Riccati equations and optimal control for infinite-dimensional linear systems

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Abstract

We generalize the standard theory on algebraic Riccati equations and optimization to infinite-dimensional well-posed linear systems, thus completing the work of George Weiss, Olof Staffans and others. We show that the optimal control is given by the stabilizing solution of an integral Riccati equation. If, e.g., the input operator is not maximally unbounded, then this integral Riccati equation is equivalent to an algebraic Riccati equation.

Our theory covers all quadratic (possibly indefinite) cost functions, but the optimal state feedback need not be well-posed unless the cost function is uniformly positive or the system is sufficiently regular. If one allows controls that do not stabilize the state, just the output, then the definition of the stabilizing solution becomes more complicated. We treat this and some other phenomena that are met also in the finite-dimensional setting but more important in the infinite-dimensional one.

A linear time-invariant system is typically governed by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

(for $t \geq 0$), $x(0) = x_0$, where the *generators* $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(H \times U, H \times Y)$ are matrices, or more generally, linear operators between Hilbert spaces (U, H, Y) of arbitrary dimensions.

Given x_0 and u, the *state* x and *output* y equal

$$x(t) = \mathscr{A}^t x_0 + \mathscr{B}^t u, \quad y = \mathscr{C} x_0 + \mathscr{D} u, \tag{1}$$
$$\mathscr{A}^t = e^{At}, \quad \mathscr{B}^t u = \int_0^t \mathscr{A}^{t-s} Bu(s)\, ds,$$
$$(\mathscr{C} x_0)(t) = C\mathscr{A}^t x_0, \; (\mathscr{D} u)(t) = C\mathscr{B}^t u + Du(t).$$

We study *Well-Posed Linear Systems (WPLSs)* ("Salamon–Weiss class"), i.e., time-invariant systems of form (1), with $\mathscr{A}^t, \mathscr{B}^t, \mathscr{C}, \mathscr{D}$ linear, bounded, compatible with each other and continuous on $H \times \mathrm{L}^2_{\mathrm{loc}}$. It follows that $\mathscr{A}$ is a C_0-semigroup and A, B, C exist to satisfy $\dot{x} = Ax + Bu$ (and $y = Cx$ when $u = 0$), but A, B, C may be unbounded. Such systems are equivalent to Lax–Phillips scattering systems and to the operator-based models of Béla Sz.-Nagy and Ciprian Foiaş [S04]. If also D exists, then the WPLS is called *regular*. [M04a]

Theorem 1 *If B is* bounded *($B \in \mathcal{B}(U, H)$), then the following are equivalent:*

(i) *For each initial state $x_0 \in H$, there exists a unique control $u : \mathbb{R}_+ \to U$ that minimizes*

$$\mathcal{J}(x_0, u) := \int_0^\infty \left(\|x(t)\|_H^2 + \|u(t)\|_U^2 \right) dt. \tag{2}$$

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(ii) *The algebraic Riccati equation (ARE)*

$$\mathcal{P}BB^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + I \qquad (3)$$

has an exponentially stabilizing[1] *solution.*

(iii) *For each $x_0 \in H$, there exists $u \in \mathrm{L}^2$ such that $\mathcal{J}(x_0,u) < \infty$.*

Any solution $\mathcal{P}$ of (ii) is unique, and the (state-feedback) control $u(t) = Kx(t)$ strictly minimizes the cost $\mathcal{J}(x_0,\cdot)$ for any $x_0 \in H$. Moreover, the minimal cost equals $\langle x_0, \mathcal{P}x_0\rangle_H$.

In fact, a solution of (3) is exponentially stabilizing iff it is nonnegative.

As above, for any *initial state* $x_0 \in H$, we want to minimize the cost $\mathcal{J}(x_0,\cdot)$ over

$$\mathcal{U}_{\mathrm{exp}}(x_0) := \{u \in \mathrm{L}^2(\mathbb{R}_+;U) \,|\, x \in \mathrm{L}^2(\mathbb{R}_+;H)\},$$

the set of *exponentially stabilizing controls*, as in Theorem 1. To cover more general cost functions, we allow for any $J = J^* \in \mathcal{B}(Y)$ in

$$\mathcal{J}(x_0,u) := \int_0^\infty \langle y(t), Jy(t)\rangle_Y \, dt. \qquad (4)$$

(Take $C := \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, $D := \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $J := \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$ get (2).)

Now we can generalize the above result for arbitrary cost functions; for simplicity, we require the *indicator* (or signature operator) $S := D^*JD$ to be uniformly positive:

Theorem 2 *If $D^*JD \geq \varepsilon I$ for some $\varepsilon > 0$ and B is bounded, then the following are equivalent:*

(i) *There is a unique minimizing control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each initial state $x_0 \in H$.*

(ii) *The ARE*

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC, \\ \quad S = D^*JD, \\ \quad K = -S^{-1}(B^*\mathcal{P} + D^*JC), \end{cases} \qquad (5)$$

has an exponentially stabilizing solution.

Any solution $\mathcal{P}$ of (ii) is unique, and the state feedback $u(t) = \overline{K}x(t)$ (a.e.) minimizes (4). The minimal cost equals $\langle x_0, \mathcal{P}x_0\rangle_H$.

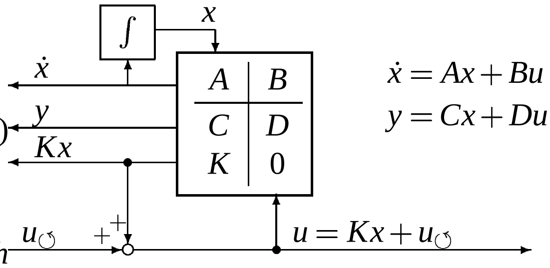

Figure 1: State feedback connection

(Here $\overline{K}x_1 := \lim_{s\to+\infty} Ks(s-A)^{-1}x_1$; we have $\overline{K} = K$ if K is bounded (e.g., if C is).) The equations are given on $\mathrm{Dom}(A), U$ and $\mathrm{Dom}(A)$. Under an external disturbance $u_{\circlearrowleft} : \mathbb{R}_+ \to U$ (i.e., $u = \overline{K}x + u_{\circlearrowleft}$), we get $\mathcal{J}(x_0,u) = \langle x_0, \mathcal{P}x_0\rangle + \langle u_{\circlearrowleft}, Su_{\circlearrowleft}\rangle$.

Next we give up the boundedness of B but require that the system is regular (i.e., that the transfer function has a weak limit at $+\infty$):

Theorem 3 *For any regular WPLS, the following are equivalent:*

(i) *There is an optimal regular state-feedback operator $K \in \mathcal{B}(\mathrm{Dom}(A), U)$;*

(ii) *The ARE*

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC, \\ \quad S = D^*JD + \lim_{s\to+\infty} \overline{B^*}\mathcal{P}(s-A)^{-1}B, \\ \quad SK = -(\overline{B^*}\mathcal{P} + D^*JC), \end{cases} \qquad (6)$$

has an exponentially stabilizing solution.

Any solution $\mathcal{P}$ of (ii) is unique, and the state feedback $u(t) = \overline{K}x(t)$ (a.e.) is optimal. For this control, the cost is given by $\mathcal{J}(x_0,u) = \langle x_0, \mathcal{P}x_0\rangle_H$.

(Here $\overline{B^*} := \lim_{s\to+\infty} B^*s(s-A^*)^{-1}$.) By u being *optimal* we mean that $\frac{d\mathcal{J}(x_0,u)}{du} = 0$ (this is a Fréchet derivative). The control u is minimizing iff $S \geq 0$. (The indefinite case gives, e.g., the maximin control of a H^∞ problem.) The optimal control is unique iff the indicator S is one-to-one.

[1] $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ s.t. (3) holds and $x \in \mathrm{L}^2$ for all $x_0 \in H$ (equivalently, $\|e^{t(A+BK)}\| \leq Me^{-\varepsilon t}$ for all $t > 0$ and some $M, \varepsilon > 0$) under the state feedback $u(t) = Kx(t)$, $K := -B^*\mathcal{P}$.

If, e.g., the input operator B is *not maximally unbounded* ($\|(s-A)^{-1}B\| \le Ms^{-1/2-\varepsilon}$ for all $s > M$), then $S = D^*JD$ and any optimal state-feedback is regular.

However, in general a regular WPLS may have an irregular optimal control, and not all WPLSs are regular. Thus, to cover all WPLSs, we must use the IRE instead of the ARE:

Theorem 4 *There is an optimal state feedback iff the following* integral Riccati equation (IRE) *has an exponentially stabilizing solution:*

$$\mathscr{K}^{t*}S\mathscr{K}^t = \mathscr{A}^{t*}\mathcal{P}\mathscr{A}^t - \mathcal{P} + \mathscr{C}^{t*}J\mathscr{C}^t, \tag{7a}$$

$$\mathscr{X}^{t*}S\mathscr{X}^t = \mathscr{D}^{t*}J\mathscr{D}^t + \mathscr{B}^{t*}\mathcal{P}\mathscr{B}^t, \tag{7b}$$

$$\mathscr{X}^{t*}S\mathscr{K}^t = -\left(\mathscr{D}^{t*}J\mathscr{C}^t + \mathscr{B}^{t*}\mathcal{P}\mathscr{A}^t\right). \tag{7c}$$

(Here $\mathscr{C}^t := \chi_{[0,t]}\mathscr{C}$, $\mathscr{D}^t := \chi_{[0,t]}\mathscr{D}\chi_{[0,t]}$ etc., where $\chi_{[0,t]}$ is the characteristic function of $[0,t]$, equivalently, the natural projection $\mathrm{L}^2_{\mathrm{loc}} \to \mathrm{L}^2([0,t];U)$ (or its adjoint, the embedding).)

If B is bounded, $C = \begin{bmatrix}\widetilde{C}\\0\end{bmatrix}$, $D = \begin{bmatrix}0\\I\end{bmatrix}$, $J = \begin{bmatrix}I&0\\0&I\end{bmatrix}$ (hence $\mathcal{J}(x_0,u) = \|u\|_2^2 + \|\widetilde{C}x\|_2^2$), then, by (5), we get $S = I$, $K = -B^*\mathcal{P}$, hence then the ARE reduces to $\mathcal{P}BB^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + \widetilde{C}^*\widetilde{C}$, or equivalently (integrate it over $[0,t]$), to (7a), which now becomes

$$\mathcal{P}x_0 = \mathscr{A}^{t*}\mathcal{P}\mathscr{A}^t x_0 + \int_0^t \mathscr{A}^{s*}(\widetilde{C}^*\widetilde{C} - \mathcal{P}BB^*\mathcal{P})\mathscr{A}^s x_0\, ds \tag{8}$$

(for all $x_0 \in H$), familiar from classical results.

The optimal control equals $u = \mathscr{X}^{-1}\mathscr{K}x_0$ (i.e., $u = \mathscr{K}x_0 + (I - \mathscr{X})u[+u_{\circlearrowleft}]$), with cost $\mathcal{J}(x_0,u) = \langle x_0, \mathcal{P}x_0\rangle[+\langle u_{\circlearrowleft}, Su_{\circlearrowleft}\rangle]$.

However, the optimal state feedback may be ill-posed (i.e., $(\mathscr{X}^t)^{-1} : u_{\circlearrowleft} \to u$ or $\mathscr{X}^t$ need not be well-defined on $\mathrm{L}^2_{\mathrm{loc}}$). Nevertheless, if there is a unique optimal control $u_{\mathrm{opt}}^{x_0}$ for each initial state $x_0 \in H$, then the map $\mathscr{K}_{\mathrm{opt}} : x_0 \mapsto u_{\mathrm{opt}}^{x_0}$ and $\mathcal{P}$ form the exponentially stabilizing solution of the $\mathscr{S}^t$-IRE, where the left-hand-side of the IRE is replaced by $\mathscr{K}_{\mathrm{opt}}^{t\,*}\mathscr{S}^t\mathscr{K}_{\mathrm{opt}}^t, \mathscr{S}^t, \mathscr{S}^t\mathscr{K}_{\mathrm{opt}}^t$.

In fact, the $\mathscr{S}^t$-IRE (as well as the IRE) is exactly the discrete-time ARE for the discretized system $\begin{bmatrix}\mathscr{A}^t & \mathscr{B}^t\\ \mathscr{C}^t & \mathscr{D}^t\end{bmatrix}$; this leads to alternative proofs. We also give frequency-domain variants of the IRE and the $\mathscr{S}^t$-IRE in [M04a].

The optimal state-feedback is well-posed iff a certain stable spectral factorization problem has a solution. If $\mathcal{J}(0,\cdot)$ is uniformly positive (e.g., as in (2)) and $\mathcal{U}_{\exp}(x_0) \neq \emptyset$ for all x_0, then this is the case. This led to the generalization to WPLS of numerous classical results on minimization, state-feedback and dynamic stabilization and coprime factorizations in [M04a].

Similar results also hold for other domains of optimization (admissible controls). E.g., for

$$\mathcal{U}_{\mathrm{out}}(x_0) := \{u \in \mathrm{L}^2(\mathbb{R}_+;U) \,|\, y \in \mathrm{L}^2(\mathbb{R}_+;Y)\},$$

the set of *output-stabilizing* controls, we must replace “exponentially stabilizing solution” by the (unique) “solution satisfying $u,y \in \mathrm{L}^2$ and $\langle \mathscr{B}^t u + \mathscr{A}_{\mathrm{opt}}^t x_0, \mathcal{P}\mathscr{A}_{\mathrm{opt}}^t x_0\rangle \to 0$, as $t \to +\infty$”. Fortunately, for the cost function $\mathcal{J} = \|u\|_2^2 + \|y\|_2^2$ (or with $\begin{bmatrix}C\\0\end{bmatrix}, \begin{bmatrix}D\\I\end{bmatrix}, I$ in place of C,D,J) that solution is the smallest nonnegative solution.

Details, corresponding LQR and H^∞ applications, discrete-time results, more detailed historical remarks etc. are given in [M04a] and [M02]. For bounded B,C, many of the above results are well known; see, e.g., [CZ94]. If we neglect the well-posedness of the state feedback and the IRE, most of the results have been established earlier under various assumptions. The well-posedness of the state feedback has been know for Pritchard–Salamon systems, and for several parabolic systems it has been established by, e.g., Irena Lasiecka, Roberto Triggiani and others; see [LT00].

For stable problems, the necessity of the ARE (6) was established by Olof Staffans [S97]; its first and third equations were independently found by Martin Weiss and George Weiss [WW97].

Reciprocal AREs

To overcome the difficulties due to unbounded generators, Ruth Curtain and Mark Opmeer have developed the reciprocal RE theory for the LQR problem assuming that $0 \notin \sigma(A)$ [OC04] [C03]. There the bounded operators $A_- := A^{-1}$, $B_- := A^{-1}B$, $C_- := -CA^{-1}$ are used in place of A,B,C — the surprising fact is that the Riccati operator $\mathcal{P}$ remains the same. E.g., the first equation in the ARE (6), when multiplied by A_-^* to the left and by A_- the right, becomes

$$K_-^* S K_- = \mathcal{P} A_- + A_-^* \mathcal{P} + C_-^* J C_-. \qquad (9)$$

Here $S = D_-^* J D_-$, $SK_- = -B_-^* \mathcal{P} - D_-^* J C_-$ (and $D_- := D + \overline{C}(0-A)^{-1}B \in \mathcal{B}(U,Y)$ or the value of the characteristic function at zero). This was generalized to arbitrary WPLSs and cost functions in [M03] (with $(A-\alpha)^{-1}$ in place of A_- and different formulas for S and K_-), leading to the first generalization of Theorem 1 and similar results to arbitrary WPLSs; see [M04b]. However the results of [M03] and [M04b] established the well-posedness of the optimal state-feedback pair only for the LQR cost function and similar ones, hence they were later partially shadowed by the IRE methods of [M04a], which provided well-posedness for any uniformly positive cost function. Nevertheless, for several purposes the reciprocal and resolvent AREs provide technically much simpler tools, which have been applied succesfully to the generalization of several results from systems with bounded B,C to general WPLSs; see, e.g. [BMS04]. Moreover, the ARE (9) is sometimes more applicable than the IRE or the ARE (6).

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