Frequency Domain Well-Posed Linear Systems

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Outline of Talk

- Time domain well-posed input/state/output systems
- Frequency domain well-posed input/state/output systems
- Intertwinement in time and frequency domain
- Compressions and dilations in time and frequency domain
- Controllability, observability, and minimality
- Work in progress

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One of the first serious attempts to do infinite-dimensional control theory was to study systems of the type

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \qquad (1)$$

$x(t) \in \mathcal{X}$ is the state,
$u(t) \in \mathcal{U}$ is the input,
$y(t) \in \mathcal{Y}$ is the output
$\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces.
The main operator A is the generator of a C_0 semigroup, but
the control operator B,
the observation operator C, and
the feed-through operator D are all bounded linear operators.
This class of systems is studied in the book (CZ95).
However, it is not really "good enough" to study boundary control systems.

“Classical” infinite-dimensional i/s/o system

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but allowing also B and C to be unbounded:
A is the generator of a C_0 semigroup,
C maps $\mathrm{dom}\,(A) \to \mathcal{Y}$ (continuous w.r.t. graph norm of A),
B maps $\mathcal{U} \to \mathcal{X}_{-1}$, where $\mathcal{X}_{-1}$ is an "extrapolation space", which contains $\mathcal{X}$ as a dense subspace,
D maps $\mathcal{U} \to \mathcal{Y}$.

This class of systems has been studied in a sequence of papers by George Weiss (the first of these appeared in 1989). (See also (Sal87) and (Šmu86).)

After a small modification (replace "regular" by "compatible") this becomes a good class for the study of boundary control systems.

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System Nodes

In the theory of "regular" and "compatible" systems the definition of the operator feed-through operator D causes some problems. One solution to this problem is to collapse the block matrix operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ into one operator, called the system node S, and to rewrite (1) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{2}$$

A classical trajectory $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ of (2) satisfies $x, \dot{x} \in C(\mathbb{R}^+; \mathcal{X})$, $u \in C(\mathbb{R}^+; \mathcal{U})$, and $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathrm{dom}\,(S)$ for all $t \in \mathbb{R}^+$.

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System Node (the "simplest" version from (Sta05))

Definition

By an operator node on a triple of Hilbert spaces $(\mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean a (possibly unbounded) linear operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ with the following properties. We denote $\mathrm{dom}\,(A) = \left\{x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right] \in \mathrm{dom}\,(S)\right\}$, define $A\colon \mathrm{dom}\,(A) \to \mathcal{X}$ by $Ax = P_{\mathcal{X}} S \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right]$, and require the following conditions to hold:

1. S is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ (with domain $\mathrm{dom}\,(S)$).
2. $P_{\mathcal{X}} S$ is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\mathcal{X}$ (with domain $\mathrm{dom}\,(S)$).
3. $\mathrm{dom}\,(A)$ is dense in $\mathcal{X}$ and $\rho(A) \neq \emptyset$.
4. For every $u \in \mathcal{U}$ there exists a $x \in \mathcal{X}$ with $\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{dom}\,(S)$.

We call S a system node if, in addition, A generates a C_0 semigroup.

Time domain well-posed i/s/o systems

Definition

An i/s/o system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where S is a "system node", is time-domain well-posed if there exists a nonnegative function η such that all classical trajectories $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ of Σ on $\mathbb{R}^+$ satisfy

$$\begin{aligned} &\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|y(s)\|^2_{\mathcal{Y}}\, ds \\ &\qquad \le \eta(t)^2 \left(\|x(0)\|^2_{\mathcal{X}} + \int_0^t \|u(s)\|^2_{\mathcal{U}}\, ds \right), \qquad t \in \mathbb{R}^+. \end{aligned}$$

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Graph form of i/s/o system

We can rewrite the i/s/o equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \tag{2}$$

in graph form to get:

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \\ x(t) \\ u(t) \end{bmatrix} \in \mathrm{graph}\,(S), \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \tag{3}$$

$$\mathrm{graph}\,(S) := \left\{ \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \;\middle|\; \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}\,(S),\ \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\}. \tag{4}$$

In this form it does not matter if S is a (single-valued) operator or a multi-valued operator, i.e., a relation.

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Graph form of i/s/o system

If S is a relation, then S maps every pair $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ into an affine subspace (which may be empty for some $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$), and the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \tag{2}$$

must be replaced by the inclusion

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Can we say anything about equations of this type?

Throughout the rest of the talk I assume that S is a closed relation (i.e., the graph of S is a closed subspace of $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{U} \\ \mathcal{Y} \end{bmatrix}$).

For this class of systems I shall not say anything about time domain well-posedness.

Instead I shall look at frequendy domain well-posedness.

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Frequency domain well-posedness

By taking (formal) Laplace transforms in the equation

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} \in S \begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \tag{5}$$

and using the fact that S is closed we get

$$\begin{bmatrix}\lambda\hat{x}(\lambda) - x_0\\ \hat{y}(\lambda)\end{bmatrix} \in S \begin{bmatrix}\hat{x}(\lambda)\\ \hat{u}(\lambda)\end{bmatrix}. \tag{6}$$

Definition

The system (5) is frequency domain well-posed if there exists at least one $\lambda \in \mathbb{C}$ such that the equation (6) defines a bounded linear everywhere defined map from $\left[\begin{smallmatrix}x_0\\ \hat{u}(\lambda)\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}\hat{x}(\lambda)\\ \hat{y}(\lambda)\end{smallmatrix}\right]$.

(In the time-domain well-posed case this condition will be true for all λ in some right half-plane.)

The node bundle

Clearly the condition (6) is (by definition) equivalent to

$$\begin{bmatrix} \lambda\hat{x}(\lambda)-x_0 \\ \hat{y}(\lambda) \\ \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \operatorname{graph}(S)\,. \qquad \textbf{(??)}$$

and this can further be rewritten in another equivalent form, namely

$$\begin{bmatrix} x_0 \\ \hat{y}(\lambda) \\ \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \mathfrak{E}(\lambda), \qquad (7)$$

where

$$\mathfrak{E}(\lambda) = \begin{bmatrix} -1_{\mathcal{X}} & 0 & \lambda & 0 \\ 0 & 1_{\mathcal{Y}} & 0 & 0 \\ 0 & 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}} \end{bmatrix} \operatorname{graph}(S)\,. \qquad (8)$$

We call $\mathfrak{E}$ the node bundle of the system. It is a subspace-valued analytic function of the complex variable λ. If $\mathcal{U} = \mathcal{Y} = \{0\}$, then $\mathfrak{E}(\lambda) = \operatorname{graph}(\lambda - A)$.

The graph representation

Lemma

The system (3) is frequency domain well-posed if and only if there exists at least one $\lambda \in \mathbb{C}$ *such that* $\mathfrak{E}(\lambda)$ *has the graph representation*

$$\mathfrak{E}(\lambda) = \operatorname{im}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) \tag{9}$$

for some bounded linear operator
$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$.

The proof is trivial.

Another graph representation

Of course, the same condition can be written directly in terms of $\operatorname{graph}(S)$ without using the node bundle $\mathfrak{E}(\lambda)$:

Lemma

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$$\operatorname{graph}(S) = \operatorname{im}\left(\begin{bmatrix} \lambda\widehat{\mathfrak{A}}(\lambda) - 1_{\mathcal{X}} & \lambda\widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) \tag{10}$$

for some operator $\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \in \mathcal{L}\left(\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}\right)$.

The proof is still trivial.

The i/s/o resolvent matrix

Definition

1. The i/s/o resolvent set $\rho_{\rm iso}(S)$ of a closed relation $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ consists of those point $\lambda \in \mathbb{C}$ for which $\mathrm{graph}\,(S)$ has a representation of the type

$$\mathrm{graph}\,(S) = \mathrm{im}\left(\begin{bmatrix} \lambda\widehat{\mathfrak{A}}(\lambda)-1_{\mathcal{X}} & \lambda\widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) \tag{10}$$

   for some operator $\widehat{\mathfrak{S}}(\lambda) = \left[\begin{smallmatrix}\widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\end{smallmatrix}\right] \in \mathcal{L}\left(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right];\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]\right)$.

2. The i/s/o resolvent matrix of S is the operator-valued function $\widehat{\mathfrak{S}}(\lambda)$ above defined for all $\lambda \in \mathrm{dom}\left(\widehat{\mathfrak{S}}(\lambda)\right) := \rho_{\rm iso}(S)$.

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$$\begin{bmatrix} -1_{\mathcal{X}} & 0 & \lambda & 0 \\ 0 & 1_{\mathcal{Y}} & 0 & 0 \\ 0 & 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}} \end{bmatrix} \mathrm{graph}\,(S) = \mathrm{im}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right), \qquad (10)$$

for some operator $\widehat{\mathfrak{S}}(\lambda) = \left[\begin{smallmatrix}\widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\end{smallmatrix}\right] \in \mathcal{L}\left(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]; \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]\right)$.

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The (standard) resolvent of a relation

By taking $\mathcal{U} = \mathcal{Y} = \{0\}$ our i/s/o resolvent becomes the "standard" resolvent of a closed relation:

Definition

1. The resolvent set $\rho(A)$ of a closed relation $A\colon \mathcal{X} \to \mathcal{X}$ consists of those point $\lambda \in \mathbb{C}$ for which $\mathrm{graph}\,(\lambda - A) := \left\{ \left[\begin{smallmatrix} \lambda x - y \\ x \end{smallmatrix}\right] \;\middle|\; x \in \mathrm{dom}\,(A)\,,\ y \in Ax \right\}$ has a representation of the type

$$\mathrm{graph}\ (\lambda - A) = \begin{bmatrix} -1_{\mathcal{X}} & \lambda \\ 0 & 1_{\mathcal{X}} \end{bmatrix} \mathrm{graph}\,(A) = \mathrm{im}\left(\begin{bmatrix} 1_{\mathcal{X}} \\ \widehat{\mathfrak{A}}(\lambda) \end{bmatrix} \right) \qquad (11)$$

   for some operator $\widehat{\mathfrak{A}}(\lambda) \in \mathcal{L}\,(\mathcal{X})$.

2. The resolvent of A is the operator-valued function $\widehat{\mathfrak{S}}(\lambda)$ above defined for all $\lambda \in \mathrm{dom}\left(\widehat{\mathfrak{S}}(\lambda)\right) := \rho(A)$.

In this case the "node bundle" is simply the graph of $\lambda - A$.

Terminology

We call:

- $\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}$ is the i/s/o resolvent matrix,
- $\widehat{\mathfrak{A}}(\lambda)$ is the s/s resolvent function,
- $\widehat{\mathfrak{B}}(\lambda)$ is the i/s resolvent function (= the "incoming wave function" or the "Gamma field"),
- $\widehat{\mathfrak{C}}(\lambda)$ is the s/o resolvent function (= the "outgoing wave function"),
- $\widehat{\mathfrak{D}}(\lambda)$ is the i/o resolvent function (= the "transfer function" or the "Weyl function").

The i/s/o resolvent identities

Theorem

The i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ *satisfies the following i/s/o resolvent identities for all* $\lambda, \mu \in \operatorname{dom}\left(\widehat{\mathfrak{S}}\right)$*:*

$$\widehat{\mathfrak{S}}(\lambda) = \widehat{\mathfrak{S}}(\mu) + (\mu - \lambda) \begin{bmatrix} \widehat{\mathfrak{A}}(\mu) \\ \widehat{\mathfrak{C}}(\mu) \end{bmatrix} \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \end{bmatrix}. \qquad (12)$$

or equivalently,

$$\begin{aligned} \widehat{\mathfrak{A}}(\lambda) - \widehat{\mathfrak{A}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{A}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{A}}(\lambda)\widehat{\mathfrak{A}}(\mu), \\ \widehat{\mathfrak{B}}(\lambda) - \widehat{\mathfrak{B}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{B}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{A}}(\lambda)\widehat{\mathfrak{B}}(\mu), \\ \widehat{\mathfrak{C}}(\lambda) - \widehat{\mathfrak{C}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{C}}(\mu)\widehat{\mathfrak{A}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{C}}(\lambda)\widehat{\mathfrak{A}}(\mu), \\ \widehat{\mathfrak{D}}(\lambda) - \widehat{\mathfrak{D}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{C}}(\mu)\widehat{\mathfrak{B}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{C}}(\lambda)\widehat{\mathfrak{B}}(\mu). \end{aligned} \qquad (13)$$

These identities imply, among others, that $\widehat{\mathfrak{S}}$ must be analytic.

- In (Opm06) Mark Opmeer uses the above i/s/o resolvent identities to define what he calls a resolvent linear system. It consists of a quadruple of operator-valued functions $\widehat{\mathfrak{S}} = \begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}$ which satisfy the i/s/o resolvent identities on some open connected subset Ω of the complex plane.
- By adding the condition that Ω contains some right half-plane and that the above functions are polynomially bounded on that half plane he gets a class of dynamical systems, which he calls integrated resolvent linear systems.
- He also defines a slightly larger class of dynamical systems that he calls distributional resolvent linear systems.

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Mark Opmeer's "Resolvent Linear Systems

- In (Opm06) Mark Opmeer uses the above i/s/o resolvent identities to define what he calls a resolvent linear system. It consists of a quadruple of operator-valued functions $\widehat{\mathfrak{S}} = \begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}$ which satisfy the i/s/o resolvent identities on some open connected subset Ω of the complex plane.
- By adding the condition that Ω contains some right half-plane and that the above functions are polynomially bounded on that half plane he gets a class of dynamical systems, which he calls integrated resolvent linear systems.
- He also defines a slightly larger class of dynamical systems that he calls distributional resolvent linear systems.

Definition

1. A $\mathcal{L}(\mathcal{X})$-valued function $\widehat{\mathfrak{A}}$ defined on some open set $\Omega \in \mathbb{C}$ is called a pseudo-resolvent if it satisfies

$$\widehat{\mathfrak{A}}(\lambda) - \widehat{\mathfrak{A}}(\mu) = (\mu - \lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{A}}(\lambda) \tag{14}$$

for all λ, $\mu \in \Omega$.

2. A $\mathcal{L}\left(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]; \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]\right)$-valued function $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$ defined on some open set $\Omega \in \mathbb{C}$ is called an i/s/o pseudo-resolvent matrix if it satisfies the i/s/o resolvent identity

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Pseudo-resolvents

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Pseudo-resolvents are resolvents!

Lemma

1. *If $\widehat{\mathfrak{A}}$ is the resolvent of a closed relation $A\colon \mathcal{X} \to \mathcal{X}$, then $\widehat{\mathfrak{A}}$ satisfies the resolvent identity* (14) *for all $\lambda, \mu \in \rho(A)$.*
2. *Conversely, if $\widehat{\mathfrak{A}}$ is a pseudo-resolvent defined on some open set $\Omega \subset \mathbb{C}$, then $\widehat{\mathfrak{A}}$ is the restriction to Ω of the resolvent of some closed relation $A\colon \mathcal{X} \to \mathcal{X}$.*
3. *A is single-valued if and only if $\widehat{\mathfrak{A}}(\lambda)$ is injective for some (and hence for all) $\lambda \in \Omega$.*
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How is all this related to "operator nodes"?

Definition

Recall: By an operator node on a triple of Hilbert spaces $(\mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean a (possibly unbounded) linear operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ with the following properties. We denote $\mathrm{dom}\,(A) = \left\{x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right] \in \mathrm{dom}\,(S)\right\}$, define $A\colon \mathrm{dom}\,(A) \to \mathcal{X}$ by $Ax = P_{\mathcal{X}} S \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right]$, and require the following conditions to hold:

1. S is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ (with domain $\mathrm{dom}\,(S)$).
2. $P_{\mathcal{X}} S$ is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\mathcal{X}$ (with domain $\mathrm{dom}\,(S)$).
3. $\mathrm{dom}\,(A)$ is dense in $\mathcal{X}$ and $\rho(A) \neq \emptyset$.
4. For every $u \in \mathcal{U}$ there exists a $x \in \mathcal{X}$ with $\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{dom}\,(S)$.

We call S a system node if, in addition, A generates a C_0 semigroup.

S operator node $\Leftrightarrow \rho_{\text{iso}}(S) \neq \emptyset$

Theorem

A linear (single-valued) operator $S\colon \begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix} \to \begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$ is an operator node if and only if $\rho_{\text{iso}}(S) \neq \emptyset$, i.e., the if and only if the system

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} \in S \begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{5}$$

is frequency domain well-posed.

- In particular, every time domain well-posed i/s/o system is automatically frequency domain well-posed. The converse is not true.
- The system (5) can be frequency domain well-posed even in the case where S is a relation.

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Example: the differentiator

Take $\mathcal{X} = \mathcal{U} = \mathcal{Y} = \mathbb{C}$,
$A = 0$, $B = 1$, $C = 1$, $D = 0$, $S = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,

$$\Sigma : \begin{cases} \dot{x}(t) = u(t), \\ y(t) = x(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0.$$

This is a integrator: $y(t) = x_0 + \int_0^t u(s)\, ds$, $t \in \mathbb{R}^+$, and the i/s/o resolvent matrix of this system is

$$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} = \begin{bmatrix} 1/\lambda & 1/\lambda \\ 1/\lambda & 1/\lambda \end{bmatrix}.$$

Let us in this system change the meaning of u and y, so that y becomes the input, and u the output. This inverted system will then be a differentiator, and it will be a system of the type

$$\begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \qquad (5)$$

for a suitable relation S.

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Example: the differentiator

It turns out that S is the purely multi-valued relation whose graph is

$$\mathrm{graph}\,(S) = \mathrm{im}\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}\right).$$

Thus,

$$\mathrm{dom}\,(S) = \{\left[\begin{smallmatrix} x \\ x \end{smallmatrix}\right] \mid x \in \mathbb{C}\}, \qquad \mathrm{mul}\,(S) = \mathrm{im}\,(S) = \{\left[\begin{smallmatrix} u \\ u \end{smallmatrix}\right] \mid u \in \mathbb{C}\}.$$

If $\begin{bmatrix} x(t) \\ y(t) \\ u(t) \end{bmatrix}$ is a trajectory of this system, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in \mathrm{dom}\,(S)$, or equivalently, $x(t) = y(t)$, and $\begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} \in \mathrm{im}\,(S)$, i.e., $\dot{x}(t) = u(t)$. Thus, $u(t) = \dot{y}(t)$. The i/s/o resolvent matrix of this system is

$$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \lambda \end{bmatrix}.$$

Outline of Talk

- Time domain well-posed input/state/output systems
- Frequency domain well-posed input/state/output systems
- **Intertwinement in time and frequency domain**
- Compressions and dilations in time and frequency domain
- Controllability, observability, and minimality
- Work in progress

Intertwinement in time domain

Definition

Let $\Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ and $\Sigma_2 = (S_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y})$ be two time domain well-posed i/s/o systems (with the same input and output spaces), and let R be a linear relation $\mathcal{X}_1 \to \mathcal{X}_2$. We say that Σ_1 and Σ_2 are intertwined by R if the following condition holds: If $\begin{bmatrix} x_1 \\ y_1 \\ u \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ u \end{bmatrix}$ are trajectories of Σ_1 and Σ_2 on $\mathbb{R}^+$, respectively (with the same input function u), and if $x_2(0) \in Rx_1(0)$, then $y_1 = y_2$ and $x_2(t) \in Rx_1(t)$ for all $t \in \mathbb{R}^+$.

The characteristic time domain operators

Notation for well-posed systems:

- $\mathfrak{A}^t$ is the map from the initial state $x_0 \in \mathcal{X}$ at time $t = 0$ to the final state $x(t) \in \mathcal{X}$ at time $t \geq 0$ when the input is zero.
- $\mathfrak{B}$ is the map from an input $u \in L^2(\mathbb{R}^-;\mathcal{U})$ with compact support into the final state $x(0) \in \mathcal{X}$ at time zero, when we take the initial state to be zero for large negative time.
- $\mathfrak{C}$ is the map from the initial state $x_0 \in \mathcal{X}$ at time $t = 0$ to the output $y \in L^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{Y})$ when the input is zero.
- $\mathfrak{D}$ is the map from an input $u \in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{U})$ whose support is bounded to the left to the output $y \in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{Y})$, when we take the initial state to be zero for large negative time.

Intertwinement in terms of characteristic operators

Theorem

The two time domain well-posed i/s/o systems Σ_1 *and* Σ_2 *are intertwined by the closed relation* R *if and only if the characteristic time domain operators of these systems satisfy:*

1. $\mathfrak{A}_2^t x_2 \in R\mathfrak{A}_1^t x_1$ *for all* $x_2 \subset Rx_1$ *and all* $t \in \mathbb{R}^+$.
2. *For all* $u \in L^2(\mathbb{R}^-;\mathcal{U})$ *with compact support we have* $\mathfrak{B}_2 u \in R\mathfrak{B}_1 u$.
3. $\mathfrak{C}_2 x_2 = \mathfrak{C}_1 x_1$ *for all* $x_2 \subset Rx_1$
4. $\mathfrak{D}_2 = \mathfrak{D}_1$.

Theorem

The two time domain well-posed i/s/o systems Σ_1 *and* Σ_2 *are intertwined by some closed relation* R *if and only if they have the same i/o map.*

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The two time domain well-posed i/s/o systems Σ_1 and Σ_2 are intertwined by the closed relation R if and only if the characteristic time domain operators of these systems satisfy:

1. $\mathfrak{A}_2^t x_2 \in R\mathfrak{A}_1^t x_1$ *for all* $x_2 \subset Rx_1$ *and all* $t \in \mathbb{R}^+$.
2. *For all* $u \in L^2(\mathbb{R}^-;\mathcal{U})$ *with compact support we have* $\mathfrak{B}_2 u \in R\mathfrak{B}_1 u$.
3. $\mathfrak{C}_2 x_2 = \mathfrak{C}_1 x_1$ *for all* $x_2 \subset Rx_1$
4. $\mathfrak{D}_2 = \mathfrak{D}_1$.

Theorem

The two time domain well-posed i/s/o systems Σ_1 and Σ_2 are intertwined by some closed relation R if and only if they have the same i/o map.

Intertwinement in frequency domain

Theorem

Let Σ_1 and Σ_2 be two time domain well-posed linear systems, with growth rates ω_1 and ω_2, respectively, let $\omega = \max\{\omega_1, \omega_2\}$, and denote $\mathbb{C}^+_\omega = \{\lambda \in \mathbb{C} \mid \Re\lambda > \omega\}$. Then Σ_1 and Σ_2 are intertwined by the closed relation R if and only if the following frequency domain conditions hold:

1. $\widehat{\mathfrak{A}}_2(\lambda)x_2 \in R\widehat{\mathfrak{A}}_1(\lambda)x_1$ *for all* $x_2 \subset Rx_1$ *and all* $\lambda \in \mathbb{C}^+_\omega$.
2. $\widehat{\mathfrak{B}}_2(\lambda)u_0 \in R\widehat{\mathfrak{B}}_1(\lambda)u_0$ *for all* $u_0 \in \mathcal{U}$ *and* $\lambda \in \mathbb{C}^+_\omega$.
3. $\widehat{\mathfrak{C}}_2(\lambda)x_2 = \widehat{\mathfrak{C}}_1(\lambda)x_1$ *for all* $x_2 \subset Rx_1$ *and all* $\lambda \in \mathbb{C}^+_\omega$.
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Outline of Talk

- Time domain well-posed input/state/output systems
- Frequency domain well-posed input/state/output systems
- Intertwinement in time and frequency domain
- **Compressions and dilations in time and frequency domain**
- Controllability, observability, and minimality
- Work in progress

Compressions and dilations in time domain

Definition

Let $\mathcal{X}_1$ be a closed subspace of $\mathcal{X}_2$, and let $\Sigma_1 = \big(S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}\big)$ and $\Sigma_2 = \big(S_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}\big)$ be two time domain well-posed i/s/s systems. We call Σ_1 the (orthogonal) compression of Σ_2 onto $\mathcal{X}_1$, and we call Σ_2 an (orthogonal) dilation of Σ_1, if the following condition holds:

- For each $x_0 \in \mathcal{X}$ and each $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{U})$, if we denote the (generalized) future trajectories of Σ_1 and Σ_2 with initial state x_0 and input function u by $\begin{bmatrix} x_1 \\ y_1 \\ u \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ u \end{bmatrix}$, respectively, then $y_1 = y_2$ and $x_1(t) = P_{\mathcal{X}_1} x_2(t)$ for all $t \in \mathbb{R}^+$.

Compressions in terms of characteristic operators

Theorem

The time domain well-posed i/s/o system Σ is the compression onto $\mathcal{X}$ of the time domain well-posed i/s/o system Σ_1 (i.e., Σ_1 is a dilation of Σ) if and only if the characteristic time domain operators of these systems satisfy:

1. $\mathfrak{A}_1^t = P_{\mathcal{X}_1}\mathfrak{A}_2^t|_{\mathcal{X}_1}$ *for all* $t \in \mathbb{R}^+$.
2. $\mathfrak{B}_1 = P_{\mathcal{X}_1}\mathfrak{B}_2$.
3. $\mathfrak{C}_1 = \mathfrak{C}_2|_{\mathcal{X}_1}$.
4. $\mathfrak{D}_1 = \mathfrak{D}_2$.

Theorem

Every dilation (and compression) can be interpreted as a special case of an intertwinement (for a suitable bounded single-valued intertwining operor R with closed domain).

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1. $\widehat{\mathfrak{A}}_1(\lambda) = P_{\mathcal{X}_1}\widehat{\mathfrak{A}}_2(\lambda)|_{\mathcal{X}_1}$ *for all* $\lambda \in \mathbb{C}^+_\omega$.
2. $\widehat{\mathfrak{B}}_1(\lambda) = \widehat{\mathfrak{B}}_2(\lambda)$ *for all* $\lambda \in \mathbb{C}^+_\omega$ *(in particular,* $\mathrm{im}\left(\widehat{\mathfrak{B}}_2(\lambda)\right) \subset \mathcal{X}_1$*).*
3. $\widehat{\mathfrak{C}}_1(\lambda) = \widehat{\mathfrak{C}}_2(\lambda)|_{\mathcal{X}_1}$ *for all* $\lambda \in \mathbb{C}^+_\omega$.
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Controllable and observable systems

Definition

Let $\Sigma = (X; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a time domain well-posed i/s/o system.

- Σ is controllable if $\operatorname{im}(\mathfrak{B})$ is dense in $\mathcal{X}$
- Σ is observable if $\ker(\mathfrak{C}) = \{0\}$.

Theorem

Let $\Sigma = (X; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a time domain well-posed i/s/o system with growth bound $\omega(\Sigma)$.

- *Σ is controllable if and only if $\bigvee_{\lambda \in \mathbb{C}^+_{\omega(\Sigma)}} \operatorname{im}\left(\widehat{\mathfrak{B}}(\lambda)\right) = \mathcal{X}$.*
- *Σ is observable if and only if $\bigcap_{\lambda \in \mathbb{C}^+_{\omega(\Sigma)}} \ker\left(\widehat{\mathfrak{C}}(\lambda)\right) = \{0\}$.*

Minimal Systems

Definition

A time domain well-posed i/s/o system Σ is minimal if it does not have any nontrivial compressions (i.e., it is not a nontrivial dilation of any other well-posed i/s/o system).

Theorem

A time domain well-posed i/s/o system Σ is minimal if and only if it is both controllable and observable.

Theorem

Every non-minimal time domain well-posed i/s/o system Σ can be compressed into a minimal time domain well-posed i/s/o system.

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Intertwinements, Dilations, Compressions for multi-valued i/s/o systems

- Above I defined the basic notions of intertwinements, dilations, compressions, controllability, observability, and minimality in the time domain, assuming time domain well-posedness, and then gave frequency domain interpretations of these notions.
- If a system is not time-domain well-posed, then the above time domain definitions are no longer valid.
- However, nothing prevents us from using the frequency domain characterizations of intertwinements, dilations, compressions, controllability, observability, and minimality as definitions of these notions. Such definitions make sense as soon as the system is frequency domain well-posed.
- This seems to work well even when the generating operator S is allowed to be multi-valued (as long as the system is frequency domain well-posed).

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Observations

- We seem to be able to prove more or less the same results in this frequency domain setting as in the standard time domain well-posed setting.
- So far we have encounterd only one major problem: We can still compress every nonminimal system into a minimal one, but we have not been able to prove that the compressed generating operator is always single-valued whenever the original generating operator S is single-valued.
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