Passive Scattering Systems

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Scattering and Inverse Scattering

This talk is about scattering.
Think about an electro-magnetic wave travelling in space which hits an object and is *scattered*, i.e., partially absorbed, and partially reflected in different directions.
Several questions can be asked about this phenomenon:

- The direct scattering problem: Compute the scattered wave when we know the object and the incident wave.
- The inverse scattering problem: Try to compute the shape and the properties of the object if we know both the incident wave and the scattered wave.

Here I do not focus specifically on either of these, but instead talk about the mathematical formulation of the scattering problem.

General Plan

- In the first half of the talk I give a short review of contraction semigroups in Hilbert spaces, to explain the background (the autonomous evolution inside the scatterer is modelled as a contraction semigroup).
- In the second half of the talk I include the incident and scattered waves, to get a passive scattering system.

PDE Formulation

Electro-magnetic waves are described by Maxwell's equations, as a connection between time derivatives (= the evolution of the system) and the space derivatives (= the variation in space) of the wave.

Let us denote the 6-dimensional vector which contains the components of the electric and magnetic field at the point ξ at time t by $x(\xi, t)$. Then the equation which describes the propagation of an electro-magnetic wave in free space is of the form

$$\begin{aligned} \frac{\partial}{\partial t} x(\xi, t) &= A x(\xi, t), \qquad \xi \in \mathbb{R}^6, \quad t \geq 0, \\ x(\xi, 0) &= \varphi(\xi), \qquad \xi \in \mathbb{R}^6, \end{aligned} \tag{1}$$

where A is a partial differential operator involving only space derivatives.

Abstract Formulation: Idea

This is too complicated. To simplify the equation we suppress the space variable ξ (= do not write it out explicitly), and replace (1) by

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{2}$$

where $\dot{x} = \frac{\partial}{\partial t} x$. We still think about x as being a function of both ξ and t, but "hide" the variable ξ.

Mathematically, $x(t)$ is a function of ξ, and it can be interpreted as a vector in an infinite-dimensional vector space $\mathcal{X}$. We call $\mathcal{X}$ the state space of the system.

The operator A on the right-hand side is an unbounded linear operator. It is not defined for all vectors $x \in \mathcal{X}$ (only those that are sufficiently differentiable). We denote the set of vectors $x \in \mathcal{X}$ for which Ax is defined by $\mathrm{Dom}\,(A)$.

Abstract Formulation:

The equation

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{2}$$

formally looks like a standard linear system of differential equations, but it is still a complicated object. Since $Ax(t)$ is not defined for all $x(t) \in \mathcal{X}$ we still need to add one more condition, namely $x(t) \in \mathrm{Dom}\,(A)$. Thus, the correct way of writing (2) is

$$\begin{aligned} x(t) &\in \mathrm{Dom}\,(A)\,, \qquad t \geq 0, \\ \dot{x}(t) &= Ax(t), \qquad t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{3}$$

Energy Preservation and Passivity: Idea

All electro-magnetic waves contain energy, and to create an electro-magnetic wave one must supply energy.

- In the absence of energy sources the the energy of the wave cannot increase. In particular, electro-magnetic waves do not appear spontaneously from nowhere.
- In the additional absence of energy absorbing materials the energy is preserved.

Terminology:

- A system is passive if the energy cannot increase.
- A system is energy preserving if the energy is preserved.
- (A system is conservative if both the system and its adjoint preserve energy.)

Mathematical Characterization of Passivity

We denote the energy of the wave x at time t by $E(x(t))$, and define $\|x(t)\|_{\mathcal{X}} = \sqrt{2E(x(t))}$, or equivalently, $E(x(t)) = \frac{1}{2}\|x(t)\|^2_{\mathcal{X}}$. (The constant $\frac{1}{2}$ is not important.)

- With the right choice of $\mathcal{X}$ we can use $\|\cdot\|_{\mathcal{X}}$ as a norm in $\mathcal{X}$, and $\mathcal{X}$ then has a natural interpretation as a Hilbert space with this norm (of L^2-type).
- The system

$$\begin{aligned} x(t) &\in \mathrm{Dom}\,(A)\,, && t \geq 0, \\ \dot{x}(t) &= Ax(t), && t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{3}$$

  is passive if $E(x(t)) \leq E(x(s))$ for $t \geq s$, or equivalently, if

$$\|x(t)\|^2_{\mathcal{X}} \leq \|x(s)\|^2_{\mathcal{X}}, \qquad t \geq s \geq 0. \tag{4}$$

- The system (3) is energy preserving if

$$\|x(t)\|^2_{\mathcal{X}} = \|x(s)\|^2_{\mathcal{X}}, \qquad t \geq s \geq 0. \tag{5}$$

Differential Characterization of Passivity

We rewrite the passivity inequality

$$\|x(t)\|_{\mathcal{X}}^2 \leq \|x(s)\|_{\mathcal{X}}^2, \qquad t \geq s \geq 0, \tag{4}$$

in the form

$$\frac{\|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2}{t-s} \leq 0, \qquad t \geq s \geq 0,$$

and let $t - s \to 0$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|_{\mathcal{X}}^2 \leq 0, \qquad t \geq 0. \tag{6}$$

Dissipative Operators

Since $\|x(t)\|_{\mathcal{X}}^2 = \langle x(t), x(t)\rangle_{\mathcal{X}}$ where $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ is the inner product in $\mathcal{X}$, we can carry out the differentiation in (6) to get

$$\begin{aligned} 0 \geq \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), x(t)\rangle_{\mathcal{X}} \\ &= \langle \dot{x}(t), x(t)\rangle_{\mathcal{X}} + \langle x(t), \dot{x}(t)\rangle_{\mathcal{X}} \\ &= 2\Re\langle x(t), \dot{x}(t)\rangle_{\mathcal{X}}, \end{aligned}$$

where the second equality follows from the rule for the derivative of a product. By (3), $\dot{x}(t) = Ax(t)$. Thus, we get

$$\Re\langle x(t), Ax(t)\rangle_{\mathcal{X}} \leq 0, \qquad x(t) \in \mathrm{Dom}\,(A)\,.$$

Definition

An operator $A\colon \mathcal{X} \to \mathcal{X}$ with domain $\mathrm{Dom}\,(A)$ is called dissipative (or anti-accretive) if

$$\Re\langle x, Ax\rangle_{\mathcal{X}} \leq 0, \qquad x \in \mathrm{Dom}\,(A)\,.$$

Dissipative Operators

Lemma

The system

$$\begin{aligned} x(t) &\in \mathrm{Dom}\,(A)\,, & t &\geq 0, \\ \dot{x}(t) &= Ax(t), & t &\geq 0, \\ x(0) &= x_0. \end{aligned} \tag{3}$$

is passive if and only if A is dissipative.

The proof in one direction was given above. The opposite direction is also easy. However, there is something missing in the above result. It does not say anything about existence of solutions of (3).

- For example, if we take $\mathrm{Dom}\,(A) = \emptyset$, then formally the above theorem applies, but (3) does not have any solutions at all.
- If we instead take $\mathrm{Dom}\,(A) = \{0\}$ and $A0 = 0$, then (3) is also passive, but it has only the zero solution $x(t) \equiv 0$.
- From physics we certainly expect the Maxwell's equations to have plenty of solutions.

Well-Posed Passive Systems

Definition

A system

$$\begin{aligned} x(t) &\in \mathrm{Dom}\,(A)\,, && t \geq 0, \\ \dot{x}(t) &= Ax(t), && t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{3}$$

is well-posed if

- $x(t)$ depends continuously on x_0 for all fixed $t \geq 0$ (trivially true if the system is passive).
- The set of all possible "smooth initial states" $x_0 \in \mathrm{Dom}\,(A)$ is "as large as possible", and all initial states x_0 are allowed if we replace "classical solutions" by "generalized solutions".

For examples, in the right setting Maxwell's equations define a well-posed passive system (like most other linear dynamical PDEs). The two "pathetic" examples on the preceding page are not well-posed.

(Almost) Without Loss of Generality: A is closed

It turn out that one can usually "without loss of generality" require the operator A to be closed.

This is a natural physical condition, which says that the graph of A is closed. If (3) is derived using "basic physical principles", then A is typically closed.

Maximal Dissipative Operators

Theorem

A closed operator $A\colon \mathcal{X} \to \mathcal{X}$ with domain $\mathrm{Dom}\,(A)$ generates a well-posed passive system (= semi-group) if and only if the following two conditions hold:

1. *A is dissipative, i.e.,*

$$\Re\langle x, Ax\rangle \leq 0, \qquad x \in \mathrm{Dom}\,(A)\,.$$

2. *A is maximal, i.e., it is not possible to enlarge $\mathrm{Dom}\,(A)$ without loosing the above property (1).*

We call an operator A satisfying (1) and (2) maximal dissipative. If, in addition, A is closed, then we call A m-dissipative.

This is a classical theorem due to Phillips (Phi59). It marked the beginning of the modern era of looking at basic partial differential equations in mathematical physics.

Non-Closed Maximal Dissipative Operators

The above theorem is not true if we remove the word closed. See counter example in (Phi59). But it can be replaced by densely defined.

Theorem

A densely defined operator $A\colon \mathcal{X} \to \mathcal{X}$ with domain $\mathrm{Dom}\,(A)$ generates a well-posed passive system ($=$ semi-group) if and only if A is maximal dissipative.

- Thus, if A is maximal dissipative and has dense domain, then it is closed.
- If A is not closed ($\Rightarrow \mathrm{Dom}\,(A)$ not dense), then the closure of A is a relation, whose multi-valued part is contained in $\mathrm{Dom}\,(A)^{\perp}$. If we "peal off" the multi-valued part, then the remainder is an operator, which is closed and maximal dissipative in $\overline{\mathrm{Dom}\,(A)}$, and hence gererates a well-posed passive system in $\overline{\mathrm{Dom}\,(A)}$.

Separating Incoming and Outgoing Data

- The above two theorems do apply to Maxwell's equations, but they does not yet give a precise description of what happens in the scattering case, where an incident wave hits an object (= the scatter) and is partially absorbed and partially reflected (= scattered) into an outgoing wave.
- To get a more precise picture of this situation we must separate the incoming wave and the outgoing wave from each other and from the state.
- In this connection boundary conditions become important, since they determine which proportion of the wave is absorbed, and which portion is reflected.

System with Input and Output

- In the new formulation we have, as before, the state $x(t) \in \mathcal{X}$, but in addition we have the incoming signal $u(t) \in \mathcal{U}$ and the outgoing signal $y(t) \in \mathcal{Y}$.
- The equation describing the dynamics of the system say that the time derivative of the state and the output are determined by the present values of the state and the input.
- This leads to an equation of the type

$$\begin{aligned} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} &\in \mathrm{Dom}\,(S)\,, && t \geq 0, \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} &= S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, && t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{7}$$

Here S maps $\mathrm{Dom}\,(S) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$. We assume "without loss of generality" that S is closed.

Energy Inequality

When we take account of the power entering the system via the input $u(t)$ and the power leaving the system via the output $y(t)$ the earlier passivity inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 \le 0, \qquad t \ge 0. \tag{6}$$

is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|x(t)\|_{\mathcal{X}}^2 + \|y(t)\|_{\mathcal{Y}}^2 \le \|u(t)\|_{\mathcal{U}}^2, \qquad t \ge 0, \tag{8}$$

where $\frac{1}{2}\|u(t)\|_{\mathcal{U}}^2$ and $\frac{1}{2}\|y(t)\|_{\mathcal{Y}}^2$ are the power carried by the incoming and outgoing signals. As before, $\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_{\mathcal{X}}^2 = 2\Re\langle x(t), \dot{x}(t)\rangle_{\mathcal{X}}$. Thus, in the new setting passivity means that

$$2\Re\langle x(t), \dot{x}(t)\rangle_{\mathcal{X}} + \|y(t)\|_{\mathcal{Y}}^2 \le \|u(t)\|_{\mathcal{U}}^2, \qquad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{Dom}\,(S)\,.$$

Scattering Dissipative Operators

Definition

I call an operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ with domain $\mathrm{Dom}\,(S)$ scattering dissipative if

$$2\Re\langle \dot{x}, x\rangle + \|y\|_{\mathcal{Y}}^2 \le \|u\|_{\mathcal{U}}^2, \quad \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{Dom}\,(S), \quad \left[\begin{smallmatrix}\dot{x}\\ y\end{smallmatrix}\right] := S\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]. \quad (9)$$

Note that

- If there is no input and no output (i.e., $\mathcal{U} = \{0\}$ and $\mathcal{Y} = \{0\}$), then scattering dissipative = dissipative in the usual sense.
- If there is no state (i.e., $\mathcal{X} = \{0\}$), then S is scattering dissipative if and only if S is a (not necessarily everywhere defined) contraction.

Main Theorem (First Version)

Theorem

A closed operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ *with domain* $\mathrm{Dom}\,(S)$ *generates a well-posed scattering passive system with input u and output y if and only if the following two conditions hold:*

1. *A is scattering dissipative, i.e.,*

$$2\Re\langle \dot{x}, x\rangle + \|y\|^2_{\mathcal{Y}} \le \|u\|^2_{\mathcal{U}}, \quad \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{Dom}\,(S)\,, \quad \left[\begin{smallmatrix}\dot{x}\\ y\end{smallmatrix}\right] := S\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]. \tag{9}$$

2. *A is maximal, i.e., it is not possible to enlarge* $\mathrm{Dom}\,(A)$ *without loosing the inequality* (9).

We call an operator S satisfying (1) and (2) maximal scattering dissipative.

This theorem was proved one year ago in (Sta12).
This characterization of a well-posed scattering system with input and output is much simpler than the one in the book (Sta05).

System Node (the "simplest" version from (Sta05))

Definition

By a system node on a triple of Hilbert spaces $(\mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean a (possibly unbounded) linear operator $S\colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ with the following properties. We denote $\mathrm{Dom}\,(A) = \left\{x \in \mathcal{X} \;\middle|\; \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathrm{Dom}\,(S)\right\}$, define $A\colon \mathrm{Dom}\,(A) \to \mathcal{X}$ by $Ax = P_{\mathcal{X}} S \begin{bmatrix} x \\ 0 \end{bmatrix}$, and require the following conditions to hold:

1. S is closed as an operator from $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ to $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ (with domain $\mathrm{Dom}\,(S)$).
2. $P_{\mathcal{X}} S$ is closed as an operator from $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ to $\mathcal{X}$ (with domain $\mathrm{Dom}\,(S)$).
3. A is the generator of a C_0 semigoup (i.e., the autonomous system is well-posed).
4. For every $u \in \mathcal{U}$ there exists a $x \in \mathcal{X}$ with $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{Dom}\,(S)$.

(The "complicated" version in (Sta05) use "extrapolation space".)

Main Theorem (Classical Version)

Theorem

An operator $S\colon \begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix} \to \begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$ *with domain* $\mathrm{Dom}\,(S)$ *generates a well-posed passive system with input* u *and output* y *if and only if* S *is a scattering dissipative system node.*

This is a much more "complicated" characterization than the new one. In the new version we have replaced conditions (2)–(4) on the preceding slide by the single word maximal.

Main Theorem (Second Version)

It is again possible to replace closed by densely defined.

Theorem

A densely defined operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ with domain $\mathrm{Dom}\,(S)$ generates a well-posed passive system with input u and output y if and only if S is maximal scattering dissipative.

The same counter example can be used to show that one cannot remove both "closed" and "densely defined" from the above two theorems.

The Cayley Transform

If A is a dissipative operator in $\mathcal{X}$ (need not be closed or maximal), then the Cayley transform of A with parameter $\alpha \in \mathbb{C}^+$ is given by

$$\mathbf{A}(\alpha) = (\overline{\alpha} + A)(\alpha - A)^{-1} = 2\Re\alpha\,(\alpha - A)^{-1} - 1_{\mathcal{X}}.$$

It has the following properties:

- $\mathbf{A}(\alpha)$ is always a contraction.
- A is closed $\Leftrightarrow$ $\mathbf{A}(\alpha)$ is closed $\Leftrightarrow$ $\mathrm{Dom}\,(\mathbf{A}(\alpha))$ is closed.
- $\mathrm{Dom}\,(\mathbf{A}(\alpha)) = \mathcal{X}$, if and only if A is closed and maximal dissipative.
- $\mathbf{A}(\alpha) + 1_{\mathcal{X}}$ is always injective (typically not surjective).
- A can be recovered from $\mathbf{A}(\alpha)$: $\mathrm{Dom}\,(A) = \mathrm{Ran}\,(\mathbf{A}(\alpha) + 1_{\mathcal{X}})$ and
$$A = \alpha 1_{\mathcal{X}} - 2\Re\alpha(\mathbf{A}(\alpha) + 1_{\mathcal{X}})^{-1}.$$

With the help of the Cayley transform Phillips (Phi59) studied dissipative operators by converting them into contractions.

The Internal Cayley Transform

If S is a scattering dissipative operator (maximal or non-maximal) mapping $\mathrm{Dom}\,(S) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$, then we define the internal Cayley transform $\mathbf{T}(\alpha)$ of A with parameter $\alpha \in \mathbb{C}^+$ by "taking the Cayley transform with parameter α of the first component, and leaving the second component untouched". It has the following properties:

- $\mathbf{T}(\alpha)$ is always a contraction $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \supset \mathrm{Dom}\,(\mathbf{T}(\alpha)) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$.
- S is closed $\Leftrightarrow$ $\mathbf{T}(\alpha)$ is closed $\Leftrightarrow$ $\mathrm{Dom}\,(\mathbf{T}(\alpha))$ is closed.
- $\mathrm{Dom}\,(\mathbf{T}(\alpha)) = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ if and only if S is closed and maximal scattering dissipative.
- S can be recovered from $\mathbf{T}(\alpha)$.

With the help of the internal Cayley transform one can study scattering dissipative operators by converting them into contractions $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$.

Application

- In recent joint study with George Weiss of Maxwell's equations we begin in (SW12) by using "the external Cayley transform" to prove an abstract result which says that an operator with a certain structure always generates a well-posed scattering passive (or conservative) system.
- In our second paper (WS12) we show that Maxwell's equations can be fit into the above structure if the conductivity, the electric permittivity, and the magnetic permeability are all constants.
- From this special case one can then get the general (non-isotropic) case (as we also show in (WS12)) by using the following lemma (see next slide):

System Transformation Lemma

Lemma

Let $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$ be Hilbert spaces, let S be a linear operator $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$, and let P a be positive self-adjoint and boundedly invertible operator in $\mathcal{X}$. Define

$$S_P = \begin{bmatrix} P & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} S \begin{bmatrix} P & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}, \qquad \mathcal{D}(S_P) = \begin{bmatrix} P^{-1} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \mathcal{D}(S).$$

Then S generates a well-posed scattering passive (or conservative) system if and only if S_P generates a scattering passive (or conservative) system.

Note that this is not a similarity transformation but a congruence transformation. The above lemma says that
scattering passivity is invariant under congruence transformations.

System Transformation Lemma

Proof.

Clearly, it suffices to prove this lemma in one direction, since the other direction then follows if we interchange S and S_P and also replace P by P^{-1}.
Suppose that S generates a well-posed scattering passive system, i.e., that S is closed and maximal scattering dissipative. Let $\left[\begin{smallmatrix} x_P \\ u \end{smallmatrix}\right] \in \mathrm{Dom}\,(S_P)$, and denote $\left[\begin{smallmatrix} \dot{x}_P \\ y \end{smallmatrix}\right] := S_P \left[\begin{smallmatrix} x_P \\ u \end{smallmatrix}\right]$. Let $x = Px_P$ and $\dot{x} = P^{-1}\dot{x}_P$. Then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{Dom}\,(S)$, and $\left[\begin{smallmatrix} \dot{x} \\ y \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$. Consequently,

$$\begin{aligned} 0 &\le 2\Re\langle \dot{x}, x\rangle + \|y\|^2_{\mathcal{Y}} - \|u\|^2_{\mathcal{U}} = 2\Re\langle P^{-1}\dot{x}_P, Px_P\rangle + \|y\|^2_{\mathcal{Y}} - \|u\|^2_{\mathcal{U}} \\ &= 2\Re\langle \dot{x}_P, x_P\rangle + \|y\|^2_{\mathcal{Y}} - \|u\|^2_{\mathcal{U}}. \end{aligned}$$

This shows that S_P is scattering dissipative. It is also easy to see that S_P is closed (since S is closed) and *maximal* scattering dissipative (since S is maximal scattering dissipative). Thus, by the main theorem, S_P generates a well-posed scattering passive system. □

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