-well-posed).

Transmission Representations

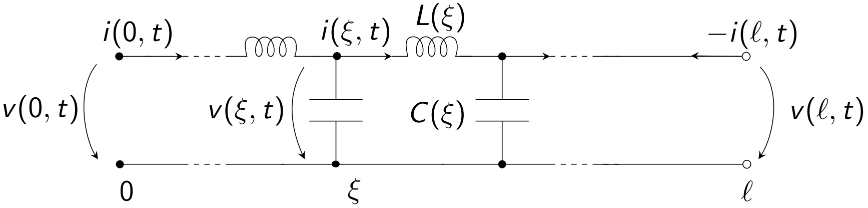
In general an arbitrary orthogonal decomposition need not be (forward or backward) i/s/o-well-posed. If it is (forward or backward) i/s/o-well-posed, then Σ again has a (forward or backward) system node representation

$$\Sigma_{i/s/o} : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \tag{24}$$

where the (forward or backward) system node S is closed and densely defined.

In this case we call $\Sigma_{i/s/o}$ a (forward or backward) transmission (chain scattering) representation of Σ.

A special case of this is the inverse scattering setting, where one uses a fundamental decomposition, but interchange the input and output spaces.



$$\frac{\partial}{\partial t}\begin{bmatrix} v(\xi,t) \\ i(\xi,t)\end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C(\xi)}\frac{\partial}{\partial \xi} \\ -\frac{1}{L(\xi)}\frac{\partial}{\partial \xi} & 0 \end{bmatrix}\begin{bmatrix} v(\xi,t) \\ i(\xi,t)\end{bmatrix}, \qquad (\xi,t)\in[0,\ell]\times\mathbb{R}^+,$$

$$w(t) = \begin{bmatrix} v(0,t) \\ i(0,t) \\ v(\ell,t) \\ -i(\ell,t)\end{bmatrix}, \qquad t\in\mathbb{R}^+,$$

$$v(\xi,0) = v_0(\xi), \quad i(\xi,0) = i_0(\xi), \qquad \xi\in[0,\ell].$$

We take $x(t) = \begin{bmatrix} v(\cdot,t) \\ i(\cdot,t)\end{bmatrix}$, $t\in\mathbb{R}^+$, and $x_0 = \begin{bmatrix} v_0(\cdot) \\ i_0(\cdot)\end{bmatrix}$.

Non-Well-Posed Input/Output Decompositions

In the impedance and transmission cases it is convenient to introduce another notion of input/state/output admissible decompositions $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ of the signal space $\mathcal{W}$ which are not i/s/o-well-posed.

Idea: The decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is i/s/o-admissible with input space $\mathcal{U}$ and output space $\mathcal{Y}$ if Σ has a generalized i/s/o transfer function with respect to this decomposition.

How do we define the generalized i/s/o transfer function?

Ordinary I/S/O Transfer function

Suppose that x, $\dot{x}$, y, and u are all Laplace transformable, with the Laplace transforms converging in the full right half-plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \Re\lambda > 0\}$, and take Laplace transforms in the i/s/o equation

$$\Sigma_{i/s/o} : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \qquad (24)$$

to get

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \qquad \lambda \in \mathbb{C}^+. \qquad (26)$$

At least in the case of a scattering representation of a passive system it is possible to solve $\left[\begin{smallmatrix}\hat{x}(\lambda)\\ \hat{y}(\lambda)\end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix}x_0\\ \hat{u}(\lambda)\end{smallmatrix}\right]$ from the identity

$$\begin{bmatrix}\lambda\hat{x}(\lambda)-x_0\\ \hat{y}(\lambda)\end{bmatrix} = S\begin{bmatrix}\hat{x}(\lambda)\\ \hat{u}(\lambda)\end{bmatrix}, \qquad \lambda\in\mathbb{C}^+. \tag{26}$$

The map $\left[\begin{smallmatrix}x_0\\ \hat{u}(\lambda)\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\hat{x}(\lambda)\\ \hat{y}(\lambda)\end{smallmatrix}\right]$ turns out to be a bounded linear operator, that we denote by $\left[\begin{smallmatrix}\widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\end{smallmatrix}\right]$. Thus,

$$\begin{bmatrix}\hat{x}(\lambda)\\ \hat{y}(\lambda)\end{bmatrix} = \begin{bmatrix}\widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\end{bmatrix}\begin{bmatrix}x_0\\ \hat{u}(\lambda)\end{bmatrix}, \qquad \lambda\in\mathbb{C}^+. \tag{27}$$

The operator $\left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$ is called the input/state/output transfer function of $\Sigma_{i/s/o}$.

Ordinary I/S/O Transfer function

In particular, if S is a bounded operator with $\mathrm{Dom}\,(S) = \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$, then S can be written in the block matrix form $S = \left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$, and we can compute the i/s/o transfer function explicitly in terms of A, B, C, and D:

$$\begin{bmatrix}\widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\end{bmatrix} = \begin{bmatrix}(\lambda - A)^{-1} & (\lambda - A)^{-1}B\\ C(\lambda - A)^{-1} & C(\lambda - A)^{-1}B + D\end{bmatrix}, \qquad \lambda \in \mathbb{C}^+. \tag{28}$$

Afternoon competition #1: Who can make the longest list of the different names that different people (such as Derkach, Malamud, Grubb, Behndt, Arlinskii, Langer, Zwart, Ran, de Snoo, Kreĭn, Weyl, Lax, Phillips, Calkin, Nevanlinna, van der Schaft, etc.) use for the four different components of the i/s/o transfer function $\left[\begin{smallmatrix}\widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\end{smallmatrix}\right]$!

Generalized I/S/O Transfer function

To get the generalized i/s/o transfer function we rewrite the identity

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \qquad \lambda \in \mathbb{C}^+. \tag{26}$$

so that it uses the generating subspace V instead of the system node S.

Generalized I/S/O Transfer function

Suppose that x, $\dot{x}$, and w are all Laplace transformable, with the Laplace transforms converging in the full right half-plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \Re\lambda > 0\}$, and take Laplace transforms in the s/s equation

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \right. \tag{1}$$

to get

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \widehat{w}(\lambda) \end{bmatrix} \in V, \qquad \lambda \in \mathbb{C}^+. \tag{29}$$

Generalized I/S/O Transfer function

Let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a direct sum decomposition of $\mathcal{W}$. The domain of the generalized i/s/o transfer function with respect to this decomposition and the function itself are defined by

$$\mathrm{Dom}\left(\begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}\right) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} \text{for all } \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \text{ there exists} \\ \text{a unique pair } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] \\ \text{such that } \begin{bmatrix} \lambda x - x_0 \\ x \\ u + y \end{bmatrix} \in V \end{array} \right\}, \tag{30}$$

$$\begin{cases} \text{For } \lambda \in \mathrm{Dom}\left(\begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}\right), \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \\ \text{where } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \text{ is given by (30).} \end{cases} \tag{31}$$

Admissible I/S/O Decomposition

Definition

The direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is (weakly) forward i/s/o-admissible with input space $\mathcal{U}$ and output space $\mathcal{Y}$ if $\mathrm{Dom}\left(\left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]\right) \cap \mathbb{C}^+ \neq \emptyset$.

Definition

The direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is (weakly) backward i/s/o-admissible with input space $\mathcal{U}$ and output space $\mathcal{Y}$ if $\mathrm{Dom}\left(\left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]\right) \cap \mathbb{C}^- \neq \emptyset$.

Boundary Triplet = Impedance Representation

Theorem

The Lagrangian decomposition $\mathcal{W} = \begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 \\ \mathcal{U} \end{bmatrix}$ is both forward and backward i/s/o-admissible for the s/s boundary control system

$$\Sigma : \begin{cases} \dot{x}(t) = iA^*x(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \geq 0; \qquad x(0) = x_0. \tag{2}$$

constructed from the boundary controlled Schrödinger equation

$$\Sigma_{i/s/o} : \begin{cases} \dot{x}(t) = iA^*x(t), \\ u(t) = \Gamma_1 x(t), \qquad t \geq 0 \\ y(t) = \Gamma_2 x(t), \\ \quad x(0) = x_0. \end{cases} \tag{25}$$

with respect to the Lagrangian decomposition $\mathcal{W} = \begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 \\ \mathcal{U} \end{bmatrix}$.

Boundary Relations = Non-Admissible Impedance Representations?

Up to now I have used the definition of a boundary triplet from Gorbachuk and Gorbachuk (1991). The title of this workshop is boundary relations, not boundary spaces.

Open Question: To what extent is it true that a boundary relation can be identified with a possibly non-admissible i/o decomposition of the signal space of a conservative boundary control s/s system?

Conjecture: To a very large extent!?

Transfer Relations

It is easy to extend the notion of a generalized i/s/o transfer function so that it becomes an i/s/o transfer relation instead.
Recall: Let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a direct sum decomposition of $\mathcal{W}$. The domain of the generalized i/s/o transfer function with respect to this decomposition and the function itself are defined by

$$\mathrm{Dom}\left(\begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}\right) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} \text{for all } \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \text{ there exists} \\ \text{a unique pair } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] \\ \text{such that } \left[\begin{smallmatrix} \lambda x - x_0 \\ x \\ u + y \end{smallmatrix}\right] \in V \end{array} \right\}, \tag{30}$$

$$\begin{cases} \text{For } \lambda \in \mathrm{Dom}\left(\begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}\right), \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \\ \text{where } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \text{ is given by (30).} \end{cases} \tag{31}$$

Definition

The (full) state/signal frequency domain behavior is the family of subspaces $\{\widehat{\mathfrak{F}}(\lambda)\}_{\lambda\in\mathbb{C}}$ of the node space $\mathfrak{K}$, where each $\mathfrak{F}(\lambda)$ is given by

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ \begin{bmatrix} x \\ x_0 \\ w \end{bmatrix} \;\middle|\; \begin{bmatrix} \lambda x - x_0 \\ x \\ w \end{bmatrix} \in V \right\}. \tag{32}$$

The definition of the generalized i/s/o transfer function $\begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}$ can be reformulated as follows:

Definition

Let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a direct sum decomposition of $\mathcal{W}$. The domain of the generalized i/s/o transfer function $\begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}$ with respect to this decomposition consists of those points $\lambda \in \mathbb{C}$ for which the state/signal frequency domain behavior $\widehat{\mathfrak{F}}(\lambda)$ is the graph of a bounded linear operator $\begin{bmatrix} 0 \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ 0 \\ \mathcal{Y} \end{bmatrix}$, and $\begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}$ is defined to be this operator.

Input/State/Output Transfer Relations

By definition, the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is non-i/s/o-admissible both in the forward and backward time directions if and only if there does not exist a single point $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ such that $\widehat{\mathfrak{F}}(\lambda)$ is the graph of a bounded linear operator $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$.

However, we can always interpret $\widehat{\mathfrak{F}}(\lambda)$ as the graph of a closed relation $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$. With this interpretation it makes sense to call this relation the i/s/o transfer relation at the point $\lambda \in \mathbb{C}$. It is defined for all $\lambda \in \mathbb{C}$.

Observe that the subspace $\widehat{\mathfrak{F}}(\lambda)$ is a state/signal invariant, i.e., it is independent of the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$.

Thus, although the s/s system Σ has many different transfer relations (corresponding to different decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$), the graphs of all possible transfer relations are the same! They are simply different representations of the i/s/o frequency domain behavior with respect to different decompositions of the signal space $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$.

External Cayley Transform and Chain Scattering Transform

Many of the standard transformations that are used in i/s/o theory can be interpreted as simple changes of i/o deocompositions $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ in the corresponding state/signal system Σ.

- The external Cayley transform (or main transformation by Derkach et al. (2006)) describes what happens when you replace an impedance representation by a scattering representation (or the other way around) of the s/s system Σ
- The chain scattering (or Potapov–Gintzburg) transform describes what happens when you replace a transmission representation by a scattering representation (or the other way around) of the s/s system Σ.

Livšic (1973) uses the name diagonal transformation for the transformation from a direct sum decomposition of $\mathcal{W}$ to a fundamental decomposition (and thus from the original i/s/o representation to a scattering representation of Σ).

Afternoon competition #2: Who can make the longest list of different names for these coordinate changes?

Discrete Time State/Signal Systems

There is also an internal Cayley Transform that can be used to map a continuous time s/s system into a discrete time s/s system and back. By using this transfrorm it is possible to convert the discrete time s/s results in Arov and Staffans (2007, 2009a,b,a) into corresponding continuous time results.

Afternoon Competition #3: How many names ....

Acknowledgement

This talk is based partly on Kurula (2009) and partly on the introduction in the new book manuscript Arov and Staffans (2011–2012).

References

D. Z. Arov and O. J. Staffans. State/signal linear time-invariant systems theory. Part II: Passive discrete time systems. *Internat. J. Robust Nonlinear Control*, 17:497–548, 2007.

D. Z. Arov and O. J. Staffans. A Kreĭn space coordinate free version of the de Branges complementary space. *J. Funct. Anal.*, 256:3892–3915, 2009c.

D. Z. Arov and O. J. Staffans. Two canonical passive state/signal shift realizations of passive discrete time behaviors. *J. Funct. Anal.*, 257:2573–2634, 2009b.

D. Z. Arov and O. Staffans. Canonical conservative state/signal shift realizations of passive discrete time behaviors. Manuscript available at http://users.abo.fi/staffans/, 2009c.

D. Z. Arov and O. J. Staffans. *Passive Linear State/Signal Systems*. 2011–2012. In preparation.

V. Derkach, S. Hassi, M. Malamud, and H. de Snoo. Boundary relations and their Weyl families. *Trans. Amer. Math. Soc.*, 358: 5351–5400, 2006.

V. I. Gorbachuk and M. L. Gorbachuk. *Boundary value problems for operator differential equations*, volume 48 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. ISBN 0-7923-0381-4. translation and revised from the 1984 Russian original.

M. Kurula. Passive continuous time state/signal systems. Manuscript available at http://users.abo.fi/mkurula/, 2009.

M. Kurula and O. J. Staffans. Well-posed state/signal systems in continuous time. *Complex Anal. Oper. Theory*, 2009. To appear.

M. S. Livšic. *Operators, Oscillations, Waves (Open Systems)*, volume 34 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1973.

O. J. Staffans. *Well-Posed Linear Systems*. Cambridge University Press, Cambridge and New York, 2005.

Generalized Input/Output Transfer Functions

Above I have looked at generalized input/state/output transfer functions, which have four components:
$\widehat{\mathfrak{A}}(\lambda) : x_0 \to \hat{x}(\lambda)$,
$\widehat{\mathfrak{B}}(\lambda) : \hat{u}(\lambda) \to \hat{x}(\lambda)$,
$\widehat{\mathfrak{C}}(\lambda) : x_0 \to \hat{y}(\lambda)$,
$\widehat{\mathfrak{D}}(\lambda) : \hat{u}(\lambda) \to \hat{y}(\lambda)$.

Often one ignores those part of this transfer function which involve the state, and only studies the input/output transfer function $\widehat{\mathfrak{D}}(\lambda) : \hat{u}(\lambda) \to \hat{y}(\lambda)$.

The generalized i/o transfer function $\widehat{\mathfrak{D}}$ can be studied directly by proceeding in the same way as before, but simply taking $x_0 = 0$ and “eliminating” the state $\hat{x}(\lambda)$.

Generalized Input/Output Transfer Functions

Let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a direct sum decomposition of $\mathcal{W}$. The domain of the generalized i/o transfer function with respect to this decomposition and the function itself are defined by

$$\mathrm{Dom}\left(\widehat{\mathfrak{D}}\right) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} \text{for each } u \in \mathcal{U} \text{ there exist} \\ \text{some } x \in \mathcal{X} \text{ and a unique} \\ y \in \mathcal{Y} \text{ such that } \left[\begin{smallmatrix} \lambda x \\ x \\ u+y \end{smallmatrix}\right] \in V \end{array} \right\}, \qquad (33)$$

$$\left\{ \begin{array}{l} \text{For } \lambda \in \mathrm{Dom}\left(\widehat{\mathfrak{D}}\right),\ \widehat{\mathfrak{D}}(\lambda) u = y, \\ \text{where } y \text{ is the unique vector in (33).} \end{array} \right. \qquad (34)$$

Frequency Domain Signal Behavior

Definition

The signal (or manifest) frequency domain behavior is the family of subspaces $\{\widehat{\mathfrak{W}}(\lambda)\}_{\lambda\in\mathbb{C}}$ of the signal space $\mathcal{W}$, where each $\mathfrak{W}(\lambda)$ is given by

$$\widehat{\mathfrak{W}}(\lambda) = \left\{ w \in \mathcal{W} \;\middle|\; \begin{bmatrix} \lambda x \\ x \\ w \end{bmatrix} \in V \text{ for some } x \in \mathcal{X} \right\}. \qquad (35)$$

The definition of the generalized i/o transfer function $\widehat{\mathfrak{D}}$ can be reformulated as follows:

Definition

Let $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ be a direct sum decomposition of $\mathcal{W}$. The domain of the generalized i/o transfer function $\widehat{\mathfrak{D}}$ with respect to this decomposition consists of those point $\lambda \in \mathbb{C}$ for which the signal frequency domain behavior $\widehat{\mathfrak{W}}(\lambda)$ is the graph of a bounded linear operator $\mathcal{U} \to \mathcal{Y}$, and $\widehat{\mathfrak{D}}(\lambda)$ is defined to be this operator.

Input/Output Admissibility

The notion of input/output admissibility (forward or backward) is defined in the same way as the notion of input/state/output admissibility, replacing the generalized i/s/o transfer function by the i/o transfer function.

The notion of an input/output transfer relation is defined in the same way as the notion of an input/state/output transfer relation by replacing the (full) state/signal frequency domain behavior by the (manifest) signal frequency domain behavior.

Clearly i/s/o-admissibility impliers i/o-admissibility.

Conjecture: For a passive s/s system these two admissibility notions are actually the same.