Reciprocal Passive Linear Time-Invariant Systems

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We develop the theory of reciprocal linear infinite-dimensional time-invariant scattering, impedance and transmission systems in discrete and continuous time. The transfer function F of such a system has the reciprocity property $F(z^*)^* = JF(z)J$, where J is a given signature operator, i.e, J is both self-adjoint and unitary. (An alternative formulation involving involutions is also avaliable.) In the case of a scattering or impedance system the operator J can be taken to be the identity, but in the transmission case we have neither $J = I$ nor $J = -I$. It has been known for a long time that in the finite-dimensional case such an "external" reciprocity symmetry is equivalent to the existence of a realization which has an analogous "internal" symmetry. Here we extend that result to passive systems with infinite-dimensional state spaces.

In the scattering case we have the following result. Let $F(z)$ be a scattering matrix of the Schur class $S(U)$, i.e., F is an operator-valued contractive holomorphic function, mapping U into itself, defined on Ω_+ (which is either the open unit disk $\mathbb{D}$ or the open right half-plane $\mathbb{C}_+$ depending on whether we work in discrete or continuous time). In addition, suppose that F has the reciprocity property $F(z^*)^* = F(z)$ for all $z \in \Omega_+$. Then it is possible to construct two special types of realizations of this scattering matrix, namely simple conservative scattering and minimal balanced passive scattering realizations, that are similar to their adjoints with similarity operators that are signature operators. In particular, the main operators of these two types of internally reciprocal realizations are similar to their adjoints with signature similarity operators. Both types of realizations are unique up to unitary similarity. We get the balanced realization by interpolating half-way between a minimal optimal and a minimal $*$-optimal realization.

Let us next discuss the discrete time impedance case the situation (as an example of a typical non-scattering result). Let F be a function of the Caratheodory-Herglotz class $C(U)$ on the open unit disk $\mathbb{D}$, i.e., F is an operator-valued function defined on $\mathbb{D}$ with the property that $F(z)+F(z)^* \geq 0$ on $\mathbb{D}$. We realize this function as the transfer function of the discrete time system

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \end{aligned} \tag{1}$$

i.e., $F(z) = zC(I - zA)^{-1}B + D$ for all $z \in \mathbb{D}$. Here x takes its values in the separable Hilbert space X (the state space), and u and y take their values in U (the input/output space). We recall that the system (1) is (impedance) *conservative* iff the operator A is unitary (i.e., $A^*A = AA^* = 1$), $C = B^*A$, and $B^*B = D+D^*$. It is *simple* iff it has no nontrivial subspace which is both unobservable and unreachable. It is well-known that a simple conservative realization of the function F always exists, ant that it is unique up to a unitary similarity transformation in X.

We say that the system (1) is *internally reciprocal* if the system (1) is similar to its adjoint and the similarity operator is a signature operator. In other words, there exists a signature operator J_X such that $A^* = J_X A J_X$ and $C^* = J_X B$. It is easy to see that if the realization (1) is simple and conservative, then it is internally reciprocal if it is *externally reciprocal*, i.e., F has the external reciprocity property $F(z^*)^* = F(z)$ for all $z \in \mathbb{D}$.

At the next step we show that among all reciprocal simple impedance conservative system (1) which realize a given reciprocal transfer function F of the Caratheodory-Herglotz class $C(U)$ there always exists a subclass which has a particularly nice structure. In this subclass of realizations it is possible to decompose X into $X = Z \times X_0 \times Z$ in such a way that the operators A, B, and J_X have the following structural decomposition with respect to this splitting:

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & J_0 & 0 \\ 0 & 0 & A_1^* \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_0 \\ A_1^* B_1 \end{bmatrix}, \quad J_X = \begin{bmatrix} 0 & 0 & I \\ 0 & J_0 & 0 \\ I & 0 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} B_1^* A_1 & B_0^* J_0 & B_1^* \end{bmatrix}.$$

Here A_1 is a unitary operator in Z whose spectrum is contained in the closed upper half-circle $\{z = e^{i\varphi} \mid 0 \leq \varphi \leq \pi\}$ and which does not have ± 1 as an eigenvalue, J_0 is a signature operator in X_0, and

$$D + D^* = 2B_1^* B_1 + B_0^* B_0.$$

This means that Σ can be written as a parallell connection of three independent subsystems $\Sigma_+ = \left[\begin{smallmatrix} A_1 & B_1 \\ B_1^* A_1 & D_1 \end{smallmatrix}\right]$, $\Sigma_0 = \left[\begin{smallmatrix} J_0 & B_0 \\ B_0^* & D_0 \end{smallmatrix}\right]$, and $\Sigma_- = \left[\begin{smallmatrix} A_1^* & A_1^* B_1 \\ B_1^* & D_1 \end{smallmatrix}\right]$, with state spaces Z, X_0, and Z, respectively (and with $D_1 = \frac{1}{2} B_1^* B_1$ and $D_0 = \frac{1}{2}(D - D^* + B_0^* B_0)$). All of these three subsystems are minimal and impedance conservative (but the full system need not be minimal).

Analogous results are obtained for other simple conservative sysstems in discrete or continuous time (scattering, impedance, or transmission), and also for minimal balanced passive systems. In particular, in the continuous time scattering case we recover a class of system which has recently been studied by Tucsnak and Weiss. In some of the cases the transfer functions are not defined in the standard affine sense, but rather as formal quotients of pairs of functions. In this way we can even treat transfer functions which are not locally bounded or otherwise not well-defined in large parts of the complex plane (including zero in the discrete time case and infinity in the continuous time case). One particular case of such a transfer function is the constant transfer function $F(z) \equiv D$, where D is a positive unbounded operator on U.

Our proofs use the new notion of a state/signal node described in the presentation “Passive and conservative infinite-dimensional linear state/signal systems” by O. Staffans.