Hilbert Spaces Contained in Quotients of Kreĭn Spaces, with Applications to Passive State/Signal Realization Theory

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Outline

PART I: Hilbert Spaces in Quotients of Kreĭn Spaces

- Maximal Nonnegative subspaces of Kreĭn spaces

- The Hilbert spaces $\mathcal{X}[\mathcal{Z}]$ and $\mathcal{X}[\mathcal{Z}^{[\perp]}]$

PART II: Passive S/S Systems

- Passive state/signal systems

- Behaviors induced by passive state/signal systems

- Passive behaviors and their realizations

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More precisely, there exist a Hilbert space inner product $(\cdot,\cdot)_{\mathfrak{K}}$ in $\mathfrak{K}$ and an operator $J \in \mathcal{B}(\mathfrak{K})$, $J = J^* = J^{-1}$ (i.e., J is both self-adjoint and unitary), such that

$$[k_1, k_2]_{\mathfrak{K}} = (k_1, Jk_2)_{\mathfrak{K}}, \quad k_1,\, k_2 \in \mathfrak{K}.$$

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The orthogonal companion $\mathcal{Z}^{[\perp]}$ to a subspace $\mathcal{Z} \subset \mathfrak{K}$ is given by

$$\mathcal{Z}^{[\perp]} = \{k \in \mathfrak{K} \mid [k, z]_{\mathfrak{K}} = 0\ \forall z \in \mathcal{Z}\}.$$

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Let $\mathcal{Z}$ be maximal nonnegative. The maximal neutral subspace $\mathcal{Z}_0$ of $\mathcal{Z}$ is given by $\mathcal{Z}_0 = \mathcal{Z} \cap \mathcal{Z}^{[\perp]}$. This is the largest neutral subspace in $\mathcal{Z}$, and also the largest neutral subspace in $\mathcal{Z}^{[\perp]}$.

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Special case: we take $\mathcal{Z}$ to be either maximal nonnegative or maximal nonpositive. Such a subspace is automatically closed (with respect to the standard quotient topology).

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This implies that $\langle \cdot, \cdot \rangle_{\mathcal{Z}}$ induces a positive (nondegenerate) inner product on the quotient space $\mathcal{Z}/\mathcal{Z}_0$. We denote this inner product by $(\cdot, \cdot)_{\mathcal{Z}/\mathcal{Z}_0}$. Thus,

$$([z_1], [z_2])_{\mathcal{Z}/\mathcal{Z}_0} := \langle z_1, z_2 \rangle_{\mathcal{Z}} = [z_1, z_2]_{\mathfrak{K}},$$

where $[z_1]$ and $[z_2]$ stand for the equivalence classes $[z_i] := z_i + \mathcal{Z}_0$, $i = 1, 2$. With this inner product $\mathcal{Z}/\mathcal{Z}_0$ becomes a pre-Hilbert space (not necessary complete).

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What does the completion of $\mathcal{Z}/\mathcal{Z}_0$ look like?

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The Completion of $\mathcal{Z}/\mathcal{Z}_0$

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The construction of the Hilbert spaces $\mathcal{X}[\mathcal{Z}]$ and $\mathcal{X}[\mathcal{Z}^{[\perp]}]$ is an abstract version of the functional construction by Louis de Branges and James Rovnyak in [dBR66].

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$$\mathcal{X}[\mathcal{Z}] = \left\{x \in \mathfrak{K}/\mathcal{Z} \;\middle|\; \|x\|_{\mathcal{X}[\mathcal{Z}]} < \infty\right\}, \tag{1}$$

where the (Hilbert space) norm $\|x\|_{\mathcal{X}[\mathcal{Z}]}$ of the equivalence class $x \in \mathfrak{K}/\mathcal{Z}$ is given by

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Compare this to the Hilbert space case: If instead $\mathcal{Z}$ is a closed subspace of a Hilbert space $\mathfrak{K}$, then the quotient norm of x in $\mathfrak{K}/\mathcal{Z}$ is given by

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Thus, the norm in $\mathcal{X}[Z]$ is simply the 'Kreĭn space version' of the quotient norm in $\mathfrak{K}/\mathcal{Z}$ when $\mathcal{Z}$ maximal nonnegative!

Definition of $\mathcal{X}[\mathcal{Z}^{[\perp]}]$ (interchange $\mathcal{Z}$ and $-\mathcal{Z}^{[\perp]}$)

$$\mathcal{X}[\mathcal{Z}^{[\perp]}] = \left\{x^\dagger \in \mathfrak{K}/\mathcal{Z}^{[\perp]} \mid \|x^\dagger\|_{\mathcal{X}[\mathcal{Z}^{[\perp]}]} < \infty\right\}, \tag{4}$$

where the (Hilbert space) norm $\|x^\dagger\|_{\mathcal{X}[\mathcal{Z}^{[\perp]}]}$ of the equivalence class $x^\dagger \in \mathfrak{K}/\mathcal{Z}^{[\perp]}$ is given by

$$\left\|x^\dagger\right\|_{\mathcal{X}[\mathcal{Z}^{[\perp]}]} = \sqrt{\sup_{k \in x^\dagger}([k,k]_{\mathfrak{K}})}. \tag{5}$$

Compare this to the Hilbert space case: If instead $\mathcal{Z}$ is a closed subspace of a Hilbert space $\mathfrak{K}$, then the quotient norm of $x^\dagger$ in $\mathfrak{K}/\mathcal{Z}^\perp$ is given by

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Thus, the norm in $\mathcal{X}[Z^{[\perp]}]$ is simply the 'Kreĭn space version' of the quotient norm in $\mathfrak{K}/\mathcal{Z}^{[\perp]}$ when $\mathcal{Z}^{[\perp]}$ maximal nonpositive!

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- $\mathcal{X}[\mathcal{Z}]$ is a subspace of $\mathfrak{K}/\mathcal{Z}$ of 'de Branges–Rovnyak' type.

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State/Signal Systems

A linear discrete time s/s (state/signal) system $\Sigma_{s/s} = (V; \mathcal{X}, \mathcal{W})$ is a dynamical system. It consists of

a state space $\mathcal{X}$ (today a Hilbert space) representing an internal memory,
a signal space $\mathcal{W}$ (today a Kreĭn space) for connections to the outside world, and
a generating subspace V of $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ which defines the dynamics.

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A trajectory $\left[\begin{smallmatrix} x(n) \\ w(n) \end{smallmatrix}\right]$, $n \in I$, on a discrete time interval I satisfies

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In order for this be be a reasonable dynamical system the generating subspace V must satisfy certain conditions. See [AS05]–[AS07c] for details.

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$$\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix}\right]_{\mathfrak{K}} = -(z_1, z_2)_{\mathcal{X}} + (x_1, x_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}.$$

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The s/s system $\Sigma_{s/s} = (V; \mathcal{X}, \mathcal{W})$ is passive if V is a maximal nonnegative subspace of $\mathfrak{K}$. (Maximal nonnegativity of V implies that $\Sigma_{s/s}$ is a 'reasonable dynamical system'.)

Input/State/Output Representations

If $\Sigma_{s/s} = (V; \mathcal{X}, \mathcal{W})$ is a passive s/s system, then by decomposing the signal space into $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ in different ways and interpreting $\mathcal{U}$ as an input space and $\mathcal{Y}$ as an output space we get standard passive i/s/o (input/state/output) systems:

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- If the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is fudamental (i.e., $\mathcal{U}$ is uniformly positive and $\mathcal{Y} = \mathcal{U}^{[\perp]}$), then we get a scattering passive i/s/o system.

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- By taking both $\mathcal{U}$ and $\mathcal{Y}$ to be neutral we get an impedance passive i/s/o system.

- By taking $\mathcal{U}$ to be a Kreĭn subspace of $\mathcal{W}$ and $\mathcal{Y} = \mathcal{U}^{[\perp]}$ we get a transmission passive i/s/o system.

Again see [AS07a]–[AS07c] for details.

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PART II: Passive S/S Systems

- Passive state/signal systems
- Behaviors induced by passive state/signal systems
- Passive behaviors and their realizations

Stable Externally Generated Trajectories

In the sequel we only consider trajectories $\begin{bmatrix} x(\cdot) \\ w(\cdot) \end{bmatrix}$ on one of the infinite discrete time intervals
$I = \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, $I = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, or $I = \mathbb{Z}^- = \{-1, -2, - \ldots\}$.

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Such a trajectory is stable if $x \in \ell^\infty(I; \mathcal{X})$ and $w \in \ell^2(I; \mathcal{W})$.

It is externally generated if the state vanishes at the left end-point:
$x(0) = 0$ in case $I = \mathbb{Z}^+$, and
$\lim_{n \to -\infty} x(n) = 0$ in case $I = \mathbb{Z}^-$ or $I = \mathbb{Z}$.
Thus, the internal memory is empty when the process starts, and the dynamics is driven purely by the signal.

Behaviors Induced by Passive S/S Systems

Every passive s/s system $\Sigma_{s/s}$ induces three types of stable behaviors, one on each of the three time intervals $I = \mathbb{Z}^+$, $I = \mathbb{Z}$, and $I = \mathbb{Z}^-$:

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The past behavior $\mathfrak{W}_{\text{past}} \subset \ell^2(\mathbb{Z}^-; \mathcal{W})$ consists of all the signal parts of all externally generated stable trajectories of $\Sigma_{s/s}$ on $\mathbb{Z}^-$.

If $\Sigma_{s/s}$ is passive (in the s/s sense), then all of these stable behaviors are 'passive' in a certain 'behavioral' sense (as will be explained below).

Definitions of $k^2(I;\mathcal{W})$, S_+, S, S_-

For each of the three time intervals $I = \mathbb{Z}^+$, $I = \mathbb{Z}$, and $I = \mathbb{Z}^-$ we turn $\ell^2(I;\mathcal{W})$ into a Kreĭn space, which we denote by $k^2(I;\mathcal{W})$, by using the indefinite inner product

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We denote the right-shift operators on $k^2(\mathbb{Z}^+;\mathcal{W})$, $k^2(\mathbb{Z};\mathcal{W})$, and $k^2(\mathbb{Z}^-;\mathcal{W})$ by S_+, S, and S_-, respectively.

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S_+ is an outgoing shift (isometry),

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Their adjoints are left-shifts: S_+^* (incoming), S^* (bilateral), and S_-^* (outgoing).

Properties of the Induced Behaviors

Theorem 2. Let $\Sigma_{s/s} = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system. Then the stable future, full, and past behaviors $\mathfrak{W}_{\rm fut}$, $\mathfrak{W}_{\rm full}$, and $\mathfrak{W}_{\rm past}$ induced by $\Sigma_{s/s}$ have the following properties:

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(iii) $\mathfrak{W}_{\text{full}}$ is a maximal nonnegative S-reducing subspace of $k^2(\mathbb{Z}; \mathcal{W})$, and, in addition, $\mathfrak{W}_{\text{full}}$ is the graph of a causal contraction $\mathfrak{D}\colon \ell^2(\mathbb{Z};\mathcal{U}) \to \ell^2(\mathbb{Z}; -\mathcal{U}^{[\perp]})$ for some fundamental decomposition $\mathcal{W} = \mathcal{U}^{[\perp]} \dotplus \mathcal{U}$ of the signal space.

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In the sequel we shall use properites (i)–(iii) above as definitions of passive future, full, and past behaviors.

Outline

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Passive Behaviors

(i) By a passive future behavior we mean a maximal nonnegative S_+-invariant subspace $\mathfrak{W}_{\mathrm{fut}}$ of $k^2(\mathbb{Z}^+;\mathcal{W})$.

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Fact 1: There is a one-to-one correspondence $\mathfrak{W}_{\rm fut} \leftrightarrow \mathfrak{W}_{\rm full} \leftrightarrow \mathfrak{W}_{\rm past}$.

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Fact 2: Also passive future and past behavior have graph representations of the type described in (iii). (The existence of such a causal graph representations is redundant in cases (i) and (ii), but not in case (iii).)

Realizations of Passive Behaviors

Question: Given a passive future behavior $\mathfrak{W}_{\rm fut}$, or a passive full behavior $\mathfrak{W}_{\rm full}$, or a passive past behavior $\mathfrak{W}_{\rm past}$, then can we always find a passive s/s system $\Sigma_{s/s}$ which induces these three behaviors ($=$ a realization)?

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Answer: Yes!

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Answer: Yes!

We get one 'canonical' class of realizations by letting the dynamics be induced by some type of left-shift, and by letting the state space be one of the de Branges–Rovnyak type spaces presented at the beginning of this talk.

The Controllable Forward Conservative Realization

The controllable forward conservative realization $\Sigma_{\rm past} = (V_{\rm past}; \mathcal{X}_{\rm past}, \mathcal{W})$ uses the fact that $\mathfrak{W}_{\rm past}$ is a maximal nonnegative subspace of $k^2(\mathbb{Z}^-; \mathcal{W})$. Let $\mathcal{L}_0$ be the maximal neutral subspace of $\mathfrak{W}_{\rm past}$.

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The state space $\mathcal{X}_{\mathrm{past}}$ of this realization is the completion of the pre-Hilbert space $\mathfrak{W}_{\mathrm{past}}/\mathcal{L}_0$, which by Theorem 1 can be identified with the subspace $\mathcal{X}[\mathfrak{W}_{\mathrm{past}}^{[\perp]}]$ of $k^2(\mathbb{Z}^-; \mathcal{W})/\mathfrak{W}_{\mathrm{past}}^{[\perp]}$.

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The dynamics of this realization is a type of (outgoing) left-shift:

- We are given an intial state $x(0)$, equal to a sequence $w(\cdot) \in \mathfrak{W}_{\rm past}$, and also a signal value $w_0 \in \mathcal{W}$ at time $n = 0$.
- The new state $x(1)$ is the left-shifted $x(0)$ filled in with w_0: $x(1) := \{\ldots, w(-2), w(-1), w_0\}$. Note that $x(1)$ may or may not belong to $\mathfrak{W}_{\rm past}$.
- The set of those $(x(1), x(0), w_0)$ for which $x(1) \in \mathfrak{W}_{\rm past}$ is dense in $V_{\rm past}$.

The Observable Backward Conservative Realization

The observable backward conservative realization $\Sigma_{\rm fut} = (V_{\rm fut}; \mathcal{X}_{\rm fut}, \mathcal{W})$ uses the fact that $\mathfrak{W}_{\rm fut}$ is a maximal nonnegative subspace of $k^2(\mathbb{Z}^+; \mathcal{W})$. Let $\mathcal{L}_0^\dagger$ be the maximal neutral subspace of $\mathfrak{W}_{\rm fut}$.

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The state space $\mathcal{X}_{\rm fut}$ of this realization is the subspace $\mathcal{X}[\mathfrak{W}_{\rm fut}]$ of $k^2(\mathbb{Z}^+; \mathcal{W})/\mathfrak{W}_{\rm fut}$, which by Theorem 1 can be identified with the completion of the pre-Hilbert space $-\mathfrak{W}_{\rm fut}^{[\perp]}/\mathcal{L}_0^\dagger$.

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The dynamics of this realization is a type of (incoming) left-shift:

- We are given an intial state $x(0)$, equal to an equivalence class $[w(\cdot)] := w(\cdot) + \mathfrak{W}_{\rm fut} \in \mathcal{X}[\mathfrak{W}_{\rm fut}]$, where $w(\cdot) \in k^2(\mathbb{Z}^+; \mathcal{W})$.
- The new state is $x(1) := [S_+^* w] := S_+^* w(\cdot) + \mathfrak{W}_{\rm fut} \in \mathcal{X}[\mathfrak{W}_{\rm fut}]$. It turns out that $x(1)$ depends not only on $x(0) = [w(\cdot)]$ but also on the value $w(0)$.
- $V_{\rm fut}$ consists of all $(x(1), x(0), w(0))$ of the type described above.

The Simple Conservative Realization

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The simple conservative realization is a certain combination of the two realizations above. It is too complicated to be described here.

Passive Input/State/Output Realization

By splitting the signal space $\mathcal{W}$ into $\mathcal{Y} = \mathcal{Y} \dot{+} \mathcal{U}$ in different ways (as described earlier) and mapping the time domain into the frequency domain with the Z-transform we get the standard de Branges–Rovnyak spaces. This gives us

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