Conservative Boundary Control Systems

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The Dynamics induced by a Boundary Triplet

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Boundary Control I/S/O System

A boundary control input/state/output system can be written in the form

$$\Sigma_{i/s/o} : \begin{cases} \dot{x}(t) = Lx(t), & \\ u(t) = \Gamma_0 x(t), & t \geq 0 \\ y(t) = \Gamma_1 x(t), & \\ \quad x(0) = x_0. & \end{cases} \tag{1}$$

$\mathcal{X}$ is the state space, $x(t) \in \mathcal{X}$, $x_0 \in \mathcal{X}$,
$\mathcal{U}$ is the input space, $u(t) \in \mathcal{U}$,
$\mathcal{Y}$ is the output space, $y(t) \in \mathcal{Y}$ (these are Hilbert spaces),
L is the main operator (always unbounded),
Γ_0 is the boundary control operator (surjective and unbounded),
Γ_1 is the observation operator (can be bounded or unbounded).

In order for these equations to generate a dynamical system we need at least the following assumptions:

- Γ_0 is surjective and strictly unbounded in the sense that $\mathrm{Ker}\,(\Gamma_0)$ is dense in $\mathcal{X}$,
- $A := L|_{\mathrm{Ker}(\Gamma_0)}$ generates a C_0 semigroup e^{At}, $t \geq 0$.

The first equation in (1) can be rewritten in the form

$$\dot{x}(t) = A_{-1}x(t) + Bu(t), \quad t \geq 0,$$

where $A_{-1} \colon \mathcal{X} \to \mathcal{X}_{-1}$ is a certain extension of A with values in an *extrapolation space* $\mathcal{X}_{-1}$, and B maps into $\mathcal{X}_{-1}$. See, e.g., Staffans (2005) for details.

A boundary control state/signal system is similar to a boundary control i/s/o system, but we no longer specify which part of the "boundary signal" $w(t) := \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$ is the input, and which part is the output. After replacing $\begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}$ by Γ we get an equation of the type

$$\Sigma : \begin{cases} \dot{x}(t) = Lx(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \geq 0; \qquad x(0) = x_0. \qquad (2)$$

$\mathcal{X}$ is the *state space*, $x(t) \in \mathcal{X}$, $x_0 \in \mathcal{X}$, $\mathcal{X}$ is a Hilbert space,
$\mathcal{W}$ is the *signal space*, $w(t) \in \mathcal{W}$, $\mathcal{W}$ is a Kreĭn space,
L is the main operator (always unbounded),
Γ is the boundary operator (also unbounded),
L and Γ have the *same domain*
$\mathrm{Dom}\,(L) = \mathrm{Dom}\,(\Gamma) = \mathrm{Dom}\,\left(\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]\right) \subset \mathcal{X}$.

Classical and Generalized Trajectories

We assume throughout that $\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]$ is closed and densely defined.

- $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a classical trajectory of Σ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$, $x(t) \in \mathrm{Dom}\left(\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]\right)$ for all $t \in \mathbb{R}^+$, $w \in C(\mathbb{R}^+; \mathcal{W})$, and (2) holds.
- $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a generalized trajectory of Σ if $x \in C(\mathbb{R}^+; \mathcal{X})$, $w \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{W})$, and there exists a sequence of classical trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ such that $x_n \to x$ uniformly on bounded intervals and $w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{W})$.

(For the moment L^1_{loc} would also be OK, but later we need L^2_{loc} in the power balance equation.)

The Kreĭn Signal Space $\mathcal{W}$

- Recall: The state space $\mathcal{X}$ is a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{X}}$ and norm $\|\cdot\|_{\mathcal{X}} = \sqrt{(\cdot,\cdot)_{\mathcal{X}}}$.
- However, the signal space $\mathcal{W}$ is a Kreĭn space, and not a Hilbert space.

Roughly speaking, a Kreĭn space $\mathcal{W}$ is a topological vector space which a (unique) indefinite inner product $[\cdot,\cdot]_{\mathcal{W}}$. It also has a Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}}$ such that

$$[w_1, w_2]_{\mathcal{W}} = (w_1, J_{\mathcal{W}} w_2)_{\mathcal{W}}, \qquad w_1,\ w_2 \in \mathcal{W}. \tag{3}$$

where $J_{\mathcal{W}}$ is a boundedly invertible self-adjoint operator in $\mathcal{W}$ (often taken to be a signature operator, i.e., $J_{\mathcal{W}} = J_{\mathcal{W}}^* = J_{\mathcal{W}}^{-1}$). However, the Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}}$ and the signature operator $J_{\mathcal{W}}$ are not unique! (One can always replace the given inner product $(\cdot,\cdot)_{\mathcal{W}}$ by another equivalent inner product, if one at the same time changes the operator $J_{\mathcal{W}}$ accordingly.)

Why Use a Kreĭn Signal Space $\mathcal{W}$?

In the sequel I shall discuss conservative boundary control state/signal systems. They satisfy the power balance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} = [w(t), w(t)]_{\mathcal{W}}, \qquad t \in \mathbb{R}^+. \tag{4}$$

- Here $\|x(t)\|^2_{\mathcal{X}}$ represents internal energy, ($=$ the Hamiltonian), and $[w(t), w(t)]_{\mathcal{W}}$ describes the energy flow from the surroundings into the system.
- The left-hand side is positive if the internal energy is increasing, and negative if the internal energy is decreasing.
- Thus, if we want to allow the energy to flow in both direction, then we must allow the right-hand side to take both positive and negative values, and we cannot replace the indefinite inner product $[\cdot,\cdot]_{\mathcal{W}}$ in $\mathcal{W}$ by a positive definite Hilbert space inner product $(\cdot,\cdot)_{\mathcal{W}}$ in $\mathcal{W}$.

The Lagrangian Identity

Recall the state equation

$$\Sigma : \begin{cases} \dot{x}(t) = Lx(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \geq 0; \qquad x(0) = x_0, \tag{2}$$

and the power balance equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|_{\mathcal{X}}^2 = [w(t), w(t)]_{\mathcal{W}}, \qquad t \in \mathbb{R}^+. \tag{4}$$

By combining these two equations we get the Lagrangian identity (or Green's formula)

$$-(Lx, x)_{\mathcal{X}} - (x, Lx)_{\mathcal{X}} + [\Gamma x, \Gamma x]_{\mathcal{W}} = 0, \qquad x \in \mathrm{Dom}\left(\begin{bmatrix} L \\ \Gamma \end{bmatrix}\right). \tag{5}$$

The left-hand side of the Lagrangian identity

$$-(Lx, x)_{\mathcal{X}} - (x, Lx)_{\mathcal{X}} + [\Gamma x, \Gamma x]_{\mathcal{W}} = 0, \qquad x \in \mathrm{Dom}\left(\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]\right) \quad ((5))$$

can be interpreted as an indefinite (Kreĭn space) inner product in the node space $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$: Define

$$\left[\left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ x_2 \\ w_2 \end{smallmatrix}\right]\right]_{\mathfrak{K}} = \left(\left[\begin{smallmatrix} z_1 \\ x_1 \\ w_1 \end{smallmatrix}\right], J_{\mathfrak{K}} \left[\begin{smallmatrix} z_2 \\ x_2 \\ w_2 \end{smallmatrix}\right]\right)_{\mathfrak{K}}, \quad J_{\mathfrak{K}} := \begin{bmatrix} 0 & -1_{\mathcal{X}} & 0 \\ -1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & J_{\mathcal{W}} \end{bmatrix}. \tag{6}$$

Then (5) says that

$$\left[\left[\begin{smallmatrix} Lx \\ x \\ \Gamma x \end{smallmatrix}\right], \left[\begin{smallmatrix} Lx \\ x \\ \Gamma x \end{smallmatrix}\right]\right]_{\mathfrak{K}} = 0, \qquad x \in \mathrm{Dom}\left(\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]\right) \tag{7}$$

The Generating Subspace

Define

$$V := \left\{ \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \in \mathfrak{K} \;\middle|\; x \in \mathrm{Dom}\left(\begin{bmatrix} L \\ \Gamma \end{bmatrix} \right) \right\}. \tag{8}$$

Then the condition

$$\left[\begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix}, \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \right]_{\mathfrak{K}} = 0, \qquad x \in \mathrm{Dom}\left(\begin{bmatrix} L \\ \Gamma \end{bmatrix} \right) \tag{7}$$

says that
V is a neutral subspace of $\mathfrak{K}$ with respect to the inner product $[\cdot,\cdot]_{\mathfrak{K}}$, i.e., all the vectors in V are orthogonal to all other vectors (including themselves) in V. Here orthogonality means that

$$w_1 \perp w_2 \Leftrightarrow [w_1, w_2]_{\mathfrak{K}} = 0.$$

The Orthogonal Companion of V

Define

$$V^{[\perp]} = \{\kappa^\dagger \in \mathfrak{K} \mid [\kappa, \kappa^\dagger]_{\mathfrak{K}} = 0 \text{ for all } \kappa \in V\}. \qquad (9)$$

We call $V^{[\perp]}$ the orthogonal companion to V.
The Lagrangian identity (5) is equivalent to the condition

$$V \subset V^{[\perp]}.$$

$$\Sigma : \begin{cases} \dot{x}(t) = Lx(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \geq 0; \qquad x(0) = x_0, \tag{2}$$

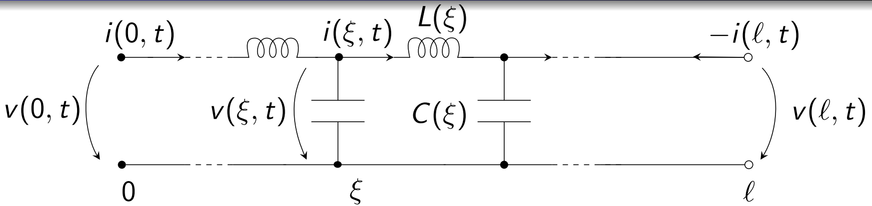
$$\left[\!\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]\!\right]_{\mathfrak{K}} = \left(\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, J_{\mathfrak{K}} \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right)_{\mathfrak{K}}, \quad J_{\mathfrak{K}} := \begin{bmatrix} 0 & -1_{\mathcal{X}} & 0 \\ -1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & J_{\mathcal{W}} \end{bmatrix}. \tag{6}$$

$$V := \left\{ \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \in \mathfrak{K} \,\middle|\, x \in \mathrm{Dom}\left(\begin{bmatrix} L \\ \Gamma \end{bmatrix} \right) \right\}. \tag{8}$$

Definition

The boundary control state/signal system Σ is conservative if the generating subspace V defined in (8) is Lagrangian, i.e., $V = V^{[\perp]}$.

A Transmission Line



$$\frac{\partial}{\partial t}\begin{bmatrix} v(\xi,t) \\ i(\xi,t)\end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C(\xi)}\frac{\partial}{\partial \xi} \\ -\frac{1}{L(\xi)}\frac{\partial}{\partial \xi} & 0 \end{bmatrix}\begin{bmatrix} v(\xi,t) \\ i(\xi,t)\end{bmatrix}, \qquad (\xi,t)\in[0,\ell]\times\mathbb{R}^+,$$

$$w(t) = \begin{bmatrix} v(0,t) \\ i(0,t) \\ v(\ell,t) \\ -i(\ell,t)\end{bmatrix}, \qquad t\in\mathbb{R}^+,$$

$$v(\xi,0) = v_0(\xi), \quad i(\xi,0) = i_0(\xi), \qquad \xi\in[0,\ell].$$

The functions $L(\cdot) > 0$ and $C(\cdot) > 0$ represent the *distributed inductance and capacitance*, respectively, of the line. For simplicity I assume that $C(\cdot)$ and $L(\cdot)$ are continuous on $[0,\ell]$.

We take $x(t) = \begin{bmatrix} v(\cdot,t) \\ i(\cdot,t)\end{bmatrix}$, $t\in\mathbb{R}^+$, and $x_0 = \begin{bmatrix} v_0(\cdot) \\ i_0(\cdot)\end{bmatrix}$.

The State Space

We take the state space $\mathcal{X}$ to be $L^2([0,\ell];\mathbb{C}^2)$ with

$$\left\| \begin{bmatrix} v(\cdot) \\ i(\cdot) \end{bmatrix} \right\|_{\mathcal{X}}^2 = \frac{1}{2}\int_0^\ell \left[C(\xi)|v(\xi)|^2 + L(\xi)|i(\xi)|^2 \right] d\xi, \qquad (10)$$

$$\left(\begin{bmatrix} v_1(\cdot) \\ i_1(\cdot) \end{bmatrix}, \begin{bmatrix} v_2(\cdot) \\ i_2(\cdot) \end{bmatrix} \right)_{\mathcal{X}} = \frac{1}{2}\int_0^\ell \left[C(\xi)v_1(\xi)\overline{v_2(\xi)} + L(\xi)i_1(\xi)\overline{i_2(\xi)} \right] d\xi. \qquad (11)$$

The operator L is given by

$$L := \begin{bmatrix} 0 & -\frac{1}{C(\xi)}\frac{\partial}{\partial \xi} \\ -\frac{1}{L(\xi)}\frac{\partial}{\partial \xi} & \end{bmatrix}, \qquad (12)$$

$$\mathrm{Dom}\,(L) := W^{1,2}([0,\ell];\mathbb{C}^2), \qquad (13)$$

where W_2^1 is the Sobolev space of functions in $L^2([0,\ell];\mathbb{C}^2)$ which have a distribution derivative in $L^2([0,\ell];\mathbb{C}^2)$.

The Boundary Operator

The boundary operator Γ has the same domain as L, and it is given by

$$\Gamma \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} = \begin{bmatrix} v(0) \\ i(0) \\ v(\ell) \\ -i(\ell) \end{bmatrix}. \tag{14}$$

It is not difficult to show that the operator $\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]$ is closed as an operator from $\mathcal{X}$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathbb{C}^4 \end{smallmatrix}\right]$ with domain
$\mathrm{Dom}\left(\left[\begin{smallmatrix} L \\ \Gamma \end{smallmatrix}\right]\right) = \mathrm{Dom}\,(L) = W^{1,2}([0,1];\mathbb{C}^2)$.
With these definitions, the transmission line becomes a special case of

$$\Sigma : \begin{cases} \dot{x}(t) = Lx(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \ge 0; \qquad x(0) = x_0, \tag{2}$$

The Lagrangian Identity

To derive the appropriate Lagrangian identity we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} &= 2\Re(x(t),\dot{x}(t))_{\mathcal{X}} \\
&= -\int_0^\ell \Re\Big[v(\xi,t)\overline{\frac{\partial}{\partial\xi}i(\xi,t)} + i(\xi,t)\overline{\frac{\partial}{\partial\xi}v(\xi,t)}\Big]\,d\xi \\
&= -\int_0^\ell \frac{\partial}{\partial\xi}\Re\Big[v(\xi,t)\overline{i(\xi,t)}\Big]\,d\xi \\
&= -\Re\Big[v(\xi,t)\overline{i(\xi,t)}\Big]_{\xi=0}^{\ell} \\
&= \Re\Big[v(0,t)\overline{i(0,t)}\Big] - \Re\Big[v(\ell,t)\overline{i(\ell,t)}\Big] \\
&= \frac{1}{2}\left(\begin{bmatrix} v(0,t) \\ i(0,t) \\ v(\ell,t) \\ -i(\ell,t)\end{bmatrix}, \begin{bmatrix} \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] & 0 \\ 0 & \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\end{bmatrix}\begin{bmatrix} v(0,t) \\ i(0,t) \\ v(\ell,t) \\ -i(\ell,t)\end{bmatrix}\right)_{\mathbb{C}^4} \\
&= (\Gamma x, J_{\mathcal{W}}\Gamma x)_{\mathbb{C}^4}, \quad J_{\mathcal{W}} = \frac{1}{2}\begin{bmatrix} \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] & 0 \\ 0 & \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\end{bmatrix}.
\end{aligned}$$

The Signal Space

The preceding identity tells us that the if we use the inner product

$$[w_1, w_2]_{\mathcal{W}} = \left(\begin{bmatrix} v_{01} \\ i_{01} \\ v_{\ell 1} \\ i_{\ell 1} \end{bmatrix}, J_{\mathcal{W}} \begin{bmatrix} v_{02} \\ i_{02} \\ v_{\ell 2} \\ i_{\ell 2} \end{bmatrix} \right), \quad J_{\mathcal{W}} = \frac{1}{2} \begin{bmatrix} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] & 0 \\ 0 & \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \end{bmatrix}. \quad (15)$$

in the signal space $\mathcal{W} = C^4$, then the generating subspace V is a neutral sbuspace of the node space $\mathfrak{K}$.

Afternoon assignment: Show that V is actually Lagrangian, and not just neutral!

Thus, the transmission line is a special case of a conservative boundary control state/signal system.

However, it can also be interpreted in terms of a boundary triplet for the operator L^*, as we shall see next.

Boundary Triplets

According to (Gorbachuk and Gorbachuk, 1991, pp. 154–155), the triple $(\Gamma_1, \Gamma_2; \mathcal{U})$ is called a boundary triplet for the closed densely defined symmetric operator A in the Hilbert space $\mathcal{X}$ with equal deficiency numbers if Γ_i, $i = 1, 2$, are linear opeartors $\mathrm{Dom}\,(A^*) \to \mathcal{U}$ and the following two conditions hold:

$$(A^* x_1, x_2)_{\mathcal{X}} - (x_1, A^* x_2)_{\mathcal{X}} = (\Gamma_1 x_1, \Gamma_2 x_2)_{\mathcal{U}} - (\Gamma_2 x_1, \Gamma_1 x_2)_{\mathcal{U}}, \quad (16)$$

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \text{ is surjective.} \quad (17)$$

Here (16) is the Lagrangian identity and (17) can be interpreted as a regularity condition (or maximality condition).

Boundary Triplet $\Rightarrow$ Boundary Control S/S System

Let $(\Gamma_1, \Gamma_2; \mathcal{U})$ be a boundary triplet for the symmetric operator A. Take $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{U}\end{smallmatrix}\right] := \mathcal{U} \times \mathcal{U}$ with the inner product

$$\left[\!\left[\begin{bmatrix} y_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ u_2 \end{bmatrix}\right]\!\right]_{\mathcal{W}} = \left(\begin{bmatrix} y_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} y_2 \\ u_2 \end{bmatrix}\right)_{\left[\begin{smallmatrix}\mathcal{U}\\ 0\end{smallmatrix}\right] \oplus \left[\begin{smallmatrix}0\\ \mathcal{U}\end{smallmatrix}\right]}. \tag{18}$$

Define $\Gamma = \left[\begin{smallmatrix}\Gamma_1\\ \Gamma_2\end{smallmatrix}\right]$ with $\mathrm{Dom}\,(\Gamma) = \mathrm{Dom}\,(A^*)$. Then

$$\Sigma : \begin{cases} \dot{x}(t) = iA^* x(t), \\ w(t) = \Gamma x(t), \end{cases} \qquad t \geq 0; \qquad x(0) = x_0, \tag{2}$$

is a conservative boundary control s/s system. See Malinen and Staffans (2006, 2007).

The notion of a *conservative boundary control s/s system is more general than the notion of a boundary triplet* in the sense of Gorbachuk and Gorbachuk (1991): There do exist conservative s/s systems which do not correspond to any boundary triplet.

The counter examples are of two types:

- The signal space $\mathcal{W}$ need not have a Lagrangian decomposition, i.e., a direct sum decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ where both $\mathcal{Y}$ and $\mathcal{U}$ are Lagrangian. A necessary and sufficient condition for the existence of a Lagrangian decomposition is that $\operatorname{ind}_+\mathcal{W} = \operatorname{ind}_-\mathcal{W}$ $(\le \infty)$. Recall that we in the case of a boundary triplet always have at least one Lagrangian decomposition, namely $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{U}\\0\end{smallmatrix}\right] \dot{+} \left[\begin{smallmatrix}0\\\mathcal{U}\end{smallmatrix}\right]$.

Boundary Control S/S System $\not\Rightarrow$ Boundary Triplet

The notion of a *conservative boundary control s/s system is more general than the notion of a boundary triplet* in the sense of Gorbachuk and Gorbachuk (1991): There do exist conservative s/s systems which do not correspond to any boundary triplet.

The counter examples are of two types:

- Even if the signal space $\mathcal{W}$ has a Lagrangian decomposition the main operator L need not be closed, and the operator $\Gamma := \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$ need not be surjective. See See Malinen and Staffans (2007) for a counter example.

According to Kurula, van der Schaft, Zwart, and Behrndt (2009), L is closed if and only if the range of Γ is closed. We may always, without loss of generality, suppose that $\mathrm{Ran}\,(\Gamma)$ is dense in $\mathcal{W}$.

Extensions

- Above we always assumed that the boundary control system Σ is conservative, i.e., that the generating subspace

$$V := \left\{ \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \in \mathfrak{K} \;\middle|\; x \in \mathrm{Dom}\left(\begin{bmatrix} L \\ \Gamma \end{bmatrix}\right) \right\}. \qquad (8)$$

  is a Lagrangian subspace of the node space $\mathfrak{K}$. There is alsa a recent theory for the case where Σ is passive instead of conservative, i.e., V is maximal nonnegative.

- There also exists a very recent theory for the case where the state/signal system is not of boundary control type, but instead of the more general type

$$\Sigma : \left\{ \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+; \qquad x(0) = x_0. \right. \qquad (19)$$

  where V is either Lagrangian (= conservative) or maximal nonnegative (= passive). See Kurula (2009); Kurula and Staffans (2009). More about this on Wednesday.

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