Chapter 2

Signal and system norms

A quantitative treatment of the performance and robustness of control systems requires the introduction of appropriate signal and system norms, which give measures of the magnitudes of the involved signals and system operators. In this chapter we give a brief description of the most important norms for signals and systems which are useful in optimal and robust control.

2.1 Signal norms

The L_2 norm

For a scalar-valued signal $v(t)$ defined for $t \geq 0$ the L_2-norm is defined as the square root of the integral of $v(t)^2$,

$$\|v\|_2 = \left(\int_0^\infty v(t)^2 dt \right)^{1/2} \tag{2.1}$$

A physical interpretation of the L_2 norm is that if $v(t)$ represents a voltage or a current, then $\|v\|_2^2$ is proportional to the total energy associated with the signal.

Recall the Laplace-transform,

$$\hat{v}(s) = \int_0^\infty v(t) e^{-st} dt \tag{2.2}$$

In analogy with (2.1), we can define an L_2-norm for the Laplace-transformed signal $\hat{v}(s)$ on the imaginary axis according to

$$\|\hat{v}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^\infty |\hat{v}(j\omega)|^2 d\omega \right)^{1/2} \tag{2.3}$$

where the factor $1/(2\pi)$ has been introduced for convenience.

By *Parseval's theorem*, the time-domain and frequency-domain L_2-norms (2.1) and (2.3) are equal,

$$\|v\|_2 = \|\hat{v}\|_2 \tag{2.4}$$

For a vector-valued signal $v(t) = [v_1(t), \ldots, v_m(t)]^T$, the L_2 norm generalizes in a natural way to

$$\|v\|_2 = \left(\sum_{i=1}^{m} \|v_i\|_2^2\right)^{1/2} = \left(\int_0^\infty \sum_{i=1}^{m} v_i(t)^2 dt\right)^{1/2} = \left(\int_0^\infty v(t)^T v(t) dt\right)^{1/2} \tag{2.5}$$

and, similarly for the Laplace-domain signal $\hat{v}(s) = [\hat{v}_1(s), \ldots, \hat{v}_m(s)]^T$,

$$\begin{aligned}
\|\hat{v}\|_2 &= \left(\sum_{i=1}^{m} \|\hat{v}_i\|_2^2\right)^{1/2} \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{i=1}^{m} |\hat{v}_i(j\omega)|^2 dt\right)^{1/2} \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{i=1}^{m} \hat{v}_i(-j\omega)\hat{v}_i(j\omega) dt\right)^{1/2} \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{v}(-j\omega)^T \hat{v}(j\omega) dt\right)^{1/2}
\end{aligned} \tag{2.6}$$

The L_p norms

The L_2-norm is a special case of the L_p-norms, defined as

$$\|v\|_p = \left(\int_0^\infty |v(t)|^p dt\right)^{1/p}, \quad p \geq 1 \tag{2.7}$$

In particular, the L_1-norm is the integral of the absolute value,

$$\|v\|_1 = \int_0^\infty |v(t)| dt \tag{2.8}$$

The L_p-norms can be defined in the Laplace domain in analogy with (2.3), but for $p \neq 2$, no relation corresponding to Parseval's theorem exists between the time-domain and s-domain norms.

Remark 2.1. The notation L in L_p refers to the fact that the integrand in (2.7) should be *Lebesgue*-integrable for the integral to exist. This is a generalization of the standard (Riemann) integral to a more general class of functions.

Remark 2.2. Function which have a bounded L_p norm belong to the linear space L_p. It is sometimes important to distinguish whether the norm is defined on the positive real axis R^+ or on the imaginary axis jR. Thus, the function $v(t)$ in (2.1) belongs to the Lebesgue space $L_2(R^+)$, and the function $\hat{v}(s)$ in (2.3) belongs to the Lebesgue space $L_2(jR)$.

The ∞-norm

As $p \to \infty$, the L_p norm tends to the so-called ∞-norm, or L_∞ norm, which can be characterized as

$$\|v\|_\infty = \max_t |v(t)| \tag{2.9}$$

supposing that the maximum exists. However, there is in general no guarantee that the maximum in (2.9) exists, and therefore the correct way is to define the L_∞ norm as the least upper bound (or *supremum*) of the absolute value,

$$\|v\|_\infty = \sup_t |v(t)| \tag{2.10}$$

Example 2.1. The function

$$f(x) = \begin{cases} x, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases} \tag{2.11}$$

has no maximum, because for any $x = x_1$, $x_1 = 1 - \epsilon$ say (where $\epsilon > 0$), one can select another value x_2, such that $f(x_2) > f(x_1)$ (take for example $x_2 = 1 - \epsilon/2$). Therefore, one instead introduces the *least upper bound* or *supremum* defined as the least number M which satisfies $M \geq f(x)$ for all x, i.e.,

$$\sup_x f(x) = \min\{M : f(x) \leq M, \text{ all } x\} \tag{2.12}$$

For the function in (2.11), $\sup_x f(x) = 1$. Notice that the minimum in (2.12) exists.

The *greatest lower bound* or *infimum* is defined in an analogous way. The function in (2.11) does not have a minimum, but its greatest lower bound is $\inf_x f(x) = -1$.

Remark 2.3. A more correct way would be to write the L_∞-norm as

$$\|v\|_\infty = \text{ess}\sup_t |v(t)| \tag{2.13}$$

indicating that points of measure zero (i.e., isolated points) are excluded when taking the supremum or maximum. However, the functions normally treated in this course are continuous, and therefore we will use the definitions (2.9) or (2.10).

2.2 System norms

2.2.1 The H_2 norm

For a stable SISO linear system with transfer function $G(s)$, the H_2-norm is defined in analogy with (2.3) as

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega\right)^{1/2} \qquad \text{(SISO)} \tag{2.14}$$

For a *multivariable* system with transfer function matrix $G(s) = [g_{kl}(s)]$, the definition generalizes to

$$\begin{aligned} \|G\|_2 &= \left(\sum_{kl} \|g_{kl}\|_2^2\right)^{1/2} \\ &= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{kl} |g_{kl}(j\omega)|^2 d\omega\right)^{1/2} \end{aligned}$$

$$= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{kl} g_{kl}(-j\omega) g_{kl}(j\omega) d\omega \right)^{1/2}$$

$$= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}[G(-j\omega)^T G(j\omega)] d\omega \right)^{1/2} \quad (2.15)$$

In the last equality, we have used the result (2.16) below.

Problem 2.1. Show that for $n \times m$ matrices $A = [a_{kl}]$ and $B = [b_{kl}]$,

$$\mathrm{tr}[A^T B] = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{kl} b_{kl} = a_{11} b_{11} + \cdots + a_{nm} b_{nm} \quad (2.16)$$

Notice in particular that in the case $A = B$, equation (2.16) gives the sum of the squares of the elements of a matrix,

$$\mathrm{tr}[A^T A] = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{kl}^2 = a_{11}^2 + \cdots + a_{nm}^2 \quad (2.17)$$

Remark 2.4 The notation H in H_2 (instead of L) is due to the fact that the function spaces which, in addition to having finite L_p norms on the imaginary axis (cf. equation (2.3)), are bounded and analytic functions in the right-half plane (i.e. with no poles in the RHP), are called *Hardy spaces* H_p. Thus, stable transfer functions belong to these spaces, provided the associated integral is finite. The spaces are called after the British pure mathematician G. H. Hardy (1877–1947).

State-space computation of the H_2 norm

Rather than evaluating the integrals in (2.14) or (2.15) directly, the H_2 norm of a transfer function is conveniently calculated in the time-domain. In particular, assume that the stable transfer function $G(s)$ has state-space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) \\ z(t) &= Cx(t) \end{aligned} \quad (2.18)$$

so that

$$G(s) = C(sI - A)^{-1} B \quad (2.19)$$

Integrating (2.18) from 0 to t gives for $z(t)$,

$$z(t) = Ce^{At} x(0) + \int_0^t H(t - \lambda) v(\lambda) d\lambda \quad (2.20)$$

where $H(t - \lambda)$ is the impulse response function defined as

$$H(\tau) = \begin{cases} Ce^{A\tau} B, & \text{if } \tau \geq 0 \\ 0, & \text{if } \tau < 0 \end{cases} \quad (2.21)$$

Problem 2.2. Verify (2.20).

It is straightforward to show that $G(s)$ is simply the Laplace-transform of the impulse response function $H(\tau)$,

$$G(s) = \int_0^\infty H(\tau) e^{-s\tau} d\tau \tag{2.22}$$

Problem 2.3. Verify (2.22).

From Parseval's theorem applied to $H(t)$ and its Laplace transform $G(s)$ it then follows that $\|G_2\|$ equals the corresponding time-domain norm of the impulse response function H,

$$\|G\|_2 = \|H\|_2 \tag{2.23}$$

where

$$\|H\|_2 = \left(\int_0^\infty \sum_{kl} h_{kl}(t)^2 dt \right)^{1/2} = \left(\int_0^\infty \mathrm{tr}[H(t)^T H(t)] dt \right)^{1/2} \tag{2.24}$$

Notice that $\|H\|_2$ is finite if the system is stable, i.e., all eigenvalues of the matrix A have strictly negative real parts. By (2.23), we can evaluate the H_2 norm in (2.15) by computing the time-domain norm in (2.24). From (2.21) and (2.24) we have

$$\|G\|_2^2 = \|H\|_2^2 = \mathrm{tr}[C \int_0^\infty e^{At} B B^T e^{A^T t} dt C^T] \tag{2.25}$$

Defining the matrix

$$P = \int_0^\infty e^{At} B B^T e^{A^T t} dt \tag{2.26}$$

equation (2.25) can be written

$$\|H\|_2^2 = \mathrm{tr}[CPC^T] \tag{2.27}$$

It turns out that the matrix P is the unique solution to the linear matrix equation

$$AP + PA^T + BB^T = 0 \tag{2.28}$$

Equation (2.28) is known as a *matrix Lyapunov equation*, due to the fact that it is associated with Lyapunov stability theory of linear systems.

The fact that P is the solution to (2.28) can be shown by observing that

$$\frac{d}{dt} e^{At} B B^T e^{A^T t} = A e^{At} B B^T e^{A^T t} + e^{At} B B^T e^{A^T t} A^T \tag{2.29}$$

and integrating both sides from 0 to ∞. The left-hand side gives

$$\int_0^\infty \frac{d}{dt} e^{At} B B^T e^{A^T t} dt = e^{At} B B^T e^{A^T t} \Big|_0^\infty = -BB^T \tag{2.30}$$

where the fact has been used that $\exp(At)$, due to stability, converges to zero as $t \to \infty$. On the right-hand side of (2.29) the definition of P gives

$$\int_0^\infty [A e^{At} B B^T e^{A^T t} + e^{At} B B^T e^{A^T t} A^T] dt = AP + PA^T \tag{2.31}$$

and thus (2.28) follows.

There are efficient numerical methods for solving the linear matrix equation (2.28). In `MATLAB`s control toolboxes (2.28) can be solved using the routine `lyap`.

Signal interpretations of the H_2 norm

It is useful to interpret the H_2 norm defined for a system transfer function in terms of the input and output signals of the system.

Consider the linear system

$$\hat{z}(s) = G(s)\hat{v}(s) \tag{2.32}$$

Assuming a SISO system, suppose that the input has the transform

$$\hat{v}(s) = 1 \tag{2.33}$$

implying that it contains equal amounts of all frequencies (because $\hat{v}(j\omega) = 1$). Then the output has the transform

$$\hat{z}(s) = G(s) \tag{2.34}$$

and according to (2.3), its L_2 norm equals

$$\|\hat{z}\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega\right)^{1/2} = \|G\|_2 \tag{2.35}$$

Hence $\|G\|_2$ can be interpreted as an average system gain taken over all frequencies.

The above result is generalized to the MIMO case by considering the effect of one input at a time. Let the kth input have a constant Laplace transform equal to one, while the other inputs are zero. Then $\hat{v}(s)$ can be written as

$$\hat{v}(s) = e_k \tag{2.36}$$

where e_k denotes the kth unit vector,

$$e_k = [0 \cdots 0 \;\; 1 \;\; 0 \cdots 0]^T \tag{2.37}$$

where the 1 is in the kth position. With this input, the output is $Z(s) = G(s)e_k$ and the square of its L_2 norm is given by

$$\begin{aligned} \|\hat{z}\|_2^2 &= \|Ge_k\|_2^2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e_k^T G(-j\omega)^T G(j\omega) e_k d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}[G(-j\omega) e_k e_k^T G(j\omega)^T] d\omega \end{aligned} \tag{2.38}$$

where we have used the identity

$$\mathrm{tr}[AB] = \mathrm{tr}[A^T B^T] \tag{2.39}$$

Now let each of the inputs in turn have constant Laplace transform, and form the sum of the squares of the L_2 norms of the resulting outputs. This gives

$$\begin{aligned}
\sum_{k=1}^{m}\left[\|\hat{z}\|_2^2 : \hat{v}(s)=e_k\right] &= \sum_{k=1}^{m}\|Ge_k\|_2^2 \\
&= \frac{1}{2\pi}\sum_{k=1}^{m}\int_{-\infty}^{\infty}\mathrm{tr}[G(-j\omega)e_ke_k^TG(j\omega)^T]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{tr}[G(-j\omega)\sum_{k=1}^{m}(e_ke_k^T)G(j\omega)^T]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{tr}[G(-j\omega)IG(j\omega)^T]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{tr}[G(-j\omega)^TG(j\omega)]d\omega \\
&= \|G\|_2^2 \qquad (2.40)
\end{aligned}$$

In the time-domain, the H_2 norm can be interpreted in a similar way by observing that a function with constant Laplace-transform is the (Dirac's) delta function with the properties

$$\delta(t) = \begin{cases}\infty, & \text{if } t=0 \\ 0, & \text{if } t\neq 0\end{cases} \qquad (2.41)$$

and

$$\int_{-\infty}^{\infty}\delta(t)dt = 1 \qquad (2.42)$$

With the input $v(t)=e_k\delta(t)$ we have the output (cf. (2.20))

$$\begin{aligned}
z(t) &= \int_0^t H(t-\lambda)e_k\delta(\lambda)d\lambda \\
&= H(t)e_k\int_0^t\delta(\lambda)d\lambda \\
&= H(t)e_k \qquad (2.43)
\end{aligned}$$

where (2.41) and (2.42) have been used. Hence

$$\begin{aligned}
\int_0^{\infty}z(t)^Tz(t)dt &= \int_0^{\infty}e_k^TH(t)^TH(t)e_kdt \\
&= \int_0^{\infty}\mathrm{tr}[H(t)e_ke_k^TH(t)^T]dt \qquad (2.44)
\end{aligned}$$

and it follows that

$$\begin{aligned}
\sum_{k=1}^{m}\left[\int_0^{\infty}z(t)^Tz(t)dt : v=e_k\delta(t)\right] &= \sum_{k=1}^{m}\int_0^{\infty}\mathrm{tr}[H(t)e_ke_k^TH(t)^T]dt \\
&= \int_0^{\infty}\mathrm{tr}[H(t)\sum_{k=1}^{m}(e_ke_k^T)H(t)^T]dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \mathrm{tr}[H(t) I H(t)^T] dt \\
&= \int_0^\infty \mathrm{tr}[H(t)^T H(t)] dt \\
&= \|H\|_2^2
\end{aligned}
\tag{2.45}
$$

It is also worth while to observe that the H_2 norm has a particularly nice interpretation in a stochastic framework. Without going into details of the (rather difficult) theory of continuous-time stochastic systems, let us note that if the input v is a (vector-valued) white noise signal with unit covariance matrix, $R_v = I$, then the sum of the stationary variances of the outputs is given by the square of the H_2 norm of the transfer function, i.e.,

$$
\lim_{t_f \to \infty} E[\frac{1}{t_f} \int_0^{t_f} z^T(t) z(t) dt] = \|G\|_2^2 \tag{2.46}
$$

Recalling that white noise contains equal amounts of all frequencies, it is understood that the characterization (2.46) is simply the stochastic counterpart of the characterizations in (2.40) and (2.45).

2.2.2 The H_∞ norm

In addition to the H_2 norm, which we have seen gives a characterization of the average gain of a system, a perhaps more fundamental norm for systems is the H_∞ norm, which provides a measure of a worst-case system gain.

The H_∞ norm for SISO systems

Consider a stable SISO linear system with transfer function $G(s)$. The H_∞ norm is defined as

$$
\|G\|_\infty = \max_\omega |G(j\omega)| \qquad \text{(SISO)} \tag{2.47}
$$

or, in the event that the maximum may not exist, more correctly as

$$
\|G\|_\infty = \sup_\omega |G(j\omega)| \qquad \text{(SISO)} \tag{2.48}
$$

See also Remark 2.3. Recalling that $|G(j\omega)|$ is the factor by which the amplitude of a sinusoidal input with angular frequency ω is magnified by the system, it is seen that the H_∞ norm is simply a measure of the largest factor by which any sinusoid is magnified by the system.

A very useful interpretation of the H_∞ norm is obtained in terms of the effect of G on the space of inputs with bounded L_2 norms. (Notice that a sinusoidal signal does not have bounded L_2 norm because the integral in (2.1) is unbounded.) Let $v(t)$ be a signal with Laplace-transform $\hat{v}(s)$ such that the L_2 norm given by (2.1) or (2.3) is bounded. Then the system output $\hat{z}(s) = G(s)\hat{v}(s)$ has L_2 norm given by (2.3) which is bounded above by $\|G\|_\infty \|\hat{v}\|_2$, because

$$
\|G\hat{v}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^\infty |G(j\omega)\hat{v}(j\omega)|^2 d\omega \right)^{1/2}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 |\hat{v}(j\omega)|^2 d\omega\right)^{1/2} \\
&\le \sup_\omega |G(j\omega)| \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{v}(j\omega)|^2 d\omega\right)^{1/2} \\
&= \|G\|_\infty \|\hat{v}\|_2
\end{aligned}
\tag{2.49}
$$

Hence

$$
\|G\|_\infty \ge \frac{\|G\hat{v}\|_2}{\|\hat{v}\|_2}, \quad \text{all } \hat{v} \ne 0 \tag{2.50}
$$

In fact there exist signals v which come arbitrarily close to the upper bound in (2.49). This can be understood by letting v be such that its Laplace-transform on the imaginary axis $\hat{v}(j\omega)$ is concentrated to a frequency range where $|G(j\omega)|$ is arbitrarily close to $\|G\|_\infty$, and with $\hat{v}(j\omega)$ equal to zero elsewhere. Then it follows that the H_∞ norm can be characterized as

$$
\|G\|_\infty = \sup\left\{\frac{\|G\hat{v}\|_2}{\|\hat{v}\|_2} : \hat{v} \ne 0\right\} \tag{2.51}
$$

Hence the H_∞ norm gives the maximum factor by which the system magnifies the L_2 norm of any input. Therefore, $\|G\|_\infty$ is also called the *gain* of the system. In operator theory, an operator norm like that in (2.51) is called an *induced norm.* The H_∞ norm is an operator norm which induced by the L_2 norm.

The H_∞ norm for MIMO systems

For multivariable systems, the H_∞ norm is defined in an analogous way. Let's first see how the definition in (2.47) could be extended to the multivariable case. The SISO gain $|G(j\omega)|$ at a given frequency should then be generalized to the multivariable case. For an $n \times m$ transfer function matrix $G(s)$, a natural way to achieve this is to introduce the maximum gain of $G(j\omega)$ at the frequency ω. For this purpose, introduce the Euclidean norm $\|v\|$ of a complex-valued vector $v = [v_1, \ldots, v_m]^T \in C^m$,

$$
\|v\| = \left(|v_1|^2 + \cdots + |v_m|^2\right)^{1/2} \tag{2.52}
$$

The maximum gain of G at the frequency ω is then given by the quantity

$$
\begin{aligned}
\|G(j\omega)\| &= \max_v \left\{\frac{\|G(j\omega)v\|}{\|v\|} : v \ne 0, v \in C^m\right\} \\
&= \max_v \left\{\|G(j\omega)v\| : \|v\| = 1, v \in C^m\right\}
\end{aligned}
\tag{2.53}
$$

In analogy with (2.47), or (2.48), the H_∞ norm of the transfer function matrix $G(s)$ is defined as

$$
\|G\|_\infty = \sup_\omega \|G(j\omega)\| \tag{2.54}
$$

where $\|G(j\omega)\|$ is given by (2.53). It will be shown below that the matrix norm $\|G(j\omega)\|$ is equal to the *maximum singular value* $\bar{\sigma}(G(j\omega))$ of the matrix $G(j\omega)$. Therefore the H_∞ norm is often expressed as

$$
\|G\|_\infty = \sup_\omega \bar{\sigma}(G(j\omega)) \tag{2.55}
$$

In analogy with (2.48), (2.54) can be interpreted in terms of the effect that the system G has on sinusoidal inputs. Let the input be a vector-valued sinusoidal given by

$$v(t) = [a_1 \sin(\omega t + \alpha_1), \cdots, a_m \sin(\omega t + \alpha_m)]^T \tag{2.56}$$

The output $z = Gv$ is then another vector-valued sinusoid with the same angular frequency ω as the input, but with components whose magnitudes and phases are transformed by the system,

$$z(t) = [c_1 \sin(\omega t + \phi_1), \cdots, c_n \sin(\omega t + \phi_n)]^T \tag{2.57}$$

Expressing the magnitude of the sinusoidal vectors in analogy with the Euclidean norm,

$$\begin{aligned} \|v\| &= \left(a_1^2 + \cdots + a_m^2\right)^{1/2} \\ \|z\| &= \left(c_1^2 + \cdots + c_n^2\right)^{1/2} \end{aligned} \tag{2.58}$$

it can be shown that (2.54) is equal to the quantity

$$\|G\|_\infty = \sup_\omega \max_{\{a_i\},\{\alpha_i\}} \left\{ \frac{\|z\|}{\|v\|} : z = Gv, v \text{ given by } (2.56) \right\} \tag{2.59}$$

Hence the H_∞ norm is the maximum factor by which the magnitude of any vector-valued sinusoidal input, as defined by (2.56), is magnified by the system. The proof of (2.59) involves a bit of algebraic manipulations, and it is left as an exercise.

A more important characterization of the H_∞ norm is in terms of the effect on signals with bounded L_2 norms. Let v denote a signal with bounded L_2 norm, and with Laplace-transform $\hat{v}(s)$. Then the L_2 norm of the system output $\hat{z}(s) = G(s)\hat{v}(s)$ is bounded in analogy with (2.49). Using (2.6),

$$\begin{aligned} \|G\hat{v}\|_2 &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{v}(-j\omega)^T G(-j\omega)^T G(j\omega)\hat{v}(j\omega) d\omega \right)^{1/2} \\ &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(j\omega)\hat{v}(j\omega)\|^2 d\omega \right)^{1/2} \\ &\leq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} [\|G(j\omega)\|\|\hat{v}(j\omega)\|]^2 d\omega \right)^{1/2} \qquad \text{(cf. (2.53))} \\ &\leq \sup_\omega \|G(j\omega)\| \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\hat{v}(j\omega)\|^2 d\omega \right)^{1/2} \\ &= \|G\|_\infty \|\hat{v}\|_2 \end{aligned} \tag{2.60}$$

Just as in the SISO case, the upper bound in (2.60) is tight. Suppose that the Laplace transform $\hat{v}(s)$ is concentrated to a frequency range where $\|G(j\omega)\|$ is arbitrary close to $\|G\|_\infty$, and with components such that $\|G(j\omega)\hat{v}(j\omega)\|/\|\hat{v}(j\omega)\|$ is arbitrary close to $\|G(j\omega)\|$. Then it is understood that (2.51) holds for the multivariable case as well, and the H_∞ norm is the operator norm induced by the L_2 norm, i.e.

$$\|G\|_\infty = \sup \left\{ \frac{\|G\hat{v}\|_2}{\|\hat{v}\|_2} : \hat{v} \neq 0 \right\} \tag{2.61}$$

The definition of the H_∞ norm above has made use of the matrix norm defined in (2.53). The characterization of this norm, and in particular its relation to the singular values of a matrix, is discussed in more detail in Section 2.2.3.

State-space computation of the H_∞ norm

The characterizations (2.51) and (2.61) provide a very useful method to evaluate the H_∞ norm by state-space formulae. Let $G(s)$ have the state-space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) \\ z(t) &= Cx(t) + Dv(t) \end{aligned} \tag{2.62}$$

so that

$$G(s) = C(sI - A)^{-1}B + D \tag{2.63}$$

By Parseval's theorem, the H_∞ norm can be characterized in the time domain as

$$\|G\|_\infty = \sup \left\{ \frac{\|z\|_2}{\|v\|_2} : v \neq 0 \right\} \tag{2.64}$$

It follows that for any $\gamma > 0$, $\|G\|_\infty < \gamma$ if and only if

$$\begin{aligned} J_\infty(G,\gamma) :=& \max_v \left[\|z\|_2^2 - \gamma^2 \|v\|_2^2 \right] \\ =& \max_v \int_0^\infty \left[z^T(t)z(t) - \gamma^2 v^T(t)v(t) \right] dt \\ <& \infty \end{aligned} \tag{2.65}$$

But the maximization problem in (2.65) can be solved using standard LQ theory, with the modification that we have a maximization problem, and a negative weight on the input v. Thus, we can check whether $\|G\|_\infty < \gamma$ holds by checking whether the LQ-type maximization problem in (2.65) has a bounded solution. This can be checked by state-space methods. We have the following characterization.

Theorem 2.1 *The system $G(s)$ with state-space representation (2.62) has H_∞-norm less than γ, $\|G\|_\infty < \gamma$, if and only if $\gamma^2 I - D^T D > 0$ and the Riccati equation*

$$A^T S_\infty + S_\infty A + (S_\infty B + C^T D)(\gamma^2 I - D^T D)^{-1}(B^T S_\infty + D^T C) + C^T C = 0 \tag{2.66}$$

has a bounded positive semidefinite solution S_∞ such that the matrix $A + B(\gamma^2 I - D^T D)^{-1}(B^T S_\infty + D^T C)$ has all eigenvalues in the left half plane.

The characterization in Theorem 2.1 can be used to determine the H_∞ norm to any degree of accuracy by iterating on γ. The characterization plays a central role in the state-space solution of the H_∞ control problem, which will be considered in later chapters.

A complete proof of the result of Theorem 2.1 is a bit lengthy, and at this stage we will not go into details. The idea is, however, straightforward. Sufficiency is proved by assuming that the Riccati equation (2.66) has a solution. It then follows from standard LQ optimal control theory that the maximum in (2.65) is bounded and is given by

$$J_\infty(G, \gamma) = x^T(0) S_\infty x(0) \tag{2.67}$$

Moreover, the maximizing disturbance input is given by

$$v_{worst}(t) = (\gamma^2 I - D^T D)^{-1}(B^T S_\infty + D^T C)x(t) \tag{2.68}$$

Necessity, i.e., that $J_\infty(G, \gamma) < \infty$ implies that the Riccati equation has a solution, is more difficult to prove. One way to do this is to consider (2.65) as the limit of a finite-horizon maximization problem from $t = 0$ to $t = T$ as $T \to \infty$. In particular, if the bound (2.65) does not hold, then the finite-horizon problem will result in a Riccati differential equation solution which blows up for some finite time t, in which case the stationary Riccati equation (2.66) has no positive semidefinite solution.

2.2.3 The singular-value decomposition (SVD)

The matrix norm in (2.53) is related to the so-called singular-value decomposition (SVD) of a matrix. The SVD has many useful applications in all kinds of multivariable analyses involving matrices, an important example being various least-squares problems. It is therefore well motivated to devote a few paragraphs to introduce the concept properly.

For simplicity, we consider first real matrices. For a real m-vector $x = [x_1, \ldots, x_m]^T \in R^m$ we define the Euclidean norm

$$\|x\| = (x^T x)^{1/2} = \left(x_1^2 + \cdots + x_m^2\right)^{1/2} \tag{2.69}$$

For a real $n \times m$ matrix A the induced matrix norm associated with the Euclidean vector norm is given by

$$\begin{aligned} \|A\| &= \max\left\{\frac{\|Ax\|}{\|x\|} : x \neq 0, x \in R^m\right\} \\ &= \max\left\{\|Ax\| : \|x\| = 1, x \in R^m\right\} \end{aligned} \tag{2.70}$$

The singular-value decomposition of a matrix is defined as follows.

Singular-value decomposition (SVD). Consider a real $n \times m$ matrix A. Let $p = \min(m, n)$. Then there exist an $m \times p$ matrix V with orthonormal columns

$$V = [v_1, v_2, \ldots, v_p], \;\; v_i^T v_i = 1, \;\; v_i^T v_j = 0 \text{ if } i \neq j, \tag{2.71}$$

an $n \times p$ matrix U with orthonormal columns,

$$U = [u_1, u_2, \ldots, u_p], \;\; u_i^T u_i = 1, \;\; u_i^T u_j = 0 \text{ if } i \neq j, \tag{2.72}$$

and a diagonal matrix Σ with non-negative diagonal elements,

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p), \;\; \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0, \tag{2.73}$$

such that A can be written as

$$A = U\Sigma V^T \tag{2.74}$$

Such a decomposition is called the singular-value decomposition of A. The non-negative scalars σ_i are the *singular values* of A, the vector u_i is the ith left singular vector, and v_j is the jth right singular vector of A.

Equation (2.74) means that the matrix A takes the vectors $v_i \in R^m, i = 1, \ldots, p$ to $u_i \in R^n$ multiplied by σ_i,

$$\begin{aligned} Av_i &= [\, u_1, \ldots, u_p \,] \,\mathrm{diag}(\sigma_1, \ldots, \sigma_p) \begin{bmatrix} v_1^T \\ \vdots \\ v_p^T \end{bmatrix} v_i \\ &= \sum_{k=1}^{p} \sigma_k u_k v_k^T v_i \\ &= \sigma_i u_i \end{aligned} \tag{2.75}$$

As both the vectors v_i and u_i have unit Euclidean norms, cf. (2.71) and (2.72), and as any vector $x \in R^m$ can be expressed in terms of a linear combination of orthonormal vectors $\{v_1, \ldots, v_p, \ldots, v_m\}$, it follows that the maximum factor by which any vector is magnified is equal to the maximum singular value σ_1, with v_1 magnified by exactly this factor. Hence the induced norm $\|A\|$ in (2.70) is equal to σ_1. The maximum singular value of a matrix is often denoted by $\bar{\sigma}(A)$, i.e., we have

$$\|A\| = \bar{\sigma}(A) = \sigma_1 \tag{2.76}$$

The singular-value decomposition generalizes trivially to *complex-valued matrices*, such as the transfer function matrix $G(j\omega)$. For a complex matrix G, the singular-value decomposition is given by

$$G = U\Sigma V^H \tag{2.77}$$

where V^H stands for the complex conjugate transpose of V (i.e., the transpose of the matrix with elements which are the complex conjugates of the elements of V). For complex matrices, the singular vectors U and V are also complex, but it is important to observe that the singular values σ_i are still *real and non-negative.*

Remark 2.5. The singular-value decomposition is important in control structure selection in multivariable control, since it defines the system gains in various directions. One speaks about the *high-gain direction* associated with the largest singular value σ_1 of G, and the *low-gain direction* associated with the smallest singular value σ_p of G.

As mentioned above, the singular-value decomposition has many important applications, for example in least-squares analysis. It is, however, not our intention to pursue these topics here. There are efficient numerical algorithms for the singular-value decomposition. An implementation is found in the program `svd` in `MATLAB`.

The singular-value decomposition is not hard to prove, and due to its importance, we give below an outline of how the result (2.74) can be shown.

Proof of (2.74). For simplicity, assume that A is square, i.e. $m = n$. This assumption is not restrictive, since A can always be extended by adding zero columns/rows in order to make it square, and the added zero columns/rows can be removed in the end. The orthonormality of the columns of V and U imply

$$V^TV = I, \;\; U^TU = I \tag{2.78}$$

As V and U are square, this is equivalent to

$$V^{-1} = V^T, \;\; U^{-1} = U^T \tag{2.79}$$

Our goal is to show that A has the decomposition (2.74). By (2.79), (2.74) is equivalent to

$$U^TAV = \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \tag{2.80}$$

Denote $\|A\| = \sigma$, and let x and y be vectors with unit Euclidean norms, $\|x\| = \|y\| = 1$, and such that $Ax = \sigma y$, i.e., the magnification $\|A\|$ is achieved for x. Introduce the $n \times n$ matrices

$$V = [x, V_1] \quad \text{and} \quad U = [y, U_1] \tag{2.81}$$

such that the columns are orthonormal, i.e., $V^TV = I$ and $U^TU = I$, and hence $V^{-1} = V^T$ and $U^{-1} = U^T$. It is always possible to determine V_1 and U_1 in such a way that this holds, although the choice is in general not unique. The matrix $A_1 = U^TAV$ then has the structure

$$\begin{aligned} A_1 &= U^TAV = \begin{bmatrix} y^T \\ U_1^T \end{bmatrix} A\,[\,x \quad V_1\,] \\ &= \begin{bmatrix} y^T \\ U_1^T \end{bmatrix} [\,\sigma y \quad AV_1\,] \\ &= \begin{bmatrix} \sigma & w^T \\ 0 & B \end{bmatrix} \end{aligned} \tag{2.82}$$

where we have used the assumption $Ax = \sigma y$, the orthonormality of U implying $y^Ty = 1$ and $U_1^Ty = 0$, and introduced the notations $w^T = y^TAV_1$ and $B = U_1^TAV_1$.

Next we will show that w in (2.82) must be zero. First notice that A and A_1 have the same norms. In order to see this notice that given any vector x, the vector x_1 defined as $x_1 = V^{-1}x$ has the same norm as x, because $\|x\|^2 = x^Tx = x_1^TV^TVx_1 = x_1^Tx_1 = \|x_1\|^2$. Moreover, $\|A_1x_1\|^2 = x_1^TA_1^TA_1x_1 = x^TV^{-T}V^TA^TUU^TAVV^{-1}x = x^TA^TAx = \|Ax\|^2$. As this holds for any x, it follows that $\|A\| = \|A_1\|$. Now, by (2.82),

$$A_1 \begin{bmatrix} \sigma \\ w \end{bmatrix} = \begin{bmatrix} \sigma^2 + w^Tw \\ Bw \end{bmatrix} \tag{2.83}$$

Hence

$$\begin{aligned} \|A_1 \begin{bmatrix} \sigma \\ w \end{bmatrix}\| &= \left((\sigma^2 + w^Tw)^2 + w^TB^TBw\right)^{1/2} \\ &\geq \sigma^2 + w^Tw \\ &= (\sigma^2 + w^Tw)^{1/2} \; \left\| \begin{bmatrix} \sigma \\ w \end{bmatrix} \right\| \end{aligned} \tag{2.84}$$

Hence $\|A_1\| \geq (\sigma^2 + w^T w)^{1/2}$. But as σ was selected such that $\sigma = \|A\| = \|A_1\|$ we must have $w = 0$. It follows that A_1 has the structure

$$A_1 = U^T A V = \begin{bmatrix} \sigma & 0 \\ 0 & B \end{bmatrix} \tag{2.85}$$

The proof of (2.80) can be completed inductively, by applying an analogous approach to B, etc.